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B. Wissenschaftliche Mitteilungen

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Some analytical approximations of stop-loss premiums

1 Introduction

Evaluation of stop-loss premiums is an important application of Risk Theory. It requires the determination of the distribution of the aggregate claims, a random variable that we will denote by S. Historically, the normal approximation was the first attempt at the evaluation of this distribution, motivated by central limit theorems. It has been promptly recognized that this approximation was not satisfactory. ≪Statistical≫ approaches were then used: one postulates that S has a properly chosen analytical probability density function with parameters ensuring a (hopefully) good fit. Other analytical approximations were developed, such as the Edgeworth expansion, Esscher and Normal power approximations, etc. Later on, with the widespread use of computers, recursive algorithms have been developed and become very popular. For a treatment of these methods see, for example, Beard *et al.* (1984), Seal (1969), Gerber (1979) or Daykin *et al.* (1994).

The aim of this paper is two-fold. Firstly, we present some analytical approximations for the net stop-loss premium. Some of them are not widely known, namely the inverse Gaussian approximation with or without translation. The more usual normal and gamma approximations are presented mainly for comparison purposes. Secondly, we suggest a technique to improve the performance of these approximations when the portfolio is small.

2 Models for S

The two most important models for the aggregate claims random variable S are the so-called *individual model* and *collective model* (see, for example, Bowers *et al.* (1986) for more details than what is provided below).

2.1 Individual model

In the individual model, the aggregate claims are the sum of the claim amounts of each individual policy of the portfolio. The risks are assumed to be independent. The following definition then results:

$$S = X_1 + X_2 + \cdots + X_n\,. \tag{1}$$

Here X_i is the claim amount of policy i and $X_1, X_2, \ldots, X_n$ are mutually independent random variables. Since the X_i's are independent, the moments of S are easily found:

$$E[S] = \sum_{i=1}^{n} E[X_i]\,, \tag{2}$$

$$\mathrm{Var}[S] = \sum_{i=1}^{n} \mathrm{Var}[X_i] \tag{3}$$

and

$$E[(S - E[S])^3] = \sum_{i=1}^{n} E[(X_i - E[X_i])^3]\,. \tag{4}$$

Furthermore, if π_k is the probability that policy k has no claims, the probability π that there are no claims in the whole portfolio is

$$\pi = \prod_{k=1}^{n} \pi_k\,. \tag{5}$$

When negative claim amounts are allowed, this probability may be different from the probability that S is equal to zero. The probability π will be used later to refine the approximations presented in the following sections.

2.2 Collective model

When one uses the collective model, one assumes that the portfolio is a mass of risks generating a number N of claims during the considered period of time. The amount of the i-th of these claims is a random variable X_i. The random variables[1] $X_1, X_2, X_3, \ldots,$ are assumed to be mutually independent with common distribution function $P(\cdot)$ and independent of N. The aggregate claims is the sum of the individual claim amounts and is defined by

$$S = X_1 + X_2 + \cdots + X_N \,. \tag{6}$$

The distribution function (d.f.), $F(\cdot)$, of S is obtained by conditioning on N and is given by the so-called *convolution* formula:

$$\begin{aligned} F(x) &= \Pr[S \leq x] \\ &= \sum_{k=0}^{\infty} \Pr[N = k] P^{*k}(x) \end{aligned} \tag{7}$$

where $P^{*k}(x)$ is the k-fold convolution of $P(\cdot)$ with itself. The first two moments can be easily obtained by conditioning. One finds that

$$E[S] = E[N] \cdot E[X] \tag{8}$$

and

$$\mathrm{Var}[S] = E[N] \cdot \mathrm{Var}[X] + E[X]^2 \cdot \mathrm{Var}[N] \tag{9}$$

where X has the same distribution as X_1. The third central moment is most easily found on a case basis using of the cumulant generating function:

$$E[(S - E[S])^3] = \frac{d^3}{dt^3} \ln E[e^{tS}] \bigg|_{t=0} = \frac{d^3}{dt^3} K_N(K_X(t)) \bigg|_{t=0} \tag{10}$$

where $K_N(t) = \ln E[e^{tN}]$ and $K_X(t) = \ln E[e^{tX}]$ are the cumulant generating functions of N and X respectively (assuming that they exist). In the collective model, the probability that there are no claims is

$$\pi = \Pr[N = 0] \,. \tag{11}$$

[1] Here, the X_i's are different from the ones in the individual model; the symbol X is reused to simplify the notation.

