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B. Wissenschaftliche Mitteilungen

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The Economics of Insurance: A Review and some Recent Developments

1 Introduction

Individuals and insurers often have to make choices under conditions of uncertainty concerning the outcome of a risk, i.e., some future random financial loss. A person bearing a risk may consider whether it is preferable or not to (partly) insure this risk, that is, to ask a third person, usually an insurance company, to assume (a part of) this risk. Insurance companies accept risks from their clients, the insureds, for a certain price called a premium. If a risk, or portfolio of risks, is too large for a company, it will pass on parts of it to one or several other companies, its reinsurers; that part which finally remains with the first company is called its retention. Therefore, an insurer may be faced with the problem of finding an optimal reinsurance program. These are but two examples of a more general class of problems where a decision-maker has to choose between several future random levels of his fortune. Of course, such choices depend on many things. In the first place, a wealthy company can clearly afford to retain more for its own account than a poor one. Secondly, it depends whether management is at all willing to take risks: a conservative manager will display little risk willingness, contrary to a courageous or even foolhardy entrepreneur. The same kind of remarks of course hold for insureds. Furthermore, both the insured and the underwriter will have to make up their mind on the premium: is the amount of premium reasonable or acceptable compared to the risk transferred?
In this paper, we present different approaches to decision making under risk. These approaches have in common that the preference relations of a decision-maker, which are qualitative in nature, follow from simple comparisons of numerical quantities to be associated to the alternative choices under consideration. The first approach is the classical expected utility theory. In this framework, a utility function u assigns a value $u(x)$ to a monetary amount of \$ x; u describes "how much the decision-maker appreciates a fortune of \$ x". Utility functions are of a subjective nature, they reflect the preferences of individuals or insurance companies. Moreover, different individuals usually have different utility functions, even if all the reasonable utility functions have to share some common properties,

such as non-decreasingness (which translates the lure of profit). Expected utility theory has greatly contributed to understanding the economics of risk and uncertainty for the past several decades. For instance, expected utility theory has been used in order to determine the optimal forms of insurance contracts and reinsurance treaties (see, for instance, Denuit and Vermandele (1998, 1999) and the references therein), optimal insurance policies in the presence of adverse selection or moral hazard, optimal insurance versus precautionary saving, and so on. This theory has also been linked to stochastic dominance relations expressing common preferences of classes of reasonable decision-makers. The second approach we will present is Yaari's (1987) dual theory for choice under risk. Yaari developed a parallel theory of risk by modifying the independence axiom of von Neumann and Morgenstern (1947). In Yaari's theory, attitudes towards risks are characterized by a distortion applied to probability distribution functions, in contrast to expected utility theory in which attitudes towards risks are characterized by a utility function of wealth. In Yaari's framework, the concept of distortion function emerges; distortion functions can be considered as the parallel to the concept of utility function in the classical expected utility theory. As in the classical expected utility approach, Yaari's theory also generates some classes of stochastic orderings termed as inverse stochastic dominance orderings by Muliere and Scarsini (1989). Our purpose is to describe and compare these two theories of decision making under risk, as well as to enlighten their applicability to actuarial problems.

The paper is organized as follows. In Section 2, we present the classical expected utility theory. Section 3 is devoted to the notion of risk aversion in this framework, while Section 4 deals with the potential use of expected utility theory in insurance business. Most results summarized in these three sections are known for a (very) long time. The reason why we provide a detailed account of these in the present paper is that they will precisely allow the reader to compare expected utility theory with Yaari's approach of choice under risk. Then, Section 5 introduces the concept of associated utility function while Section 6 gives a first extension of expected utility theory, namely the approach based on integral stochastic orderings. In Section 7, we examine a somewhat new application of expected utility theory in actuarial sciences, namely in the study of the extreme dependence structures among correlated risks. Section 8 introduces the distorted expected hypothesis and details Yaari's dual theory for choice under risk. Sections 9, 10, 11 and 12 try to stress the differences and similarities of the two approaches. Finally, Section 13 expands on the application presented in Section 7. The very last Section 14 briefly presents a new approach mixing the two theories. The latter will certainly be applied in actuarial sciences.

The present paper is of pedagogical nature and is far from being exhaustive. The interested reader will find in the references a lot of material to continue his investigations. We also mention the papers by Gerber (1987) and Gerber and Pafumi (1998) reviewing expected utility and related topics in an actuarial setting.

2 The Expected Utility Hypothesis

Consider a decision-maker who has to choose between two uncertain future incomes modeled by the random variables X and Y. One possible methodology for making a choice among these two alternatives consists in computing their respective expectations and then select the income with the highest expectation. This simple valuation method has already been challenged by Nicholas Bernoulli as early as 1728. He posed the following problem: "A fair coin is tossed repeatedly until it lands heads. The income you receive is equal to 2^n if the first head appears on the n-th toss. How much are you willing to pay for this game?". Assuming that the coin is fair, it is easy to verify that the expected income of the gamble is equal to infinity. It has been noticed, however, that although the expected income is infinite, the maximum amount almost all decision-makers would pay to take part in the game is finite and even moderate. This seemingly paradox is known in the literature as the St. Petersburg paradox.

Gabriel Cramer (1728) and Daniel Bernoulli (1738) proposed to solve this paradox by stating that decision-makers do not base their decisions under risk on simply comparing the expectations of the incomes under consideration. Since the value of money does not solve the St. Petersburg paradox, they suggested to adopt the moral value of money as a standard of judgment. More precisely, they introduced the concept of "utility" and hypothesized that a decision-maker possesses a utility function u such that the utility (or moral value) of having a fortune of \$ x is given by $u(x)$. If the decision maker has to choose between two uncertain future incomes X and Y, he will prefer the one which leads to the higher expected utility of the future fortune. Hence, if the decision-maker's initial fortune is w, then the decision-maker is willing to play the coin tossing game for a price P if, and only if, the following inequality is satisfied:

$$u(w) \leq \sum_{n=1}^{+\infty} u(w - P + 2^n)\frac{1}{2^n}.$$

The latter inequality expresses that the decision-maker will enter the game only if the expected utility of playing the game is greater than the utility of not taking part in it. Cramer proposed that the utility $u(x)$ of a fortune of \$ x is given by the square root $\sqrt{x}$ of this fortune. Bernoulli on the other hand, suggested a logarithmic utility function $u(x) = \log x$. As an illustration, for an initial wealth of \$ 10 000 and a logarithmic utility function, the decision-maker is only willing to pay \$ 14.25 to play the St. Petersburg game, although the expected gain is infinite.

Let us associate to each random variable X its cumulative distribution function F_X, defined as $F_X(x) = P[X \leq x]$, $x \in \mathbb{R}$. In other words, $F_X(x)$ represents the probability that the random variable X assumes a value that is less than or equal to a point x of the real line $\mathbb{R}$. In the sequel, we will always assume that the distribution functions of the random variables under consideration are known. Hence, the only risk when considering a future random income is assumed to be the uncertainty about the particular outcome, not the uncertainty about its distribution function. A decision-maker is said to base his preferences on the "expected utility hypothesis" if he acts in order to maximize his expected utility. This means that there exists a real-valued function u which asserts the decision-maker's utility-of-wealth to each fortune. For all uncertain future fortunes X and Y we have that the decision-maker with utility u will prefer Y over X (denoted as $X \preceq_u Y$ in the remainder of the paper) if, and only if,

$$E[u(X)] = \int_{x=-\infty}^{+\infty} u(x)\, dF_X(x) \leq E[u(Y)] = \int_{x=-\infty}^{+\infty} u(x)\, dF_Y(x), \quad (2.1)$$

provided that the expectations exist. In other words, he will prefer fortune Y over X if the expected utility of Y exceeds the expected utility of X. A decision-maker with utility function u is said to be indifferent between X and Y (denoted as $X \approx_u Y$ in the remainder) if, and only if, equality holds in (2.1); that is, if $X \preceq_u Y$ and $Y \preceq_u X$ simultaneously hold. Since two random fortunes X and Y with the same distribution function have the same expected utility, the decision-maker will be indifferent between them.

According to von Neumann and Morgenstern (1947), expected utility theory is built up from five axioms describing rational behaviour of decision-makers. Let the binary relation $\trianglelefteq$ defined on the space of all (distribution functions of) random fortunes be such that $X \trianglelefteq Y$ if, and only if, Y is preferred over X and $X \bowtie Y$ if, and only if, the decision-maker is indifferent between X and Y. Now, consider the following axioms:

1. Axiom EU1: if X and Y are identically distributed then $X \bowtie Y$;
2. Axiom EU2: $\trianglelefteq$ is reflexive, transitive and connected;
3. Axiom EU3: $\trianglelefteq$ is continuous with respect to L_1-convergence;
4. Axiom EU4: if $F_X \geq F_Y$ then $X \trianglelefteq Y$;

5. Axiom EU5: if $X \trianglelefteq Y$ and if the distribution function of $\widetilde{X}_p$ and $\widetilde{Y}_p$ are given by

$$F_{\tilde{X}_p}(x) = pF_X(x) + (1-p)F_Z(x), \quad x \in \mathbb{R},$$

and

$$F_{\tilde{Y}_p}(x) = pF_Y(x) + (1-p)F_Z(x), \quad x \in \mathbb{R},$$

for an arbitrary distribution function F_Z, then $\widetilde{X}_p \trianglelefteq \widetilde{Y}_p$ for any $p \in [0, 1]$.

