D. Kurzmitteilungen

CARMEN RIBAS, MARC J. GOOVAERTS and JAN DHAENE*, Leuven

A note on the stop-loss preserving property of Wang's premium principle

1 Introduction

A premium calculation principle is a rule that assigns a non-negative real number, the net premium, to each insured risk. Each premium principle induces a total order for all risks, ranking risks with low premium below risks with higher premium. A natural requirement for a premium principle is that the order obtained this way should closely correspond to the well-known stochastic orders between risks. Therefore, a desirable property for a premium principle is that preserves stochastic order and stop-loss order, see e.g. Goovaerts et al. (1990) or Kaas et al. (1994).
In the actuarial literature several premium principles have been presented, see e.g. Goovaerts et al. (1984). Most of these premium principles have interpretations within the framework of expected utility.
Wang (1996) introduced a new class of premium principles which can be interpreted within the framework of Yaari's (1987) dual theory of choice under risk.
In this paper we will investigate the stop-loss preserving property of Wang's class of premium principles in this dual setting. In Wang (1996), a proof is given for this property. However, as shown by Hürlimann (1998), the original proof contains an error. Dhaene et al. (1997) give a general proof for the stop-loss order preserving property of the class of Wang's premium principles. As they point out, other proofs are possible for less general but still realistic situations. In this paper, we will derive a proof for the case that the distribution functions involved only have a finite number of crossing points. Hürlimann (1998) also gives a (more complicated) proof for this special case, based on the Hardy-Littlewood transform.
Although proofs are available for the general case, the straightforward and elementary proof presented here (which is valid in a restricted but still realistic environment), is more suited for pedagogical purposes.

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2 Wang's premium principle

For a risk X (i. e. a non-negative real valued random variable with finite mean), we denote its decumulative distribution function (ddf) by S_X:

$$S_X(x) = \Pr(X > x) \qquad 0 \leq x < \infty$$

Within the framework of Yaari's (1987) dual theory of choice under risk the concept of"distortion function" emerges. It can be considered as the parallel to the concept of"utility function" in utility theory.

Definition 1. *A distortion function g is a non- decreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$.*

Wang (1996) proposes to compute the risk-adjusted premium of a risk X as a "distorted" expectation of X:

$$H_g(X) = \int_0^\infty g[S_X(x)]\, dx$$

for some concave distortion function g.
A distortion function g will said to be concave if for each y in $[0,1]$ there exist real numbers a_y and b_y and a line $l(x) = a_y x + b_y$, such that $l(y) = g(y)$ and $l(x) \geq g(x)$ for all x in $[0,1]$. As $l(y) = g(y)$ we find that $l(x) = a_y(x-y)+g(y)$. Hence $l(x) \geq g(x)$ can be written as:

$$g(x) - g(y) \leq a_y(x-y) \qquad \text{for all } x, y \text{ in } [\,0,1\,]\,.$$

This inequality will be used later for proving some of our results.

3 Stop-loss preserving property of Wang's class of premium principles

We say that risk X precedes risk Y in stop-loss order, written $X \leq_{sl} Y$, if their stop-loss premiums are ordered uniformly.
A desirable property of Wang's class of premium principles is that it preserves stop-loss order, i. e., $X \leq_{sl} Y \Rightarrow H_g(X) \leq H_g(Y)$. A proof of this result can be found in Wang (1996). Unfortunately, Wang's proof contains an error, as is shown by Hürlimann (1998).

Hürlimann (1998) presents a proof of the stop-loss order preserving property of Wang's class of premium principles, when the distribution functions of X and Y only cross finitely many times. His proof is based on a characterization of stop-loss order in terms of the Hardy-Littlewood transform and stochastic dominance.

In the following theorem, we also consider the case of two distribution functions which only cross finitely many times. We present a new and simpler proof for the stop-loss preserving property in this case.

Theorem 2. *Suppose that X and Y are risks for which $S_X - S_Y$ has only finitely many sign changes. If $X \leq_{sl} Y$, then $H_g(X) \leq H_g(Y)$ for any distortion function g which is concave in $[0,1]$.*

Proof. If $S_X - S_Y$ has no sign changes, then we must have that $S_X(n) \leq S_Y(n)$ for all $n \geq 0$, which implies that $H_g(X) \leq H_g(Y)$.