It is also possible to use models which are hybrids. For example, Kaas *et al.* (1988) have suggested to split portfolios to separate the ≪dangerous≫ risks from the standard risks. One would then use an individual model for the riskiest part and a collective model for the other. Independence of the risks is, as always, assumed.
In the rest of this paper, we will sometimes use the following symbols to simplify the notation:

$$\mu \equiv E[S]\,, \tag{12}$$
$$\sigma^2 \equiv \mathrm{Var}[S] \tag{13}$$

and

$$\gamma \equiv E[(S-\mu)^3]\,, \tag{14}$$

whatever the model for S is.

3 The net stop-loss premium

Under a stop-loss reinsurance contract with deductible $d \geq 0$, the amount assumed by the reinsurer is a random variable I_d which is the excess of the aggregate claims over the deductible or zero if S is less than or equal to d. Then, we can write:

$$I_d = (S-d)_+ = \begin{cases} 0 & \text{if} \quad S \leq d\,, \\ S-d & \text{if} \quad S > d\,. \end{cases} \tag{15}$$

The net stop-loss premium is the expected value of I_d, that is

$$E[I_d] = \int_{-\infty}^{\infty} (x-d)_+ \, dF(x) = \int_{d}^{\infty} (x-d)\, dF(x)\,. \tag{16}$$

An alternative expression for the net stop-loss premium is

$$E[I_d] = \int_{d}^{\infty} [1-F(x)]\, dx \tag{17}$$

which can be obtained by integration by parts.

4 Approximations

In this section, we present five approximations for the distribution of the aggregate claims. The first three are well-known: the normal approximation and the approximation by a gamma distribution with or without translation. The last two, the approximation by an inverse Gaussian distribution with or without translation are not as well known but we show that they can be of value especially when the distribution is ≪dangerous≫. In Section 6, we present a simple technique to improve on these approximations when the insurance portfolio is small.

4.1 Normal approximation

The oldest approximation for the distribution of the aggregate claims is the normal approximation. It is justified by the Central Limit Theorem: if the portfolio grows without bounds then the distribution of $(S - E[S])/(\mathrm{Var}[S])^{1/2}$ tends to the standard normal distribution. The approximation consists in assuming that S has a normal distribution with expectation $E[S]$ and variance $\mathrm{Var}[S]$. The normal probability density function (p.d.f.) is

$$\varphi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty, \tag{18}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are location and scale parameters, respectively When $\mu = 0$ and $\sigma = 1$, we have the standard normal distribution and we simply write $\varphi(x)$ for its p.d.f. The standard normal distribution function is denoted by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy. \tag{19}$$

Under the above assumption, the expectation and the variance of S are

$$E[S] = \mu, \tag{20}$$

and

$$\mathrm{Var}[S] = \sigma^2. \tag{21}$$

Obtaining the net stop-loss premium is a simple exercise of integration requiring only an appropriate change of variable; starting with the definition (16), one finds that

$$E[I_d] = \sigma \cdot \varphi\left(\frac{d-\mu}{\sigma}\right) + (\mu - d) \cdot \left[1 - \Phi\left(\frac{d-\mu}{\sigma}\right)\right]. \tag{22}$$

If the moments of S are given, the parameters of the normal distribution are obtained immediately from (20) and (21). The normal p.d.f. being symmetric, its third central moment vanishes. Thus, it is then not possible to take into account the typical asymmetry of the distribution of S.

4.2 Gamma approximation

The gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ has the following p.d.f.:

$$g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0. \tag{23}$$

The corresponding d.f. will be denoted by $G(x; \alpha, \beta)$. If we assume that S has a gamma distribution then we have

$$E[S] = \frac{\alpha}{\beta}, \tag{24}$$

$$\mathrm{Var}[S] = \frac{\alpha}{\beta^2}, \tag{25}$$

and

$$E[(S - E[S])]^3 = \frac{2\alpha}{\beta^3}. \tag{26}$$

It should be noted that the third central moment is always positive.
The formula for the net stop-loss premium is readily obtained from formula (16):

$$E[I_d] = \frac{\alpha}{\beta}[1 - G(d; \alpha + 1, \beta)] - d[1 - G(d; \alpha, \beta)]. \tag{27}$$