It has been shown that, if the axioms EU1–EU5 are satisfied then there must exist a utility function u such that

$$X \trianglelefteq Y \iff X \preceq_u Y \quad \text{and} \quad X \bowtie Y \iff X \approx_u Y,$$

see, e.g., Fishburn (1982).
It seems natural that an individual always prefers more wealth to less wealth. Therefore, the utility function u is always assumed to be non-decreasing. Usually, u is almost everywhere differentiable, so that the lure of gain is expressed by the condition $u^{(1)} \geq 0$ (where $u^{(s)}$ denotes the s-th derivative of the function u).
Remark that a decision maker's utility function needs only to be determined up to positive linear transformations. This follows from the fact that the utility function u^* defined by

$$u^*(x) = au(x) + b, \quad x \in \mathbb{R}, \tag{2.2}$$

for real constants $a > 0$ and b leads to the same preference structure as the utility function u (in such a case, u^* is said to be equivalent to u). Hence, it is always possible to standardize a utility function u, for example by requiring that

$$u(x_0) = 0 \quad \text{and} \quad u^{(1)}(x_0) = 1, \tag{2.3}$$

for a particular point $x_0 \in \mathbb{R}$.
The considerations above indicate that a general theory of insurance must, or at least could, be based on the utility concept. This has in fact been recognized for a long time. In 1834, Barrois constructed a very complete theory of fire insurance, based on the particular utility function $u(x) = \log(x)$, originally used by Bernoulli. It must, however, be admitted that the modern use of the utility concept in insurance literature is due to the results provided by von Neumann and Morgenstern. The expected utility theory became popular after these authors developed their axiomatic approach of it in 1947. Borch enlightened the relevance of the expected utility theory in order to solve problems in insurance; his works

were collected in two books published in 1974 and 1990. As Trowbridge (1989) pointed out, utility theory forms the philosophical basis of actuarial sciences, and yet this subject is seldom mentioned in the actuarial literature beyond chapter 1 of Bowers *et al.* (1997). For more details concerning expected utility, we refer interested readers to Huang and Litzenberger (1988), Schmidt (1998) and Panjer et al. (1998), as well as to the references therein.

3 Risk Aversion and Expected Utility

An important concept in utility theory, which is in accordance with rational behavior of insurance managers, is the notion of risk aversion. A decision-maker is said to be risk averse when his utility function is concave on its domain. If u is assumed to be almost everywhere twice differentiable, this reduces to $u^{(2)} \leq 0$. Remark that risk aversion induces some smoothness property on u, since a concave function is necessarily continuous. One way to justify the concavity assumption is to remark that it implies that the marginal utility $u^{(1)}$ is a decreasing function of wealth, or equivalently, that the increase of utility resulting from a gain of \$ Δ, $u(x+\Delta) - u(x)$, is a decreasing function of the wealth x. But this is rather an attitude towards wealth than an attitude towards risk. Therefore, the usual explanation of the meaning of risk aversion is provided by Jensen's inequality. The latter states that, given any concave function u, the inequality

$$E[u(X)] \leq u(E[X]) \tag{3.1}$$

holds for all random fortunes X. Therefore, a risk averse decision-maker always prefers a certain fortune to a random fortune income with the same expected value. As a special case of (3.1), we find that a risk averse person is never willing to accept (or is indifferent to) any actuarially fair gamble (i.e. a gamble with zero expected payoff).

To a given twice differentiable utility function u, one can associate a function r defined as

$$r(x) = \frac{-u^{(2)}(x)}{u^{(1)}(x)} = -\frac{d}{dx}\ln(u^{(1)}(x)) \tag{3.2}$$

called the risk aversion function. It is easily seen that $r \geq 0$ for any profit-seeking risk averse decision-maker. The Arrow-Pratt measure of absolute risk aversion (3.2) measures the local propensity to insure under the utility function u. Requiring that a decision-maker has a decreasing $r(\cdot)$ means that his risk premium is larger the larger the risks, i.e., the amount of money he is willing to pay in order to replace a random loss with its expected value is a decreasing function of his initial wealth, for all possible loss. Remark that in the framework of expected utility, the agent's attitude towards risk and the agent's attitude towards wealth are forever bonded together (since they are both derived from the characteristics of u): risk aversion and diminishing marginal utility of wealth are synonymous. Nevertheless, risk aversion expresses an attitude towards risk while decreasing marginal utility expresses an attitude towards wealth. In the dual theory of choice under risk proposed by Yaari (1987), we will see that these two notions are kept separate from each other.

In addition to the risk aversion function, one also defines the risk tolerance. For a twice differentiable utility function u, the function

$$\tau(x) = \frac{-u^{(1)}(x)}{u^{(2)}(x)} = \frac{1}{r(x)}$$

is called the risk tolerance function (see, e.g., Panjer *et al.* (1998), page 161). The assumption that u translates lure of profit and risk aversion (i.e. $u^{(1)} \geq 0$ and $u^{(2)} \leq 0$) implies that $\tau(x) \geq 0$. This auxiliary measure of risk has been used by Gerber and Shiu (2000) in order to study the optimal dynamic investment strategy for allocating assets in a pension plan.

If u is replaced by an equivalent function u^* satisfying (2.2) then the risk aversion function remains unchanged, i.e. $r^* \equiv r$. Moreover, if u is standardized about x_0 (as in (2.3)), u can be expressed in terms of r as

$$u(x) = \int_{x_0}^{x} \exp\Big(-\int_{x_0}^{\xi} r(\eta)\, d\eta \Big)\, d\xi\,. \tag{3.3}$$

The certainty equivalent of a random fortune X, denoted as $\mathrm{CE}[X]$, is defined as the root of the following equation:

$$u(\mathrm{CE}[X]) = E[u(X)]\,. \tag{3.4}$$

The certainty equivalent $\mathrm{CE}[X]$ of a random fortune X is therefore defined by the condition that the decision-maker is indifferent between receiving X or the fixed amount $\mathrm{CE}[X]$. From (3.1) together with (3.4), we find that the inequality

$$\mathrm{CE}[X] \leq E[X] \tag{3.5}$$

holds for any non-decreasing and concave utility function u. Therefore, a profit-seeking risk averse decision-maker always prefers certainty to uncertainty, even

if the certain income is (to a certain amount) less than the expected uncertain income.
The risk premium of X is defined as the difference between the expected fortune and the certainty equivalent, i.e.

$$\text{RP}[X] = E[X] - \text{CE}[X]. \tag{3.6}$$

As a consequence of (3.4), we have by definition that

$$u(E[X] - \text{RP}[X]) = E[u(X)]\,. \tag{3.7}$$

Formula (3.7) can be interpreted as follows: the risk premium $\text{RP}[X]$ is equal to the amount of money the individual is willing to pay in order to get a certain fortune $E[X]$ rather than the random fortune X, i.e. to replace uncertainty by certainty.

4 Expected Utility and Insurance

The central notion in actuarial mathematics is the notion of risk. A risk can be described as an event solely due to the whims of fate that may or may not take place, and that brings about some financial loss. It always contains an element of uncertainty: either the moment of its occurrence (like in life insurance), either its occurrence itself, or the nature and severity of its consequences (like in third party liability automobile insurance).
The actuary models these risks with the aid of random variables. The latter represent the random amounts of money the insurance company will have to pay out in order to indemnify the policyholder and/or the third party for the consequences of the occurrence of the insured risk. These random variables may generally be assumed non-negative with bounded support (as the upper limit to the financial loss for which the insurance company underwrites is, in most of the cases, fixed by the contract, or obtained *via* reinsurance techniques). Henceforth, we consider that a risk X is a non-negative random variable with a finite mean, representing a future financial loss.
Suppose that a profit-seeking risk averse decision-maker faces a risk X. Suppose that an insurer is willing to accept the risk X for an amount of premium P. In other words, in return for a premium P, the insurer is willing to bear the financial consequences of the claims produced by X. Let u be the utility function of the decision-maker and w be his initial wealth. We assume that the development of the decision maker's fortune during the insurance period is not influenced by any other factors than the risk and the insurance premium. According to the

expected utility hypothesis, the person is only willing to underwrite the insurance if $w - X \preceq_u w - P$, i.e.

$$u(w - P) \geq E[u(w - X)]. \tag{4.1}$$

Obviously, (4.1) is satisfied for $P = 0$. Let P^M be the supremum of all premiums P satisfying (4.1). We tacitly assume that P^M is finite (if this were not the case then the risk X should be so terrible that the risk holder is willing to pay any premium to be insured and $P^M = +\infty$). From the monotonicity and the continuity of the utility function, we find that P^M satisfies

$$u(w - P^M) = E[u(w - X)], \tag{4.2}$$

which is equivalent to $w - P^M \approx_u w - X$. Therefore, P^M is the amount of premium for which the decision-maker is indifferent between insurance and no insurance. Moreover, the inequality

$$P^M \geq E[X] \tag{4.3}$$

follows from (3.1). In conclusion, a risk averse decision-maker is willing to pay more than his expected loss to get insured. It is straightforward to verify that the risk premium the person is willing to pay, is given by

$$\mathrm{RP}[w - X] = P^M - E[X], \tag{4.4}$$

where the utility function is strictly increasing.
Let us now examine the viewpoint of a profit-seeking risk averse insurer. Assume that the insurer has a utility function $\widetilde{u}$ and an initial fortune $\widetilde{w}$. The insurer is willing to insure the risk at a premium P if $\widetilde{w} \preceq_{\widetilde{u}} \widetilde{w} + P - X$, i.e.

$$\widetilde{u}(\widetilde{w}) \leq E[\widetilde{u}(\widetilde{w} + P - X)]. \tag{4.5}$$

Formula (4.5) means that the insurer will write the policy only if his expected utility with the contract (right-hand side member) is greater than or equal to his utility without the contract (left-hand side member). Here, we have made the (unrealistic) assumption that the insurer only takes into account his initial fortune and the risk X to determine his future random fortune. Let P^m be the infimum of all premiums that satisfy the inequality (4.5); it fulfills

$$\widetilde{u}(\widetilde{w}) = E[\widetilde{u}(\widetilde{w} + P^m - X)], \tag{4.6}$$

which is equivalent to $\widetilde{w} \approx_{\widetilde{u}} \widetilde{w} + P^m - X$ and possesses an obvious intuitive explanation. From Jensen's inequality (3.1) and the monotonicity of the utility function we find that

$$P^m \geq E[X]. \tag{4.7}$$

Hence, the insurer will require a premium that is greater than or equal to the expected claim amount for covering the risk X.
Finally, we can conclude that an insurance policy is only feasible if the amount of premium relating to the contract, P say, satisfies the following inequalities

$$P^m \leq P \leq P^M \tag{4.8}$$

since such a premium fulfills the expected utility requirements (4.1) and (4.5) of both parties. In insurance practice, $P - E[X]$ is usually called the safety loading and it has, *inter alia*, to compensate the random fluctuations of the observed claims with respect to the expected claims.