Now consider the case that S_X and S_Y have at least one crossing point. We denote the crossing points by $c_1, c_2, \ldots c_n$ with $n \geq 1$ and $0 = c_0 < c_1 < c_2 < \cdots < c_n$. Let g be a distortion function which is concave in $[0,1]$. Then we have that for each y in $[0,1]$, there exists a real number a_y such that,

$$g(x) - g(y) \leq a_y(x - y)$$

for all x in $[0,1]$. Further, because g is non-decreasing and concave, a_y is a non-negative. non-increasing function of y.

By substituting $S_X(x)$ and $S_Y(x)$ for x and y in the above inequality, we find

$$g(S_X(x)) - g(S_Y(x)) \leq a_{S_Y(x)}(S_X(x) - S_Y(x)) \quad \text{for all } x \geq 0\,.$$

Remark that $a_{S_Y(x)}$ is a non-decreasing function of x.

As $X \leq_{sl} Y$, we must have that $S_X(x) \leq S_Y(x)$ for all $x \geq c_n$. Thus, we have

$$\begin{aligned}
\int_{c_n}^{\infty} [\,g(S_X(x)) - g(S_Y(x))\,]\, dx &\leq \int_{c_n}^{\infty} a_{S_Y(x)}\, [\,S_X(x) - S_Y(x)\,]\, dx \\
&\leq a_{S_Y(c_n)} \int_{c_n}^{\infty} [\,S_X(x) - S_Y(x)\,]\, dx \\
&\leq 0\,.
\end{aligned}$$

We have that $S_X(x) \geq S_Y(x)$ in the interval $[c_{n-1} - c_n]$. Hence,

$$\begin{aligned}
\int_{c_{n-1}}^{\infty} [g(S_X(x)) - g(S_Y(x))]\, dx &\leq \int_{c_{n-1}}^{c_n} a_{S_Y(x)} [S_X(x) - S_Y(x)]\, dx \\
&\quad + \int_{c_n}^{\infty} a_{S_Y(x)} [S_X(x) - S_Y(x)]\, dx \\
&\leq a_{S_Y(c_n)} \int_{c_{n-1}}^{\infty} [S_X(x) - S_Y(x)]\, dx \\
&\leq a_{S_Y(c_{n-1})} \int_{c_{n-1}}^{\infty} [S_X(x) - S_Y(x)]\, dx \\
&\leq 0 .
\end{aligned}$$

Continuing this procedure, we find that $X \leq_{sl} Y$ implies

$$\begin{aligned}
\int_{c_{n-j}}^{\infty} [g(S_X(x)) - g(S_Y(x))]\, dx &\leq a_{S_Y(c_{n-j})} \int_{c_{n-j}}^{\infty} [S_X(x) - S_Y(x)]\, dx \\
&\leq 0 , \qquad \text{for } j = 0, 1, 2, \ldots, n .
\end{aligned}$$

The case $j = n$ leads to the desired result. □

We say that risk Y is more dangerous than risk X, written $X \leq_D Y$, if $E[X] \leq E[Y]$, and moreover the distribution functions of X and Y only cross once. As order in dangerousness implies stop-loss order, we find the following corollary to Theorem 1.

Corollary 3. *If $X \leq_D Y$, then $H_g(X) \leq H_g(Y)$ for all distortion functions g which are concave in $[0, 1]$.*

In the following theorem, we consider the case of two risks that are uniformly bounded.

Theorem 4. *Consider two risks X and Y with finite support $[0, b]$. If $X \leq_{sl} Y$, then $H_g(X) \leq H_g(Y)$ for any distortion function g which is concave in $[0, 1]$.*

Proof. As stop-loss order is the transitive (stop-loss) closure of order in dangerousness, see e. g. Müller (1996), the result follows from the corollary and the dominated convergence theorem. □

Remark that this proof for the stop-loss order preserving property is not valid if X and Y are not uniformly bounded, because the dominated convergence theorem can not be applied in this case. A proof for this general case can be found in Dhaene et al. (1997) or in Hürlimann (1998).

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Carmen Ribas, Marc J. Goovaerts and Jan Dhaene
K.U. Leuven
Minderbroeddersstrat, 5.
B-3000 Leuven
Belgium