When the first two moments of S are given, one finds the parameters of the approximating gamma distribution to be, according to (24) and (25):

$$\alpha = \frac{E[S]^2}{\mathrm{Var}[S]} \tag{28}$$

and

$$\beta = \frac{E[S]}{\mathrm{Var}[S]} \,. \tag{29}$$

4.3 *Translated gamma approximation*

If a random variable Y has a gamma distribution then $Y + x_0$ has a translated gamma distribution. The density function of $Y + x_0$, denoted here by $g^t(\cdot)$, is, of course, given by

$$g^t(x) = g(x - x_0)\,, \qquad x > x_0\,. \tag{30}$$

The associated distribution function is simply $G(x - x_0; \alpha, \beta)$. The first three moments of this distribution are given by

$$E[S] = \frac{\alpha}{\beta} + x_0\,, \tag{31}$$

$$\mathrm{Var}[S] = \frac{\alpha}{\beta^2}\,, \tag{32}$$

and

$$E[(S - E[S])^3] = \frac{2\alpha}{\beta^3}\,. \tag{33}$$

Of course, the central moments have not changed, only the expected value is modified by the translation of the gamma distribution.
Since the gamma distribution is only translated to the right by x_0, then, for $d > x_0$, the net stop-loss premium in the translated case is, according to (27),

$$E[I_d] = \frac{\alpha}{\beta}[1 - G(d - x_0; \alpha + 1, \beta)] + (x_0 - d)[1 - G(d - x_0; \alpha, \beta)]\,, \qquad d > x_0\,. \tag{34}$$

The case $d < x_0$ cannot happen in practice: the reinsurer would pay everything with probability one!
For a given portfolio with known first three moments, the parameters of the translated gamma distribution are obtained by solving the system of equations given by (31), (32) and (33). This gives the following formulas:

$$\alpha = \frac{4\,\mathrm{Var}[S]^3}{E[(S - E[S])^3]^2}\,, \tag{35}$$

$$\beta = \frac{2\,\mathrm{Var}[S]}{E[(S - E[S])^3]} \tag{36}$$

and

$$x_0 = E[S] - \frac{2\,\mathrm{Var}[S]^2}{E[(S - E[S])^3]}\,. \tag{37}$$

To apply this approximation, it is required that the distribution of S be positively skewed. This is always the case in practice.

4.4 *Inverse Gaussian approximation*

The inverse Gaussian distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ has the following p.d.f.:

$$ig(x) = \frac{\alpha}{\sqrt{2\pi\beta}} x^{-\frac{3}{2}} e^{-\frac{(\beta x - \alpha)^2}{2\beta x}}\,, \qquad x > 0\,. \tag{38}$$

For $\alpha > 1$, the shape of this distribution is very similar to the one of a gamma distribution with the same parameters. But contrarily to the gamma distribution, its mode is always positive, even when $0 < \alpha < 1$. Its distribution function will be denoted by $IG(x; \alpha, \beta)$ and can be expressed in terms of the standardized normal distribution function:

$$IG(x;\alpha,\beta) = \Phi\left(\frac{-\alpha}{\sqrt{\beta x}} + \sqrt{\beta x}\right) + e^{2\alpha}\Phi\left(\frac{-\alpha}{\sqrt{\beta x}} - \sqrt{\beta x}\right), \quad x > 0\,. \tag{39}$$

If we assume that S has such a distribution, its moments are as follows:

$$E[S] = \frac{\alpha}{\beta}\,, \tag{40}$$

$$\mathrm{Var}[S] = \frac{\alpha}{\beta^2} \tag{41}$$

and

$$E[(S-\mu)^3] = \frac{3\alpha}{\beta^3}\,. \tag{42}$$

With the current parametrization, comparison of (24)–(26) with (40)–(42) shows that the inverse Gaussian has the same expected value and variance as the gamma distribution, but also that it has always a higher third central moment than the gamma distribution.

If we assume that S has an inverse Gaussian distribution, the basic formula (16) for the net stop-loss premium yields, after a change of variable and some calculations,

$$\begin{aligned} E[I_d] = {} & \left(\frac{\beta}{\alpha} - d\right)\left[1 - \Phi\left(\frac{-\alpha}{\sqrt{\beta d}} + \sqrt{\beta d}\right)\right] \\ & + \left(\frac{\alpha}{\beta} + d\right) e^{2\alpha} \Phi\left(\frac{-\alpha}{\sqrt{\beta d}} - \sqrt{\beta d}\right). \end{aligned} \tag{43}$$

An alternative proof is based on the verification that the derivative with respect to d of the expression on the right hand side of (43) equals $-[1 - IG(d;\alpha,\beta)]$, see (17).

