

Zeitschrift: Mitteilungen / Schweizerische Aktuarvereinigung = Bulletin / Association Suisse des Actuaires = Bulletin / Swiss Association of Actuaries

Herausgeber: Schweizerische Aktuarvereinigung

Band: - (1998)

Heft: 2

Rubrik: Kurzmitteilungen

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 23.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

D. Kurzmitteilungen

CARMEN RIBAS, MARC J. GOOVAERTS and JAN DHAENE*, Leuven

A note on the stop-loss preserving property of Wang's premium principle

1 Introduction

A premium calculation principle is a rule that assigns a non-negative real number, the net premium, to each insured risk. Each premium principle induces a total order for all risks, ranking risks with low premium below risks with higher premium. A natural requirement for a premium principle is that the order obtained this way should closely correspond to the well-known stochastic orders between risks. Therefore, a desirable property for a premium principle is that preserves stochastic order and stop-loss order, see e. g. Goovaerts et al. (1990) or Kaas et al. (1994).

In the actuarial literature several premium principles have been presented, see e. g. Goovaerts et al. (1984). Most of these premium principles have interpretations within the framework of expected utility.

Wang (1996) introduced a new class of premium principles which can be interpreted within the framework of Yaari's (1987) dual theory of choice under risk.

In this paper we will investigate the stop-loss preserving property of Wang's class of premium principles in this dual setting. In Wang (1996), a proof is given for this property. However, as shown by Hürlimann (1998), the original proof contains an error. Dhaene et al. (1997) give a general proof for the stop-loss order preserving property of the class of Wang's premium principles. As they point out, other proofs are possible for less general but still realistic situations. In this paper, we will derive a proof for the case that the distribution functions involved only have a finite number of crossing points. Hürlimann (1998) also gives a (more complicated) proof for this special case, based on the Hardy-Littlewood transform.

Although proofs are available for the general case, the straightforward and elementary proof presented here (which is valid in a restricted but still realistic environment), is more suited for pedagogical purposes.

* J. Dhaene and M. J. Goovaerts would like to thank for the financial support of Onderzoeksfonds K. U. Leuven (grant OT/97/6) and F. W. O. (grant "Actuarial ordering of dependent risks").
C. Ribas would like to thank for the financial support of CIRIT (grant 1997BEAI200107).

2 Wang's premium principle

For a risk X (i.e. a non-negative real valued random variable with finite mean), we denote its decumulative distribution function (ddf) by S_X :

$$S_X(x) = \Pr(X > x) \quad 0 \leq x < \infty$$

Within the framework of Yaari's (1987) dual theory of choice under risk the concept of "distortion function" emerges. It can be considered as the parallel to the concept of "utility function" in utility theory.

Definition 1. A distortion function g is a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$.

Wang (1996) proposes to compute the risk-adjusted premium of a risk X as a "distorted" expectation of X :

$$H_g(X) = \int_0^\infty g[S_X(x)] dx$$

for some concave distortion function g .

A distortion function g will said to be concave if for each y in $[0, 1]$ there exist real numbers a_y and b_y and a line $l(x) = a_y x + b_y$, such that $l(y) = g(y)$ and $l(x) \geq g(x)$ for all x in $[0, 1]$. As $l(y) = g(y)$ we find that $l(x) = a_y(x - y) + g(y)$. Hence $l(x) \geq g(x)$ can be written as:

$$g(x) - g(y) \leq a_y(x - y) \quad \text{for all } x, y \text{ in } [0, 1].$$

This inequality will be used later for proving some of our results.

3 Stop-loss preserving property of Wang's class of premium principles

We say that risk X precedes risk Y in stop-loss order, written $X \leq_{sl} Y$, if their stop-loss premiums are ordered uniformly.

A desirable property of Wang's class of premium principles is that it preserves stop-loss order, i.e., $X \leq_{sl} Y \Rightarrow H_g(X) \leq H_g(Y)$. A proof of this result can be found in Wang (1996). Unfortunately, Wang's proof contains an error, as is shown by Hürlimann (1998).

Hürlimann (1998) presents a proof of the stop-loss order preserving property of Wang's class of premium principles, when the distribution functions of X and Y only cross finitely many times. His proof is based on a characterization of stop-loss order in terms of the Hardy-Littlewood transform and stochastic dominance.

In the following theorem, we also consider the case of two distribution functions which only cross finitely many times. We present a new and simpler proof for the stop-loss preserving property in this case.

Theorem 2. *Suppose that X and Y are risks for which $S_X - S_Y$ has only finitely many sign changes. If $X \leq_{sl} Y$, then $H_g(X) \leq H_g(Y)$ for any distortion function g which is concave in $[0, 1]$.*

Proof. If $S_X - S_Y$ has no sign changes, then we must have that $S_X(n) \leq S_Y(n)$ for all $n \geq 0$, which implies that $H_g(X) \leq H_g(Y)$.

Now consider the case that S_X and S_Y have at least one crossing point. We denote the crossing points by c_1, c_2, \dots, c_n with $n \geq 1$ and $0 = c_0 < c_1 < c_2 < \dots < c_n$.

Let g be a distortion function which is concave in $[0, 1]$. Then we have that for each y in $[0, 1]$, there exists a real number a_y such that,

$$g(x) - g(y) \leq a_y(x - y)$$

for all x in $[0, 1]$. Further, because g is non-decreasing and concave, a_y is a non-negative, non-increasing function of y .