5 The Associated Utility Function

Let u be a non-decreasing utility function. Then, the function v defined by

$$v(x) = -u(-x), \quad x \in \mathbb{R}, \tag{5.1}$$

is also a non-decreasing utility function. If u is the utility function of a decision-maker, then v will be said to be the associated utility function of the decision-maker under consideration. Moreover, it is easy to verify that

$$u \text{ is convex} \iff v \text{ is concave.}$$

Hence, saying that a decision-maker with utility function u is risk averse is equivalent to saying that his associated utility function v is convex.
Let us now consider a loss $X \geq 0$ almost surely, or equivalently, an income $-X$. We have that

$$E[u(w - X)] = -E[v(-w + X)]. \tag{5.2}$$

Hence, we get the following equivalence for losses X and Y (which are almost surely nonnegative)

$$E[u(w - X)] \geq E[u(w - Y)]$$
$$\iff E[v(-w + X)] \leq E[v(-w + Y)]. \tag{5.3}$$

In terms of the associated utility function, the expected utility hypotheses states that a loss X is preferred over a loss Y if, and only if,

$$E[v(-w + X)] \leq E[v(-w + Y)], \tag{5.4}$$

i.e.,

$$w - Y \preceq_u w - X \iff X - w \preceq_v Y - w\,. \tag{5.5}$$

Remark that in the expected utility approach one considers the expected utility of random variables describing fortune, income or wealth. In actuarial sciences, the key study objects are risks, i.e., negative incomes or losses. One could interpret a risk X as a negative income $-X$, and then compute the expected utility of $w - X$. Equivalently, one can compute the expected associated utility of $X - w$. The utility function u expresses the utility associated to wealth. The associated utility function v is in fact a "pain function", $v(x)$ expresses the pain associated with debt x. This interpretation makes it clear that risk aversion can be expressed as decreasing marginal utility of wealth or equivalently, increasing marginal pain of debt.

6 Stochastic Orderings Among Risks

The utility concept may be considered indispensable in theoretical work on insurance, but it does not seem to have found many applications to insurance practice. One explanation of this apparent paradox may be that presidents and executives of insurance companies find it difficult to specify the utility function which represents their preference-ordering over the set of attainable profit distributions. Another explanation may be that the expected utility model oversimplifies reality. A lot of decision problems in an insurance company involve a choice among probability distributions, but it is not certain that these decisions or choices can be studied in isolation. In simple terms, any decision may depend on the whole situation of the company, and this situation may again depend on the choices which are expected to be available in the future.

The main criticism addressed to expected utility theory is that ordering of risks depends on a subjective utility function, unknown to an objective observer. In most practical situations, it is indeed extremely difficult to find an explicit expression for a decision-maker's utility function u. Therefore, several authors suggested to focus on the common preferences shared by all the members of classes of reasonable decision-makers, and this gave rise to the theory of the integral stochastic orderings. Note that whereas each individual of the class totally orders the risk, albeit differently, their common preferences only generate a partial ordering. The application of the stochastic ordering concept in decision theory began about forty years ago (see Allais (1953), Quirk and Saposnik (1962) and Fishburn (1964)).

The preferences shared by all the decision-makers whose utility function satisfies certain reasonable conditions constitute a partial order of all risks, which can be represented as an integral stochastic ordering. Such an order is useful because it gives information about the risk-preferences of an actuary based on the distribution of the risk alone, not on the actual utility function, except that it satisfies some general conditions. More precisely we consider that a risk X is dominated by another one, Y say, when

$$X \preceq_v Y \quad \text{for all } v \text{ in a class } \mathcal{F}, \tag{6.1}$$

where the class $\mathcal{F}$ contains all the "reasonable" pain functions v. Henceforth, we always assume that if $x \mapsto v(x) \in \mathcal{F}$, then also $x \mapsto v(w+x) \in \mathcal{F}$. Dealing with such a class enables us to assume, without loss of generality, that the initial wealth w equals 0.

During the last two decades, the interest of the actuarial literature in the stochastic orderings has been growing to such a point that they become one of the most important tools to compare the riskiness of different random situations. The reader interested in actuarial applications of stochastic orderings is referred to the books by Goovaerts, Kaas, Van Heerwaarden and Bauwelinckx (1990) and by Kaas, Van Heerwaarden and Goovaerts (1994). For a general overview of this topic in applied probability, see Shaked and Shanthikumar (1994).

A first possibility consists in considering all the profit-seeking decision-makers. We then obtain the stochastic dominance $\preceq_{st}$ defined as

$$X \preceq_{st} Y \iff X \preceq_v Y$$
$$\text{for all the pain functions } v \text{ such that } v^{(1)} \geq 0. \tag{6.2}$$

Note that

$$X \preceq_{st} Y \iff -Y \preceq_u -X$$
$$\text{for all the utility functions } u \text{ such that } u^{(1)} \geq 0.$$

In other words, given two risks X and Y, saying that $X \preceq_{st} Y$ means that the loss X is preferred over the loss Y by all the profit-seeking decision-makers. It is worth mentioning that

$$X \preceq_{st} Y \iff F_X(x) \geq F_Y(x) \quad \text{for all } x \in \mathbb{R},$$

so that $X \preceq_{st} Y$ means that the probability that X assumes small values (i.e. less than x) is always greater than the corresponding probability for Y. Intuitively, X is thus "smaller" than Y. The stochastic dominance is usually termed as the first-degree stochastic dominance in economics (see, e.g., Levy (1992)).

A second possibility consists in further assuming that the decision-makers are risk averse. We then get the stop-loss order $\preceq_{s\ell}$ defined as

$$X \preceq_{s\ell} Y \iff X \preceq_v Y \tag{6.3}$$

for all the pain functions v such that $v^{(1)}, v^{(2)} \geq 0$.

Again,

$$X \preceq_{s\ell} Y \iff -Y \preceq_u -X$$

for all the utility functions u such that $u^{(1)} \geq 0\,,\ u^{(2)} \leq 0$

so that, given two risks X and Y, saying that $X \preceq_{s\ell} Y$ means that the loss X is preferred over the loss Y by all the risk averse profit-seeking decision-makers. The name stop-loss order comes from the following characterization of this stochastic order relation: given two risks X and Y,

$$X \preceq_{st} Y \iff E(X-d)_+ \leq E(Y-d)_+ \quad \text{for all } d \geq 0\,, \tag{6.4}$$

that is, X is smaller than Y in the stop-loss sense when the stop-loss premiums for X are smaller than the corresponding ones for Y, for any level d of the deductible. One intuitively feels that Y will be considered as more dangerous than X by all the "reasonable" decision-makers. The stop-loss order is widely used by the actuaries. It can be considered as a dual version of the well-known second-degree stochastic dominance of the economists (see, e.g., Levy (1992)), and is usually termed as the increasing convex order in the mathematical literature (see, e.g., Shaked and Shanthikumar (1994)).

As pointed out earlier, the stochastic dominance and the stop-loss order are now widely used for many application purposes in actuarial sciences. In an attempt to generalize these orderings, Goovaerts *et al.* (1990) and Denuit, Lefèvre and Shaked (1998) introduced respectively the higher degree stop-loss orders and the higher degree convex orders. The s-th degree stop-loss order between risks X and Y is defined as follows:

$$X \preceq_{s-s\ell} Y \iff X \preceq_v Y \tag{6.5}$$

for all the pain functions v such that $v^{(1)}, v^{(2)}, \ldots, v^{(s+1)} \geq 0\,.$

It is easily seen that, given two risks X and Y,

$$X \preceq_{st} Y \iff X \preceq_{0-s\ell} Y \quad \text{and} \quad X \preceq_{s\ell} Y \iff X \preceq_{1-s\ell} Y\,.$$

We also have that

$$X \preceq_{s-s\ell} Y \iff -Y \preceq_u -X$$

for all the utility functions u such that

$$u^{(1)} \geq 0, u^{(2)} \leq 0\,, \ldots, (-1)^s u^{(s+1)} \geq 0\,.$$

Given two risks X and Y, saying that $X \preceq_{s-s\ell} Y$ means that the loss X is preferred over the loss Y by all the decision-makers with non-decreasing utility functions with first s derivatives of alternating signs. Therefore, $\preceq_{s-s\ell}$ may be regarded as a dual of the $(s-1)$th-degree stochastic dominance in economics (see, e.g., Levy (1992)).
Strengthenings of the orderings $\preceq_{s-s\ell}$ have been recently proposed by Denuit, Lefèvre and Shaked (1998) and termed as the s-convex orders. These are obtained by requiring, in addition, that the first moments of the risks X and Y to be compared coincide. More precisely, given two risks X and Y with finite first s moments, the ordering $\preceq_{s-s\ell,=}$ is defined as

$$X \preceq_{s-s\ell,=} Y \iff \begin{cases} X \preceq_{s-s\ell} Y \,, \\ EX^k = EY^k \quad \text{for } k = 1, 2, \dots s \,. \end{cases} \tag{6.6}$$