Formula (43) is difficult to evaluate accurately when α is «large». The problem comes from the last summand but it can be easily circumvented by the use of an asymptotic development for $1-\Phi(\cdot)$ (see the appendix for this development).

For a given portfolio with known first two moments, the parameters of the inverse Gaussian distribution are obtained from the equations given by (40) and (41). It results the following formulas which are the same as the ones for the gamma distribution:

$$\alpha = \frac{E[S]^2}{\mathrm{Var}[S]} \tag{44}$$

and

$$\beta = \frac{E[S]}{\mathrm{Var}[S]}\,. \tag{45}$$

4.5 Translated inverse Gaussian approximation

If a random variable Y has an inverse Gaussian distribution, then $Y+x_0$ has a translated inverse Gaussian distribution. The density function of $Y+x_0$, denoted here by $ig^t(\cdot)$, is, of course, given by

$$ig^t(x) = ig(x - x_0)\,, \qquad x > x_0\,. \tag{46}$$

The associated distribution function is simply $IG(x - x_0; \alpha, \beta)$. The first three moments of this distribution are given by

$$E[S] = \frac{\alpha}{\beta} + x_0\,, \tag{47}$$

$$\mathrm{Var}[S] = \frac{\alpha}{\beta^2}\,, \tag{48}$$

and

$$E[(S - E[S])^3] = \frac{3\alpha}{\beta^3}\,. \tag{49}$$

Again, only the expected value is modified by the addition of x_0.
For the same reasons as in the gamma case, the net stop-loss premium for the translated inverse Gaussian is, for $d > x_0$, only a translation by x_0 of the corresponding function in the non-translated case:

$$\begin{aligned} E[I_d] = {} & \left(\frac{\alpha}{\beta} + x_0 - d\right)\left[1 - \Phi\left(\frac{-\alpha}{\sqrt{\beta(d - x_0)}} + \sqrt{\beta(d - x_0)}\right)\right] \\ & + \left(\frac{\alpha}{\beta} - x_0 + d\right)e^{2\alpha}\Phi\left(\frac{-\alpha}{\sqrt{\beta(d - x_0)}} - \sqrt{\beta(d - x_0)}\right). \end{aligned} \tag{50}$$

Like in the translated gamma case, d is never less than x_0.
For a given portfolio with known first three moments, the parameters of the approximating translated gamma distribution are obtained by solving the system of equations given by (47), (48) and (49). The formulas for the parameters are

$$\alpha = \frac{9\,\mathrm{Var}[S]^3}{E[(S - E[S])^3]^2}\,, \tag{51}$$

$$\beta = \frac{3\,\mathrm{Var}[S]}{E[(S - E[S])^3]} \tag{52}$$

and

$$x_0 = E[S] - \frac{3\,\mathrm{Var}[S]^2}{E[(S - E[S])^3]} \,. \tag{53}$$

5 Numerical illustrations

We present three numerical applications of the approximations discussed in the previous sections. In the first two examples, a compound Poisson model is assumed. The third example is based on the pension fund of Held (1982); an individual model is used for that case.

5.1 Case: Gamma/Poisson $\lambda = 10$

For this example we suppose that S has a compound Poisson distribution with parameter $\lambda = 10$ and that the distribution of an individual claim amount has a gamma distribution with parameters $\alpha = 2$ and $\beta = 2/1000$. It follows that

$$\mu = E[S] = \lambda E[X] = \lambda \frac{\alpha}{\beta} = 10\,000 \,,$$

$$\sigma^2 = \mathrm{Var}[S] = \lambda E[X^2] = \lambda \frac{\alpha(\alpha+1)}{\beta^2} = 1.5 \times 10^7$$

and

$$\gamma = E[(S-\mu)^3] = \lambda E[X^3] = \lambda \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^2} = 3.0 \times 10^{10} \,.$$

The coefficient of variation is $\sigma/\mu = 0.3873$ and the coefficient of skewness is $\gamma/\sigma^3 = 0.5164$.
The parameters of the five distributions suggested as approximations to the real distribution of S are shown in Table 1. The formulas used are those found at the end of subsections 4.1 to 4.5.