By substituting $S_X(x)$ and $S_Y(x)$ for x and y in the above inequality, we find

$$g(S_X(x)) - g(S_Y(x)) \leq a_{S_Y(x)}(S_X(x) - S_Y(x)) \quad \text{for all } x \geq 0.$$

Remark that $a_{S_Y(x)}$ is a non-decreasing function of x .

As $X \leq_{sl} Y$, we must have that $S_X(x) \leq S_Y(x)$ for all $x \geq c_n$. Thus, we have

$$\begin{aligned} \int_{c_n}^{\infty} [g(S_X(x)) - g(S_Y(x))] dx &\leq \int_{c_n}^{\infty} a_{S_Y(x)} [S_X(x) - S_Y(x)] dx \\ &\leq a_{S_Y(c_n)} \int_{c_n}^{\infty} [S_X(x) - S_Y(x)] dx \\ &\leq 0. \end{aligned}$$

We have that $S_X(x) \geq S_Y(x)$ in the interval $[c_{n-1} - c_n]$. Hence,

$$\begin{aligned}
 \int_{c_{n-1}}^{\infty} [g(S_X(x)) - g(S_Y(x))] dx &\leq \int_{c_{n-1}}^{c_n} a_{S_Y(x)} [S_X(x) - S_Y(x)] dx \\
 &\quad + \int_{c_n}^{\infty} a_{S_Y(x)} [S_X(x) - S_Y(x)] dx \\
 &\leq a_{S_Y(c_n)} \int_{c_{n-1}}^{\infty} [S_X(x) - S_Y(x)] dx \\
 &\leq a_{S_Y(c_{n-1})} \int_{c_{n-1}}^{\infty} [S_X(x) - S_Y(x)] dx \\
 &\leq 0.
 \end{aligned}$$

Continuing this procedure, we find that $X \leq_{sl} Y$ implies

$$\begin{aligned}
 \int_{c_{n-j}}^{\infty} [g(S_X(x)) - g(S_Y(x))] dx &\leq a_{S_Y(c_{n-j})} \int_{c_{n-j}}^{\infty} [S_X(x) - S_Y(x)] dx \\
 &\leq 0, \quad \text{for } j = 0, 1, 2, \dots, n.
 \end{aligned}$$

The case $j = n$ leads to the desired result. \square

We say that risk Y is more dangerous than risk X , written $X \leq_D Y$, if $E[X] \leq E[Y]$, and moreover the distribution functions of X and Y only cross once. As order in dangerousness implies stop-loss order, we find the following corollary to Theorem 1.

Corollary 3. *If $X \leq_D Y$, then $H_g(X) \leq H_g(Y)$ for all distortion functions g which are concave in $[0, 1]$.*

In the following theorem, we consider the case of two risks that are uniformly bounded.

Theorem 4. *Consider two risks X and Y with finite support $[0, b]$. If $X \leq_{sl} Y$, then $H_g(X) \leq H_g(Y)$ for any distortion function g which is concave in $[0, 1]$.*

Proof. As stop-loss order is the transitive (stop-loss) closure of order in dangerousness, see e. g. Müller (1996), the result follows from the corollary and the dominated convergence theorem. \square

Remark that this proof for the stop-loss order preserving property is not valid if X and Y are not uniformly bounded, because the dominated convergence theorem can not be applied in this case. A proof for this general case can be found in Dhaene et al. (1997) or in Hürlimann (1998).

References

- Dhaene, J., Wang, S., Young, V. and Goovaerts, M.J. (1997). "Comonotonicity and maximal stop-loss premiums". Research Report 9730, Departement Toegepaste Economische Wetenschappen, K.U. Leuven, submitted.
- Goovaerts, M.J., De Vylder, F. and Haezendonck, J. (1984). *Insurance premiums*. North-Holland, Amsterdam, New York, Oxford.
- Goovaerts, M.J., Kaas, R., van Heerwaarden, A. and Bauwelinckx, T. (1990). *Effective actuarial methods*. Insurance Series, Vol 3., North-Holland, Amsterdam, New York, Oxford, Tokyo.
- Hürlimann, W. (1998). "On stop-loss order and the distortion pricing principle", *ASTIN Bulletin* 28, 119–134.
- Kaas, R., van Heerwaarden, A. and Goovaerts, M.J. (1994). *Ordering of actuarial risks*. Education Series 1, CAIRE. Brussels.
- Müller, A. (1996). "Ordering of risks: a comparative study via stop-loss transforms", *Insurance: Mathematics and Economics* 17, 215–222.
- Müller, A. (1997). "Stop-loss order for portfolios of dependent risks". *Insurance: Mathematics and Economics*, to appear.
- Wang, S. (1995). "Insurance pricing and increased limits ratemaking by proportional hazard transforms". *Insurance: Mathematics and Economics* 17, 43–54.
- Wang, S. (1996). "Premium calculation by transforming the layer premium density", *ASTIN Bulletin* 26, 71–92.
- Wang, S. and Dhaene, J. (1998). "Comonotonicity, correlation order and premium principles". *Insurance: Mathematics and Economics*, 22 (3), 235–242.
- Yaari, M.E. (1987). "The dual theory of choice under risk", *Econometrica* 55, 95–115

Carmen Ribas, Marc J. Goovaerts and Jan Dhaene
K. U. Leuven
Minderbroedersstrat, 5.
B-3000 Leuven
Belgium