The relation $\preceq_{s-s\ell,=}$ is termed as the convex order of degree $s+1$ (since it is closely related to the cones of the convex functions of degree $s+1$; for an overview about this concept of generalized convexity, the reader is referred e.g. to Pecaric, Proschan and Tong (1992), or to Roberts and Varberg (1973). The orderings $\preceq_{s-s\ell,=}$ have been applied to insurance problems by Denuit, De Vylder and Lefèvre (1999) and Denuit (1998, 1999).
In economics, the analysis of investor's behavior is typically confined to first, second and third stochastic dominance. The insurers are assumed to have increasing utility of wealth (i.e. their utility function has to satisfy $u^{(1)} \geq 0$). The class of risk averse insurers is defined by adding the stipulation $u^{(2)} \leq 0$, while the addition of decreasing absolute risk aversion implies that $u^{(3)} \geq 0$ (a non-negative third derivative is, of course, a necessary but not a sufficient condition for decreasing absolute risk aversion).
A special interest in only the first three derivatives of the utility function has probably been driven by the common analysis of utility functions which is based on developing a Taylor series expansion for u, truncating and then taking the expected value of the truncated series. Nevertheless, the three-moments approach to the ranking of two risks is not in general consistant with the ranking based on expected utility theory (see, e.g., Levy (1992), Kang (1994) and Brockett and Kahane (1992)), except in some very particular cases (as for cubic polynomial utility functions).
Now, the standard stochastic dominance, stop-loss and convex orders take into account the first and second derivative of the pain function v (i.e., they only express lure of gain and risk aversion). Therefore, it seems natural to consider stochastic orderings as $\preceq_{2-s\ell,=}$ taking the third derivative into account and translating thus decreasing risk aversion. In economics, third-degree stochastic

dominance has been considered e.g. in Whitmore (1970) and Fishburn (1985); the interested reader is referred to the review paper by Levy (1992) for more details.

When used to compare pairs of risks with equal means and variances, $\preceq_{2-s\ell,=}$ expresses the common preferences of all the risk-averse profit-seeking decision-makers who are afraid of positive skewness (i.e., they prefer risks left-asymmetric). On the other hand, $\preceq_{2-s\ell}$ expresses the same preferences among risks with possible different first two moments.

De Villiers (1997) got the following interesting result about third degree stochastic dominance with equal mean and variance (which is in fact equivalent to $\preceq_{2-s\ell,=}$): there exists a non-decreasing, twice differentiable convex pain function ψ such that, given two risks X and Y, if $X \preceq_{2-s\ell,=} Y$ then X will be preferred over Y by all the rational individuals at least as risk averse as the one with utility function ψ (i.e. by all those with a pain function v that can be expressed as $v = \varphi \circ \psi$ for some non-decreasing and convex function φ). In other words, there exist a pain function ψ such that

$$X \preceq_{2-s\ell,=} Y \iff X \preceq_v Y \text{ for all } v \text{ more risk averse than } \psi\,.$$

Such a result thus relies on the second degree stochastic dominance with respect to a function considered by Meyer (1977).

Let us now recall the following well-known characterization of $\preceq_{s-s\ell}$. Therefore, let us introduce the iterated right-tail distributions of a risk X as follows: put $S_X^{[0]} \equiv 1 - F_X$ and then define recursively the k-th iterated right-tail distributions $S_X^{[k]}$ of X by

$$S_X^{[k+1]}(x) = \int_{t=x}^{+\infty} S_X^{[k]}(t)\,dt\,, \quad x \in \mathbb{R}\,.$$

Then, the following equivalence holds:

$$X \preceq_{s-s\ell} Y \iff \begin{cases} EX^k \leq EY^k & \text{for } k = 1, 2, \dots s\,, \\ S_X^{[s]}(x) \leq S_Y^{[s]}(x) & \text{for all } x \in \mathbb{R}\,. \end{cases} \tag{6.7}$$

A similar result for $\preceq_{s-s\ell,=}$ is easily deduced from (6.6).

To end with, Denuit, Lefèvre and Shaked (1998) proved that there is an easy sufficient condition of crossing-type for $\preceq_{s-s\ell,=}$. Indeed, if F_X and F_Y cross each other exactly s times with F_X surpassing F_Y after the last crossing, and if

$$EX^k = EY^k \quad \text{for } k = 1, 2, \dots, s\,,$$

then $X \preceq_{s-s\ell,=} Y$ holds true. Of course, this is only a sufficient condition and does not cover all the cases. But when the sufficient condition is indeed fulfilled, it is easily detected.

7 Mutually Exclusive Risks

The framework of expected utility theory and stochastic orderings has been recently used by several authors in order to investigate the consequences of a possible dependence among the risks of a given portfolio. We provide hereafter a brief summary of the techniques employed to study what happens when the independence assumption is no more reasonable (see also Section 13). The present section is mainly based on the ideas contained in Dhaene and Denuit (1999) and Dhaene and Goovaerts (1996, 1997); for related results, see also Denuit and Lefèvre (1997), Denuit, Lefèvre and Mesfioui (1999a, b), Müller (1997), Bäuerle and Müller (1998) and Ribas, Goovaerts and Dhaene (1998).

Consider the individual model of risk theory where the aggregate claim S of the portfolio is modeled as the sum of the claims relating to the individual risks $X_1, X_2, \ldots, X_n$ i.e.

$$S = \sum_{i=1}^{n} X_i \,.$$

In many situations, individual risks are correlated since they are subject to the same claim generating mechanism or are influenced by the same economic/physical environment. In traditional risk theory, individual risks are usually assumed to be independent, mainly because the mathematics for correlated risks are less tractable. Consequently, the aggregate claims distribution and the stop-loss premiums for the portfolio are evaluated under the independence assumption. In order to investigate the effect of correlation on stop-loss premiums when the assumption of mutual independence of the individual risks no longer holds, one possibility is to determine the safest and the worst dependence structures, in the sense that they are those which generate the smallest and the largest stop-loss premiums for any given retention level.

In order to formalize the problem, let $F_1, F_2, \ldots, F_n$ be univariate cumulative distribution functions and consider the Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ consisting of all n-dimensional cumulative distribution functions $F_{\mathbf{X}}$ (or equivalently of all the n-dimensional random vectors $\mathbf{X} = (X_1, X_2, \ldots, X_n)$) possessing

$F_1, F_2, \ldots F_n$ as marginal cumulative distribution functions, i.e.

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) &\equiv F_{\mathbf{X}}(x_1, x_2, \ldots, x_n) \\ &= P[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n], \\ &\qquad \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \end{aligned}$$

and

$$\lim_{x_i \to +\infty \ \forall i \neq j} F_{\mathbf{X}}(\mathbf{x}) = F_j(x_j), \quad \mathbf{x} \in \mathbb{R}^n.$$

We restrict ourselves to (cumulative distribution functions of) non-negative random variables with finite expectations, further called multivariate risks; **X** takes on the n risks of the portfolio under interest. In other words, we assume that the marginal distributions are known but the structure of the dependence among the X_i's in the portfolio is unknown.
We have that for all **X** in $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ the following inequality holds:

$$M_n(\mathbf{x}) \le F_{\mathbf{X}}(\mathbf{x}) \le W_n(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n, \tag{7.1}$$

where W_n is usually referred to as the Fréchet upper bound of $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ and is defined by

$$W_n(\mathbf{x}) = \min\{F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\}, \quad \mathbf{x} \in \mathbb{R}^n,$$

while M_n is usually referred to as the Fréchet lower bound of $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ and is defined by

$$M_n(\mathbf{x}) = \max\left\{\sum_{i=1}^{n} F_i(x_i) - n + 1, 0\right\}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Remark that W_n is reachable in $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$. Indeed, given a random variable U uniformly distributed on $[0, 1]$, it can be shown that W_n is the cumulative distribution function of the vector

$$(F_1^{-1}(U), F_2^{-1}(U), \ldots, F_n^{-1}(U)) \in \mathcal{R}_n(F_1, F_2, \ldots, F_n), \tag{7.2}$$

where the generalized inverses of the F_i's are defined as

$$F_i^{-1}(u) = \inf\{x \in \mathbb{R} \mid F_i(x) \ge u\}, \quad u \in [0, 1], \ i = 1, 2, \ldots, n,$$

with the convention that $\inf \emptyset = +\infty$. On the contrary, when $n \ge 3$, M_n is not always a cumulative distribution function anymore (see, e.g., Tchen (1980, Section 4.2)). The following necessary and sufficient condition for M_n to be a cumulative distribution function in $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ can be found e.g. in Joe (1997, Theorem 3.7).

Proposition 7.1 *A necessary and sufficient condition for M_n to be a cumulative distribution function in $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ is that either*

1. $\sum_{j=1}^{n} F_j(x_j) \leq 1$ *whenever* $0 < F_j(x_j) < 1$, $j = 1, 2, \ldots, n$; *or*
2. $\sum_{j=1}^{n} F_j(x_j) \geq n - 1$ *whenever* $0 < F_j(x_j) < 1$, $j = 1, 2, \ldots, n$.