Table 1 Parameters used in the approximations
Case Gamma/Poisson $\lambda = 10$

Approximations	Parameters				
	μ	σ^2	α	β	x_0
Normal	10^4	1.5×10^7			
Gamma			$20/3$	$2/3 \times 10^{-3}$	
Translated gamma			15	0.001	−5000
Inverse Gaussian			$20/3$	$2/3 \times 10^{-3}$	
Translated inverse Gaussian			33.75	0.0015	−12 500

Table 2 Approximations of the net stop-loss premiums expressed as a percentage of the ≪exact≫ values
Case Gamma/Poisson $\lambda = 10$

			% of exact stop-loss premium				
d	$F(d)$	Exact Stop-loss premium	Normal	Gamma	Translated gamma	Inverse Gaussian	Translated inverse Gaussian
13 000	0.79071	556.30	87.50	104.57	99.66	109.04	99.39
14 000	0.84918	377.41	80.29	108.11	99.80	117.19	99.61
15 000	0.89435	250.22	71.83	112.73	100.08	128.15	100.05
16 000	0.92796	162.25	62.48	118.58	100.51	142.62	100.75
17 000	0.95211	102.97	52.70	125.85	101.12	161.52	101.77
18 000	0.96893	64.02	43.01	134.79	101.93	186.12	103.15
19 000	0.98031	39.02	33.88	145.71	102.99	218.18	104.95
20 000	0.98779	23.34	25.72	158.95	104.29	260.05	107.21
21 000	0.99258	13.71	18.77	174.97	105.88	315.05	110.00

Table 2 shows the ≪exact≫ net stop-loss premiums in column 3 and the approximate stop-loss premiums from the approximations in the subsequent columns. A very fine discretization has been used to compute the ≪exact≫distribution of S with the so-called Panjer's algorithm (Panjer (1981)). Table 2 gives also the values of the distribution function of S at the selected deductibles. It can be seen that the normal approximation is very poor while the (nontranslated) gamma and inverse Gaussian distribution give reasonable estimates, at least for the smallest values of d presented. The translated versions of the gamma and inverse Gaussian approximations are, respectively, within 6 % and 10 % of the true values.

5.2 *Case: Gamma/Poisson* $\lambda = 100$

This second example is related to the first one. We make the same assumptions on the distribution of S except for the Poisson parameter which is now 10 times larger than before, that is $\lambda = 100$.
The expected value of S, its variance and third central moment are simply 10 times larger than the corresponding values in the first example:

$$\mu = E[S] = 100\,000\,,$$
$$\sigma^2 = \mathrm{Var}[S] = 1.5 \times 10^8$$

and

$$\gamma = E[(S-\mu)^3] = 3.0 \times 10^{11}$$

The coefficient of variation is $\sigma/\mu = 0.1225$ and the coefficient of skewness is $\gamma/\sigma^3 = 0.1633$. The latter coefficient indicates that we are getting closer to normality than in the first example; nevertheless, this value is still too large to let us expect a good performance of the normal approximation. The normal distribution being a limiting case of both the gamma and the inverse Gaussian distribution (with or without translation), we can expect the precision of these approximations to be better than in the first example. The parameters of the five distributions suggested as approximations to the real distribution of S are shown in Table 3.

Table 3 Parameters used in the approximations
Case Gamma/Poisson $\lambda = 100$

Approximations	Parameters				
	μ	σ^2	α	β	x_0
Normal	10^5	1.5×10^8			
Gamma			200/3	$2/3 \times 10^{-3}$	
Translated gamma			150	0.001	−50 000
Inverse Gaussian			200/3	$2/3 \times 10^{-3}$	
Translated inverse Gaussian			337.5	0.0015	−125 000

Table 4 shows the approximate net stop-loss premiums expressed as a percentage of the exact ones found in the third column. As expected, all approximations have improved in precision. The precision of the translated gamma and translated inverse Gaussian approximations is almost spectacular! This is (partly) due to the fact that the ≪true≫ underlying distribution is not really dangerous. The three other approximations are much less successful in dealing with the very end of the right tail of the distribution. Finally, one should note the very significant (and positive) effect of the introduction of the location parameter, x_0, on the inverse Gaussian approximation.