Let us assume that $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ fulfills the condition

$$\sum_{i=1}^{n} q_i \leq 1 \text{ where } q_i = 1 - F_i(0)\,,\ \ i = 1, 2, \ldots, n\,. \tag{7.3}$$

According to Proposition 7.1 (2), (7.3) is a sufficient condition for the lower Fréchet bound M_n to be a proper cumulative distribution function in $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$. The study of Fréchet spaces satisfying (7.3) has some actuarial relevance (see Dhaene and Denuit (1999), as well as Hu and Wu (1999)).

Let us introduce the notion of mutually exclusive risks. Roughly speaking, the risks $X_1, X_2, \ldots X_n$ are said to be mutually exclusive when at most one of them can be different from zero. More formally the risks $X_1, X_2, \ldots, X_n$ are said to be mutually exclusive (or, equivalently, the multivariate risk $\mathbf{X}$ is said to possess this property) when

$$P[X_i > 0, X_j > 0] = 0 \quad \text{for all } i \neq j\,.$$

Examples of mutually exclusive risks abound in actuarial sciences: think for instance of the present value of the benefit associated with a whole life insurance A_x (which can be decomposed as $A_x = A^1_{x;\,\overline{k|}} + {}_{k|}A_x$ where the benefit functions associated with $A^1_{x;\,\overline{k|}}$ and ${}_{k|}A_x$ are mutually exclusive). Other examples are a term insurance with doubled capital in case of accidental death, a n-year endowment insurance (with payment in case of death and survival), a franchise deductible where the risks taken by the insured and the insurer are respectively given by

$$X_1 = \begin{cases} X & \text{if } X \leq d\,, \\ 0 & \text{otherwise} \end{cases} \quad \text{and } X_2 = \begin{cases} 0 & \text{if } X \leq d\,, \\ X & \text{otherwise.} \end{cases}$$

Dhaene and Denuit (1999) proved the following characterization of mutual exclusivity, which relates this notion to the Fréchet lower bound.

Proposition 7.2 *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ satisfying (7.3). The multivariate risk $\boldsymbol{X} \in \mathcal{R}_n(F_1, F_2, \ldots, F_n)$ is mutually exclusive if, and only if, $F_{\boldsymbol{X}} \equiv M_n$.*

Consider a decision-maker with a non-decreasing and concave utility function u, and faced with a risk S which is the sum of the n mutually exclusive risks $X_1, X_2, \ldots, X_n$. It can be shown that the distribution function of $X_1 + X_2 + \cdots + X_n$ can be expressed as

$$F_{X_1+X_2+\cdots+X_n}(x) = \sum_{i=1}^{n} F_{X_i}(x) + 1 - n\,.$$

Let w be the decision maker's initial capital. The expected utility of the random fortune under consideration is then given by

$$\begin{aligned} E[u(w - X_1 - X_2 - \cdots - X_n)] &= \int_{x=0}^{\infty} u(w-x)\, dF_{X_1+X_2+\cdots+X_n}(x) \\ &= \sum_{i=1}^{n} \int_{x=0}^{\infty} u(w-x)\, dF_{X_i} \\ &= \sum_{i=1}^{n} E[u(w - X_i)]\,. \end{aligned}$$

This means that the expected utility related to bearing a risk which is a sum of mutually exclusive risks is equal to the sum of the individual expected utilities involved. This linearity property for the expected utility enables us to state the following result.

Proposition 7.3 *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ satisfying (7.3). Let S_1 and S_2 be two aggregate claims of the form $S_1 = X_1 + X_2 + \cdots + X_n$ and $S_2 = Y_1 + Y_2 + \cdots + Y_n$, where $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathcal{R}_n(F_1, F_2, \ldots, F_n)$ and $\boldsymbol{X}$ is mutually exclusive. Then, $S_1 \preceq_{s\ell} S_2$ holds.*

In other words, in a Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$ such that (7.3) is fulfilled, the mutually exclusive risks lead to the safest portfolio, in the sense that this kind of mutual dependency leads to the smallest stop-loss premiums. Therefore, the portfolio consisting of mutually exclusive risks is preferred by all the profit-seeking risk averse decision-makers over all the other portfolio's with the same marginal structure.

8 The Distorted Expectation Hypothesis

Although the expected utility model has been the main framework for analyzing decisions under risk, there is experimental evidence that individual's preferences are not always based on the expected utility hypothesis. A famous experiment in this context is known as the Allais (1953) paradox. Let us represent (the distribution of) an income X by a vector $(x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$ where $x_1, x_2, \ldots, x_n$ are the possible values of the income (measured in millions of dollars, for instance) and $p_1, p_2, \ldots, p_n$ are the associated probabilities, i.e. $p_i = P[X = x_i]$, $i = 1, 2, \ldots, n$. Consider the following incomes X, Y, V and W defined as $X = (1, 1)$, $Y = (5, 0.1;\ 1, 0.89;\ 0, 0.01)$, $V = (1, 0.11;\ 0, 0.89)$ and $W = (5, 0.1;\ 0, 0.9)$. Now, suppose that the decision maker's preferences can be represented by an increasing utility function u. Without loss of generality, we can assume that $u(5) = 1$ and $u(0) = 0$. If X is preferred over Y then $u(1) > 0.1 + 0.89u(1)$ and, therefore $0.11u(1) > 0.1$ which, in turn, implies that V is preferred over W. However, empirical studies reveal that many people tend to prefer X over Y and W over V. People with these preferences cannot take their decisions under uncertainty based on the expected utility hypothesis.

Motivated by the empirical evidence that individuals often tend to violate the expected utility hypothesis, several researchers have developed alternative theories of choice under risk which are able to explain the observed patterns of behavior. A review of such models, usually termed as "non-expected utility" or "generalizations of expected utility", is given in Sugden (1997) or Schmidt (1998). The ideas that will be developed hereafter originate from Yaari (1987), see also Roëll (1987) and Schmeidler (1989). Yaari's "dual theory of choice under risk" turns out to be a special case of Quiggin's (1982) "anticipated utility theory".

Let us associate to each random variable X its decumulative distribution function S_X, defined as $S_X(x) = P[X > x]$, $x \in \mathbb{R}$, which gives the probability that the random variable X exceeds some point $x \in \mathbb{R}$. Consider a decision-maker with a future random fortune equal to X. The expectation of X can be written as

$$E[X] = -\int_{x=-\infty}^{0} [1 - S_X(x)]\, dx + \int_{x=0}^{+\infty} S_X(x)\, dx\,.$$

Under the "distorted expectations hypothesis" it is assumed that each decision-maker has a distortion function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Instead of using the tail probabilities $S_X(x)$, the decision-maker uses the distorted tail probabilities $f(S_X(x))$. In order to express preferences, a fortune X is valued at its "distorted expectation" $H_f[X]$ defined as

$$H_f[X] = -\int_{x=-\infty}^{0} [1 - f(S_X(x))]\, dx + \int_{x=0}^{+\infty} f(S_X(x))\, dx\,. \tag{8.1}$$

If the fortune is non-negative with probability one, then we find

$$H_f[X] = \int_{x=0}^{+\infty} f(S_X(x))\,dx = \int_{p=0}^{1} S_X^{-1}(p)\,df(p)\,, \tag{8.2}$$

where

$$S_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid S_X(x) \le p\}\,, \quad 0 \le p \le 1\,,$$

with $S_X^{-1}(1) = 0$. The function f is called a distortion because it distorts the probabilities $S_X(x)$ before calculating a generalized expected value. In expected utility theory, one has (by substituting p for $F_X(x)$ in (2.1)) that

$$E[u(X)] = \int_{p=0}^{1} u(S_X^{-1}(p))\,dp\,, \tag{8.3}$$

so that the expressions (8.2) and (8.3) are rather similar from the mathematical point of view but are really distinct from a philosophical point of view: under the expected utility hypothesis, the possible amounts of fortune are adjusted by a utility function while, under the distorted expectation hypothesis, the tail probabilities are adjusted.
Increasing x will lead to a smaller tail probability $S_X(x)$. It is a desirable property that increasing the fortune x will also lead to a smaller distorted tail probability $f(S_X(x))$. Therefore, we will always assume that the distortion functions are non-decreasing. Remark that the distorted tail function cannot necessarily be interpreted as a tail function associated to some random variable, so that the distorted expectation is not necessarily the expectation of some random variable.
Under the "distorted expectations hypothesis" the following preference rule is used: if the decision-maker has a distortion function f, then a fortune Y is preferred over a fortune X (denoted as $X \ll_f Y$ in the remainder of the paper) if, and only if,

$$H_f[X] \le H_f[Y]\,. \tag{8.4}$$

The hypothesis of the dual theory is that agents will choose among random variables so as to maximize the distorted expectation of their fortune. In both theories, the preference relations of a decision-maker are thus modeled by comparisons of numerical quantities associated to the choices under consideration; compare (8.4) to (2.1). Finally, a decision-maker with distortion function f is said

to be indifferent between X and Y (denoted as $X \propto_f Y$ in the remainder) if, and only if, equality holds in (8.4).
Yaari's theory can be considered as a dual theory of choice under risk in the sense that it uses the concept of "distortion function" as opposed to "utility function" in utility theory. Starting from an axiomatic setting different from the one of utility theory, Yaari (1987) showed that there must exist a distortion function f such that the decision-maker will prefer Y to X (or be indifferent between them) if, and only if, (8.4) holds. Yaari's axiomatic setting differs from von Neumann and Morgenstern's one by the independence axiom EU5. Instead of requiring independence with respect to probability mixtures of risky prospects, Yaari (1987) required independence with respect to direct mixing of payments of risky prospects. More precisely, the axiom that gives rise to the dual theory of choice under risk is as follows :

> Axiom DU5: given two random variables X and Y such that Y is preferred to X, $\widetilde{Y}_p$ is preferred to $\widetilde{X}_p$ for any $p \in [0,1]$, where the inverse decumulative distribution functions of $\widetilde{X}_p$ and $\widetilde{Y}_p$ are respectively given by
>
> $$pS_X^{-1} + (1-p)S_Z^{-1} \text{ and } pS_Y^{-1} + (1-p)S_Z^{-1},$$
>
> for an arbitrary decumulative distribution function S_Z.