Table 4 Approximations of the net stop-loss premiums expressed as a percentage of the ≪exact≫ values
Case Gamma/Poisson $\lambda = 100$

			% of exact stop-loss premium				
d	$F(d)$	Exact Stop-loss premium	Normal	Gamma	Translated gamma	Inverse Gaussian	Translated inverse Gaussian
110 000	0.79560	1505.50	94.98	102.33	99.97	105.58	99.95
115 000	0.88744	728.38	89.65	105.05	100.02	112.40	100.04
120 000	0.94438	320.62	82.10	109.23	100.14	123.23	100.24
125 000	0.97531	128.36	72.49	115.28	100.35	139.75	100.61
130 000	0.99013	46.78	61.29	123.67	100.67	164.45	101.19

5.3 Pension fund of Held (1982)

In this third example, we utilize the pension fund PK-230 of Held (1982). We consider only positive amounts at risk, so there are no negative claim amounts. Therefore, negative risk sums in the data have been set equal to zero. Further, we assume the use of an individual model. The resulting ≪portfolio≫ can be considered small since its expected number of claims is about 1.23.

The moments, computed directly from the adjusted data, are

$$\mu = E[S] = 66478.19\,,$$
$$\sigma^2 = \mathrm{Var}[S] = 7.041421 \times 10^9$$

and

$$\gamma = E[(S-\mu)^3] = 1.119695 \times 10^{15}.$$

The coefficient of variation is $\sigma/\mu = 1.2623$ and the coefficient of skewness $\gamma/\sigma^3 = 1.895$. Clearly, the distribution of the aggregate claims is far from normality.

Table 5 shows the parameters of the approximating distributions. It should be noted that the shape of the gamma distribution and the one of the translated gamma distribution are completely different: the mode is zero for the former and greater than zero (bell shape) for the latter.

Table 5 Parameters used in the approximations
Pension fund PK-230

	Parameters				
Approximations	μ	σ^2	α	β	x_0
Normal	66 478.19	7.041421×10^9			
Gamma			0.627622	9.44101×10^{-6}	
Translated gamma			1.117463	1.25976×10^{-5}	−22 226.5
Inverse Gaussian			0.627622	9.44101×10^{-6}	
Tr. inverse Gaussian			2.514293	1.88963×10^{-5}	−66 578.9

Table 6 gives the exact net stop-loss premiums and the approximations, in addition to the distribution function for six different values of the deductible *d*. The ≪true≫ probability function has been calculated by brute force convolution. Using the variant of Kornya's algorithm suggested by Dufresne (1996) would lead to the very same results.

Not surprisingly, the normal approximation gives very bad results: the (true) distribution *is* dangerous. The non-translated gamma and inverse Gaussian approximations end up with significant estimation errors. Their translated versions provide satisfactory approximations.

This pension fund being small (from a modeling point of view at least), the distribution has a substantial (and disturbing) probability mass at zero. The approximations have a hard time coping with this spike at zero. In the next section, we suggest a technique to get rid of it.

Table 6 Approximations of the net stop-loss premiums expressed as a percentage of the ≪exact≫ values
Pension fund PK-230

			% of exact stop-loss premium				
d	$F(d)$	Exact Stop-loss premium	Normal	Gamma	Translated gamma	Inverse Gaussian	Translated inverse Gaussian
280 000	0.97145	2230.10	6.56	136.15	102.62	176.16	103.17
290 000	0.97523	1963.16	5.10	139.57	103.05	185.51	104.21
300 000	0.97843	1729.71	3.91	142.98	103.38	195.30	105.20
360 000	0.98965	814.74	0.61	164.70	104.55	267.35	111.31
370 000	0.99070	715.94	0.43	169.36	105.13	283.29	112.91
380 000	0.99181	628.10	0.30	174.46	105.88	300.81	114.74

6 Improving the approximations

When the portfolio is small, there is usually a non-negligible probability that there are no claims in the given period of time. This implies that the distribution has a probability mass at zero and it can be expected that approximations based on continuous p.d.f. would benefit from its removal, complete or partial. We present such a technique below.