Note that $(\alpha S_X^{-1} + (1-\alpha)S_Z^{-1})^{-1}$ can also be seen as a sort of mixture for decumulative distribution functions S_X and S_Z. In other words, instead of independence being postulated for convex combinations which are formed along the probability axis, independence is postulated in Yaari's theory for convex combinations which are formed along the payment axis. Under axioms EU1–EU4 and DU5, Yaari (1987) showed that for each decision-maker, there exists a distortion function f such that X is preferred over Y if, and only if, (8.4) is satisfied.
The function f is unique up to a positive affine transformation and can therefore be normalized (in the spirit of (2.2)–(2.3) above). It is easy to prove (see, e.g., Yaari (1987), Proposition 2) that, for real constants $a > 0$ and b,

$$H_f[aX + b] = aH_f[X] + b. \tag{8.5}$$

This means that under the distorted expectations hypothesis, the preferences are invariant up to positive linear transformations: if a fortune Y is preferred over a fortune X, then the same preference holds for a positive linear transformation of X and Y. As an important consequence, we find that the preferences of the decision-maker are independent of initial wealth.

Note that in utility theory, an agent has to be risk neutral (i.e. this agent's preferences always rank random variables by comparing their means, and therefore u is increasing linear) in order to have

$$X \preceq_u Y \iff aX + b \preceq_u aY + b \qquad \text{for all } a > 0, b \in \mathbb{R},$$

whereas

$$X \ll_f Y \iff aX + b \ll_f aY + b \qquad \text{for all } a > 0, b \in \mathbb{R},$$

holds in the dual theory without any further assumption on the distortion function. As quoted above, behavior which is inconsistent with expected utility theory has been experimentally observed, and often such a behavior is called "paradoxical". Yaari (1987) showed that a behavior which is "paradoxical" under expected utility theory is, in many cases, entirely consistent with the dual theory. However, this does not mean that the dual theory is exempt of paradox. On the contrary, for each "paradox" of expected utility theory, one can usually construct a "dual paradox" for the dual theory.
To end with, let us point out some properties of the distorted expectations. In addition to (8.5), Wang (1996) got the following properties of H_f

1. if $f(p) \geq p$ for all $p \in [0,1]$ then
$$H_f[x] \geq EX\,;$$
2. for concave f, $H_f[X] \geq EX$ and
$$H_f[X+Y] \leq H_f[X] + H_f[Y]\,;$$
3. for convex f, $H_f[X] \leq EX$ and
$$H_f[X+Y] \geq H_f[X] + H_f[Y]\,.$$

9 Risk Aversion and Distorted Expectations

We have seen above that the notion of risk aversion plays a crucial role in the economics of insurance. We now describe how this notion can be translated in the framework of distorted expected utility.
Under the distorted expectations hypothesis, a decision-maker is said to be risk averse if his distortion function is convex. This comes from the fact that a convex distortion function satisfies

$$f(p) \leq p\,, \qquad p \in [0,1]\,,$$

and hence

$$f(S_X(x)) \leq S_X(x), \qquad x \in \mathbb{R}.$$

This means that a risk averse decision-maker systematically underestimates his tail probabilities related to levels-of-fortune, which is a prudent attitude. As we immediately find for convex f that

$$H_f[X] \leq E[X] = H_f[E[X]], \tag{9.1}$$

we see that a risk averse decision-maker will always prefer a certain fortune to a random fortune with the same expected value. Therefore, the philosophy of risk aversion is similar in the two theories; see (9.1) and (3.1). A decision-maker is said to be risk neutral if $f(p) = p$. In this case, the distorted expectation hypothesis coincides with comparing expected values. The notion of risk neutrality is therefore very similar in the two approaches.

In theory, we could also define a certainty equivalent for any fortune X as the certain fortune for which the decision-maker is indifferent between choosing this amount and the random fortune X. Hence, CE$[X]$ is determined as the root of the equation

$$H_f[\mathrm{CE}[X]] = H_f[X],$$

which is similar to (3.4). We find that the certainty equivalent is equal to $H_f[X]$:

$$\mathrm{CE}[X] = H_f[X].$$

Note that in expected utility theory, CE$[X]$ was implicitly defined by (3.4) but no explicit expression was available in general. On the other hand, formula (3.5) still holds in Yaari's framework. As in (3.6), the risk premium of X, denoted by RP$[X]$, is then defined by

$$\mathrm{RP}[X] = E[X] - H_f[X].$$

As

$$H_f[X] = H_f[E[X] - \mathrm{RP}[X]],$$

we find that the risk premium is the amount that the decision-maker is willing to pay in order to get a random fortune replaced by its expectation.

10 Distorted Expectations and Insurance

Suppose that a risk averse decision-maker faces a risk X and that an insurer is willing to accept the risk X in return for a premium P. Let the distortion function of the decision-maker be given by f, while his initial wealth equals a certain fixed amount w. As earlier, we assume that the development of the decision-maker's fortune during the insurance period is not influenced by any other factors than the risk and the insurance premium. Under the distorted expectation hypothesis, the person is only willing to underwrite the insurance if $w - X \ll_f w - P$, i.e.

$$H_f[w - P] \geq H_f[w - X]. \tag{10.1}$$

The maximal premium P^M the person is willing to pay is the largest value of P satisfying (10.1); it is the root of the equation

$$w - P^M = H_f[w - X], \tag{10.2}$$

which is equivalent to $w - P^M \propto_f w - X$. Since $H_f[w - X] \leq w - E[X]$, we find that P^M has to satisfy

$$P^M \geq E[X].$$

As for the expected utility theory, we can conclude that the risk averse insured is willing to pay more than the expected loss for being covered.
Let us now examine the viewpoint of the risk averse insurer. Assume that the insurer has a distortion function $\tilde{f}$ and an initial fortune $\widetilde{w}$. The insurer is willing to assume the risk at a premium P if $\widetilde{w} \ll_f \widetilde{w} + P - X$, i.e.

$$H_{\tilde{f}}[\widetilde{w}] \leq H_{\tilde{f}}[\widetilde{w} + P - X]. \tag{10.3}$$

Let P^m be the infimum of all premiums that satisfy the inequality (10.3); it has to fulfill

$$\widetilde{w} = H_{\tilde{f}}[\widetilde{w} + P^m - X],$$

which reduces to $\widetilde{w} \propto_{\tilde{f}} \widetilde{w} + P^m - X$. As

$$H_{\tilde{f}}[\widetilde{w} + P^m - X] \leq \widetilde{w} + P^m - E[X],$$

we find that P^m has to be such that

$$P^m \geq E[X].$$

Hence, the risk averse insurer will require a premium that is greater than or equal to the expected claim amount.
Finally, we can conclude that an insurance policy is only feasible if the premium P satisfies the following inequalities:

$$P^m \le P \le P^M$$

since such a premium satisfies the expected utility requirements (10.1)–(10.3) of both parties. In conclusion, we could say that the two theories lead to a similar analysis of the microeconomy of insurance business. It is worth mentioning that in distortion theory we find explicit expressions of P^M and P^m as

$$P^M = -H_f[-X] \quad \text{and} \quad P^m = -H_{\tilde{f}}[-X].$$

11 The Associated Distortion Function

Let f be the distortion function involved in (8.4); f is thus used in order to compare different levels of fortunes and can be considered as an income distortion function. To each income distortion function f, we associate a function g defined as

$$g(p) = 1 - f(1-p), \qquad 0 \le p \le 1; \tag{11.1}$$

it is easily seen that g is also a distortion function, i.e. g is a non-decreasing function, defined on the interval $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$ (compare (11.1) to (5.1)). If f is the distortion function of a decision-maker, then g will be said to be the associated distortion function of the decision-maker under consideration. Remark that

$$f \text{ convex} \iff g \text{ concave}.$$

Hence, saying that a decision-maker with distortion function f is risk averse is equivalent to saying that his associated distortion function g is concave.
Let us now consider an income $-X$, or equivalently, a loss X. We have that

$$H_f[-X] = -\int_{x=-\infty}^{0} [1 - f(S_{-X}(x))]\,dx + \int_{x=0}^{+\infty} f(S_{-X}(x))\,dx.$$

In terms of the associated distortion function g, we find

$$H_f[-X] = -H_g[X],$$

which is similar to (5.2). Hence, we have

$$H_f[-Y] \leq H_f[-X] \iff H_g[X] \leq H_g[Y].$$

In terms of the associated distortion function, the distorted expectations hypothesis states that a risk X is preferred over a risk Y, if and only, if

$$H_g[X] \leq H_g[Y]$$

i.e.,

$$-Y \ll_f -X \iff X \ll_g Y.$$

Remark that whereas f has to be interpreted as an income distortion function, the associated distorsion function g can be regarded as a loss distortion function. Risk aversion means either that the tail function of a random wealth is underestimated (by use of f) or, equivalently, that the tail function of a random loss is overestimated (by use of g). Moreover, remark that if we use the distortion function approach then we have to consider the random variables as income variables (a loss X is equal to an income $-X$). If we use the associated distortion function approach, however, then the random variables involved have to be considered as loss-variables.
If X is a risk, i.e. a non-negative loss, then we find from (8.1) that

$$H_g[X] = \int_{x=0}^{\infty} g(S_X(x))\,dx.$$

Wang (1996) suggested to compute the risk-adjusted premium $\pi[X]$ of a risk X as a distorted expectation of X, i.e. $\pi[X] = H_g[X]$ with the distortion function g such that $g(p) \geq p$ for all $p \in [0,1]$ (which is true when g is concave). Wang's class of premium principles is therefore strongly connected with Yaari's dual theory for choice under risk. It is also related to recent developments in non-additive measure theory; see Denneberg (1997). Actuarial applications of Wang's premium principles can be found in Wang and Dhaene (1997) and Dhaene, Wang, Young and Goovaerts (1997). Recently, Wang, Young and Panjer (1997) proposed an axiomatic approach to characterize insurance prices in a competitive market setting. They determined some properties that should hold for a reasonable premium principle and proved that if these are fulfilled then the premium principle that the insurer should use is uniquely determined and turns out to be a principle belonging to Wang's class. A similar problem is addressed in Goovaerts and Dhaene (1998). They consider a less general axiomatic setting, leading to an easier-to-prove characterization of Wang's premium principles.

12 Stochastic Orderings and Distorted Utility Theory

One possible criticism against Yaari's dual theory, as against expected utility theory, is that orderings of risks depends on a subjective distortion function, unknown to an objective observer. As in expected utility, if one is interested in how a collection of decision-makers orders risks, then the resulting ranking will be a partial ordering, dual to those introduced in Section 6. We summarize in this section the results obtained by Wang and Young (1998) and Dhaene, Wang, Young and Goovaerts (1997).

We have seen above that, within the framework of expected utility theory, stochastic dominance of two risks is equivalent to saying that one risk is preferred over another by all the profit-seeking decision-makers. A similar interpretation exists within the framework of Yaari's theory of choice under risk. Indeed, it is possible to prove that, given two risks X and Y,

$$X \preceq_{st} Y \iff H_g[X] \leq H_g[Y] \tag{12.1}$$

for all non-decreasing distortion functions g.

In other words, a risk X is smaller than a risk Y in the stochastic dominance sense if, and only if, X is preferred over Y by all the decision-makers with non-decreasing associated distortion function. A result in the same vein holds for the stop-loss order, i.e. given two risks X and Y, one can show that

$$X \preceq_{s\ell} Y \iff H_g[X] \leq H_g[Y] \tag{12.2}$$

for all non-decreasing concave associated distortion functions g.

Within the framework of utility theory, we have seen that the stop-loss ordering of two risks is equivalent to saying that one risk is preferred over another by all the profit-seeking risk averse decision-makers. Within the framework of Yaari's theory of choice under risk, $X \preceq_{s\ell} Y$ holds if, and only if, all the decision-makers with a non-decreasing and concave associated distortion function prefer the risk X.

Therefore, stochastic dominance and stop-loss order have a common interpretation in both theories of choice under risk. In view of (12.1) and (12.2), Wang and Young (1998) suggested to define the dual s-th degree stop-loss order between risks X and Y as follows:

$$X \preceq_{s-s\ell*} Y \iff X \ll_g Y \tag{12.3}$$

for all g such that $g^{(1)} \geq 0, g^{(2)} \leq 0, \ldots, (-1)^s g^{(s+1)} \geq 0$.

It can be proven that

$$X \preceq_{0-s\ell} Y \iff X \preceq_{0-s\ell*} Y \iff X \preceq_{st} Y\,, \tag{12.4}$$

$$X \preceq_{1-s\ell} Y \iff X \preceq_{1-s\ell*} Y \iff X \preceq_{s\ell} Y\,, \tag{12.5}$$

and a natural question that arises is whether the equivalence

$$X \preceq_{s-s\ell} Y \iff X \preceq_{s-s\ell*} Y \tag{12.6}$$

holds true for $s \geq 3$. Quite surprisingly, (12.6) is not true in general, as shown in Wang and Young (1998), Example 4.8. Orderings $\preceq_{s-s\ell*}$ are in fact those introduced by Muliere and Scarsini (1989). In other words, the classes $\{\preceq_{s-s\ell},\ s \in \mathbb{N}\}$ and $\{\preceq_{s-s\ell*},\ s \in \mathbb{N}\}$ coincide for $s = 0$ and $s = 1$ but turn out to be really distinct for $s \geq 3$.

A characterization in the spirit of (6.7) still holds for $\preceq_{s-s\ell*}$. To be more specific, put $S_X^{[0]*} \equiv S_X^{-1}$ and define recursively the k-th iterated inverse decumulative distribution $S_X^{[k]*}$ of X by

$$S_X^{[k+1]*}(p) = \int_{\xi=0}^{p} S_X^{[k]*}(\xi)\, d\xi\,, \quad p \in [0,1]\,,$$

as well as the k-th inverse moment of X as

$$H_{g_k}[X] = \int_{x=0}^{+\infty} \{1 - (1 - S_X(x))^k\}\, dx\,,$$

with $g_k[p] = 1 - (1-p)^k$. Wang and Young (1998) then proved that

$$X \preceq_{s-s\ell*} Y \iff \begin{cases} H_{g_k}[X] \leq H_{g_k}[Y] & \text{for } k = 1, 2, \ldots, s\,, \\ S_X^{[s]*}(p) \leq S_Y^{[s]*}(p) & \text{for all } p \in [0,1]\,. \end{cases} \tag{12.7}$$

As in (6.6), we may define the strengthening $\preceq_{s-s\ell*,=}$ of $\preceq_{s-s\ell*}$ obtained by requiring the equality of the first s inverse moments of the risks X and Y to be compared, i.e.

$$X \preceq_{s-s\ell*,=} Y \iff \begin{cases} X \preceq_{s-s\ell*} Y\,, \\ H_{g_k}[X] = H_{g_k}[Y] & \text{for } k = 1, 2, \ldots, s\,. \end{cases} \tag{12.8}$$

Wang and Young (1998), Proposition 4.9, proved the following crossing condition, which is very similar to the one for $\preceq_{s-s\ell,=}$. Namely, if the numbers of crossings of S_X and S_Y is equal to s, with S_Y surpassing S_X after the last crossing, and

$$H_{g_k}[X] = H_{g_k}[Y] \quad \text{for } k = 1, 2, \ldots, s,$$

then $X \preceq_{s-s\ell*,=} Y$. Using the sufficient condition of crossing type for $X \preceq_{s-s\ell,=} Y$, Denuit, De Vylder and Lefèvre (1998) deduced the extremal distributions for $\preceq_{s-s\ell,=}$ in moment spaces. A point of interest for future research should be whether the extremal distributions with respect to $\preceq_{s-s\ell*,=}$ could be determined using Wang and Young's crossing result.

13 Mutually comonotonic risks

As in Section 7, assume we are faced with a case where the independence hypothesis of the individual claim amounts may be regarded as unrealistic. For instance, consider individual risks of an earthquake risk portfolio located in the same geographic area: these are correlated since individual claims are contingent on the occurrence and severity of the same earthquake.

As another example, think of a bond portfolio: individual bond default experience may be conditionally independent for given market conditions but the underlying economic environment (e.g., interest rates) may affect all individual bonds in a similar way.

In order to model such a situation, the notion of comonotonicity has been introduced. It is defined as follows. The risks $X_1, X_2, \ldots, X_n$ are said to be mutually comonotonic (or equivalently, the multivariate risk $\mathbf{X}$ is said to possess this property) when there exists a random variable Z and non-decreasing functions $\phi_1, \phi_2, \ldots, \phi_n$: $\mathbb{R} \to \mathbb{R}^+$ such that $\mathbf{X}$ is distributed as the vector $(\phi_1(Z), \phi_2(Z), \ldots, \phi_n(Z))$.

The economic meaning of comonotonicity is as follows: when two random variables are comonotonic, then it can be said that neither of them is a hedge against the other. The variability of one is never tempered by counter-variability of the other. The interpretation for more than two risks immediately follows from the equivalence

$$\begin{aligned} &(X_i, X_j) \text{ are comonotonic for all } i \neq j \\ \iff\ &(X_1, X_2, \ldots, X_n) \text{ are comonotonic.} \end{aligned}$$

From the definition, we see that comonotonic risks can be considered as "common monotonic" in the sense that such risks are not able to compensate each other. See Denneberg (1997) for an extensive theory about this topic.

Now, let us stress the meaning of comonotonicity in insurance business. Consider for example an insurance company that gives compensation for the damages caused by catastrophes like hurricanes or earthquakes. In this case, it is realistic to assume that there is a real random variable Z, which gives the measure for the magnitude of the catastrophe, and the individual risks are non-decreasing functions ϕ_i of the magnitude Z of the catastrophe. In such a case, the random vector **X** is clearly comonotonic.
The modified independence axiom DU5 giving rise to Yaari's dual theory for choice under risk can be stated in a way that makes its economic content clear using comonotonic random variables. Indeed, this axiom requires that, (X_1, X_2, X_3) being comonotonic,

$$X_1 \ll_g X_2 \;\Rightarrow\; pX_1 + (1-p)X_3 \ll_g pX_2 + (1-p)X_3$$

for all $p \in [0,1]$. The equivalence follows from the fact that if X_1 and X_3 are comonotonic then $pX_1 + (1-p)X_3$ has inverse decumulative distribution function $pS_{X_1}^{-1} + (1-p)S_{X_3}^{-1}$. Note that we are dealing with ordinary convex combinations of random variables and that $pX_1 + (1-p)X_3$ is not a probability mixture. Dual independence requires therefore the direction of preference to be retained under mixing of payments, provided hedging is not involved. In an insurance context, suppose that a reinsurer has to choose between the following portfolio's of insurance risks: portfolio 1 consists of X_1 in proportion p with the remainder of the portfolio being made up with risks X_3; portfolio 2 consists of risks X_2 in proportion p and the remainder made up with risk X_3. Also assume that the reinsurer considers X_1 as less risky than X_2. If X_3 is a hedge against risk X_2, then the reinsurer may decide that portfolio 2 is preferable to portfolio 1 even though X_2 is riskier than X_1 without the presence of X_3. However, if the risks X_1, X_2 and X_3 are comonotonic, X_3 will not be a hedge against X_1 or X_2, and in Yaari's approach, the reinsurer will choose portfolio 1 over portfolio 2, for the same premium scheme.
The following well-known result characterizes comonotonicity with the aid of the Fréchet upper bound. It can be seen as a dual of Proposition 7.2 relating to mutually exclusive risks.

Proposition 13.1 *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$. The multivariate risk **X** is comonotonic if, and only if, $F_{\mathbf{X}} \equiv W_n$.*

The concept of comonotonicity can be explained in terms of Monte Carlo simulation. From (7.2) together with Proposition 13.1, the risks $X_1, X_2, \ldots, X_n$ are comonotonic if, and only if, **X** is distributed as the vector $(F_1^{-1}(U), F_2^{-1}(U), \ldots, F_n^{-1}(U)$ for U being any uniformly distributed random variable on $[0,1]$.

Hence, in order to simulate comonotonic risks, one needs to generate only one sample of random uniform numbers and insert them in the F_i^{-1}'s to get a sample of the X_i's. By contrast, if the X_i's were independent, then one needs to generate n independent samples of random uniform numbers and then insert them in $F_1, F_2, \ldots, F_n$, respectively.

Recall that X and Y are positively perfectly correlated if, and only if, there exist real numbers $a > 0$ and b such that $Y = aX + b$, except, perhaps, for values of X with zero probability. It follows immediately that perfect correlation of X and Y implies

$$P[X \leq x, Y \leq y] = \min(F_X(x), F_Y(y)) = W_2(x, y),$$

so that positively perfectly correlated risks are also comonotonic by Proposition 13.1. Hence comonotonicity appears as an extension of the concept of positive perfect correlation. This extension is very useful in order to analyze insurance business. Consider a risk X and split it as follows:

$$X_1 = \begin{cases} X & \text{if } X \leq d, \\ d & \text{otherwise,} \end{cases} \quad \text{and} \quad X_2 = \begin{cases} 0 & \text{if } X \leq d, \\ X - d & \text{otherwise.} \end{cases}$$

Then, X_1 can be interpreted as the part of total claims generated by X to be covered by the primary insurer and X_2 the part to be covered by the reinsurer in a stop-loss treaty. It follows that X_1 and X_2 are not perfectly correlated since one cannot be written as a linear function of the other. However, since X_1 and X_2 are non-decreasing functions of the original risk X, they are comonotonic. More generally, we can say that most risk sharing schemes (between insurer and reinsurer, or between insured and insurer) lead to partial risks that are comonotonic. The only restriction that has to hold is that both risk sharing partners have to bear more (or at least as much) if the underlying total claims increases. To be specific, let the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ describe the indemnity benefit associated to some insurance agreement; $\varphi(x)$ is the amount paid by the insurance company to the policyholder if a loss of amount x occurs. It is usual to restrict φ to be a non-decreasing function. In such a case, X and $\varphi(X)$ are comonotonic risks. Similarly, one often restricts φ to increase at a slower rate than the underlying loss (if φ is piecewise differentiable, this condition reduces to $\varphi^{(1)} \leq 1$). As a consequence, $X - \varphi(X)$ and X are comonotonic risks. Examples of insurance contracts that satisfy both restrictions are

1. deductible coverage: $\varphi(x) = \max(x - d, 0)$ for some $d \geq 0$;
2. coinsurance: $\varphi(x) = \alpha x$ for some $\alpha \in [0, 1]$;

3. coverage with a maximal limit: $\varphi(x) = \min(x, d)$ for some $d \geq 0$;
4. as well as coverages combining the three forms above.

Under the expected utility hypothesis, we found that the expected utility was additive for mutually exclusive risks. Under the distorted expectations hypothesis, a similar result holds for comonotonic risks. More precisely, consider a decision-maker with an associated distortion function g, who bears the mutual comonotonic risks $X_1, X_2, \ldots, X_n$. Then, we find

$$\begin{aligned} H_g[X_1 + X_2 + \cdots + X_n] &= \int_{x=0}^{\infty} g[S_{X_1+X_2+\cdots+X_n}(x)]\, dx \\ &= \int_{p=0}^{1} S^{-1}_{X_1+X_2+\cdots+X_n}(p)\, dg(p)\,. \end{aligned}$$

Now, since the inverse decumulative distribution of the sum $X_1 + X_2$ satisfies

$$S^{-1}_{X_1+X_2}(p) = S^{-1}_{X_1}(p) + S^{-1}_{X_2}(p)\,, \quad p \in [0,1]\,,$$

when the risks X_1 and X_2 are comonotonic (see, e.g., Wang (1996)), we get

$$\begin{aligned} H_g[X_1 + X_2 + \cdots + X_n] &= \sum_{i=1}^{n} \int_{p=0}^{1} S^{-1}_{X_i}(p)\, dg(p) \\ &= \sum_{i=1}^{n} H_g[X_i]\,. \end{aligned}$$

This means that the dual distorted expectation operator is linear for mutually comonotonic risks.

Proposition 13.2 *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$. Let S_1 and S_2 be two aggregate claims of the form $S_1 = X_1 + X_2 + \cdots + X_n$ and $S_2 = Y_1 + Y_2 + \ldots + Y_n$, where **X**, **Y** $\in \mathcal{R}_n(F_1, F_2, \ldots, F_n)$ and **Y** is mutually comonotonic. Then, for any non-decreasing and concave associated distortion function g, $S_1 \ll_g S_2$ holds, i.e., $S_1 \preceq_{s\ell} S_2$.*

In other words, in a Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$, the mutually comonotonic risks lead to the most dangerous portfolio, in the sense that this kind of mutual dependency leads to the largest stop-loss premiums. Therefore, when a profit seeking risk averse decision-maker can choose between elements of a given Fréchet space, he will never choose the mutually comonotonic risks.

14 The Rank-Dependent Expected Utility Hypothesis

In this last section, we briefly present a theory which combines the expected utility and the distorted expected utility assumptions, to a certain extent. For more details, we refer the interested reader e.g. to Chateauneuf, Cohen and Meilijson (1997).

Under the rank-dependent expected utility model, a decision-maker is characterized by a non-negative utility function u (that plays the role of utility on certainty) in conjunction with a distorted function f (that plays the role of a probability perception function). Such a decision-maker prefers the fortune Y to the fortune X if, and only if,

$$H_f^u[X] \leq H_f^u[Y], \tag{14.1}$$

where $H_f^u[X]$ is defined as

$$\begin{aligned} H_f^u[X] &= -\int_{x=-\infty}^{+\infty} u(x)\, df(S_X(x)) \\ &= \int_{t=0}^{+\infty} f(P[u(X) > t])\, dt\,. \end{aligned}$$

It is easy to see that if $f(v) = v$, we get the expected utility model. There is a huge literature about these topics in economics, with applications to insurance problems. See, for instance, Landsberger and Meilijson (1990, 1994a, b). For related stochastic orderings, see, e.g., Chateauneuf, Cohen and Meilijson (1996).

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Abstract

The present paper is devoted to different methods of choice under risk in an actuarial setting. The classical expected utility theory is first presented, and its drawbacks are underlined. A second approach based on the so-called distorted expectation hypothesis is then described. It will be seen that the well-known stochastic dominance as well as the stop-loss order have common interpretations in both theories, while defining higher degree stochastic orders leads to different concepts. The aim of this paper is to emphasize the similarities of the two approaches of choice under risk as well as to point out their major differences.

Résumé

Ce papier traite de différentes méthodes de choix de risques dans un contexte actuariel. La théorie classique de l'utilité moyenne est présentée d'abord et ses désavantages sont soulignés. Ensuite une approche alternative basée sur l'hypothèse de l'espérance altérée est décrite. On montre que tant la dominance stochastique que l'ordre stop-loss possèdent une interprétation commune dans les deux théories, alors que la définition d'ordres stochastiques de degrés plus élevés conduit à des concepts différents. Le but de ce papier est de souligner les similarités et les différences des deux aproches de choix de risques.

Zusammenfassung

Der vorliegende Artikel befasst sich mit verschiedenen Methoden der Entscheidungsfindung unter Risiko in einem aktuariellen Umfeld. Zuerst wird die klassische Theorie des erwarteten Nutzens vorgestellt und ihre Nachteile werden unterstrichen. Ein alternativer Zugang basiert auf der sogenannten Hypothese der verzerrten Erwartung. Es wird gezeigt, dass sowohl die bekannte stochastische Dominanz als auch die Stop-Loss-Ordnung gemeinsame Interpretationen in beiden Theorien haben, während man durch die Definition höhergradiger stochastischer Ordnungen zu anderen Konzepten gelangt. Das Ziel dieses Artikels ist es, sowohl die Ähnlichkeiten der beiden beschriebenen Zugänge zur Entscheidungsfindung unter Risiko zu betonen, wie auch ihre hauptsächlichsten Unterschiede aufzuzeigen.