6.1 Case of non-negative claims

If the claim amounts are non-negative, the aggregate claims amount will be zero only if there are no claims. The probability of this event is known in our models:

$$\Pr[S = 0] = F(0+) - F(0-) = \pi\,. \tag{54}$$

We can condition on S to isolate the probability mass at zero:

$$F(x) = \Pr[S \le x \mid S = 0]\ \Pr[S = 0] + \Pr[S \le x \mid S > 0]\ \Pr[S > 0]\,. \tag{55}$$

Let $\varepsilon(x)$ denote the degenerate distribution at zero, that is

$$\varepsilon(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ 1 & \text{if } x \ge 0\,. \end{cases} \tag{56}$$

Using (54) and (56), (55) can be rewritten as follows:

$$\begin{aligned} F(x) &= \pi\varepsilon(x) + (1-\pi)\ \Pr[S \le x \mid S > 0] \\ &= \pi\varepsilon(x) + (1-\pi)\widetilde{F}(x) \end{aligned} \tag{57}$$

where $\widetilde{F}(x) = \Pr[S \le x \mid S > 0]$. If we denote by $\widetilde{S}$ the random variable $S \mid S > 0$, then $\widetilde{F}(\cdot)$ is its distribution function. Formula (57) shows that the distribution function $F(\cdot)$ can be written as a mixture of two distributions functions. The Beekman-Bowers approximation also uses such a representation in the context of the evaluation of ruin probabilities (see Beekman (1969)).

The idea now consists in applying the approximations presented in the Section 4 to $\widetilde{F}(\cdot)$ instead of $F(\cdot)$. It remains to determine the central moments of $\widetilde{S}$ in terms of those of S.

We first note that $d\widetilde{F}(x) = dF(x)/(1-\pi)$, for $x > 0$. The required expected value $\widetilde{\mu}$, variance $\widetilde{\sigma}^2$ and third central moment $\widetilde{\gamma}$ of $\widetilde{S}$ are obtained by isolating the probability mass at zero, as described below. We start with the relation

$$\begin{aligned} E[S^k] &= 0^k\pi + (1-\pi)E[\widetilde{S}^k] \\ &= (1-\pi)E[\widetilde{S}^k]\,, \qquad \text{for } k = 1, 2, 3, \ldots \end{aligned} \tag{58}$$

obtained by conditioning on the events $S = 0$ and $S > 0$. Then the expectation of $\widetilde{S}$ is

$$\widetilde{\mu} = E[\widetilde{S}] = \frac{E[S]}{1-\pi}\,. \tag{59}$$

We proceed in a similar way for the variance:

$$\begin{aligned}\sigma^2 = \mathrm{Var}[S] &= E[(S-\mu)^2] \\ &= \pi\mu^2 + (1-\pi)E[(\widetilde{S}-\mu)^2]\,.\end{aligned} \tag{60}$$

If we replace $\widetilde{S} - \mu$ by $(\widetilde{S} - \widetilde{\mu}) + (\widetilde{\mu} - \mu)$ in (60), we find that

$$\sigma^2 = \pi\mu^2 + (1-\pi)\{\widetilde{\sigma}^2 - (\widetilde{\mu}-\mu)^2\}$$

which yields, after some manipulations,

$$\widetilde{\sigma}^2 = \frac{\sigma^2}{1-\pi} - \pi\widetilde{\mu}^2\,. \tag{62}$$

Finally, for the third central moment, if we apply the same technique we see that

$$\gamma = E[(S-\mu)^3] = -\mu^3\pi + (1-\pi)E[(\widetilde{S}-\mu)^3]\,. \tag{63}$$

Again, we replace $\widetilde{S} - \mu$ by $(\widetilde{S} - \widetilde{\mu}) + (\widetilde{\mu} - \mu)$ in (63) and we find that

$$\gamma = -\mu^3\pi + (1-\pi)\{\widetilde{\gamma} + 3\widetilde{\sigma}^2(\widetilde{\mu}-\mu) + (\widetilde{\mu}-\mu)^3\} \tag{64}$$

and, finally, we obtain

$$\widetilde{\gamma} = \frac{\gamma}{1-\pi} - 3\pi\widetilde{\mu}\widetilde{\sigma}^2 + \pi(1-2\pi)\widetilde{\mu}^3\,. \tag{65}$$

Of course, formulas (59), (61) and (64) are sufficient to determine the desired moments; formulas (62) and (65) are simply more explicit.
We now return to the problem of determining the net stop-loss premiums. Since

$$\begin{aligned}E[(S-d)_+] &= E[(S-d)_+ \mid S=0]\ \Pr[S=0] \\ &\quad + E[(S-d)_+ \mid S>0]\ \Pr[S>0]\,, \qquad d \geq 0\,,\end{aligned} \tag{66}$$

it follows that

$$\begin{aligned} E[I_d] &= (1-\pi)E[(\widetilde{S}-d)_+] \\ &= (1-\pi)E[\tilde{I}_d] \end{aligned} \tag{67}$$

with $E[\tilde{I}_d] = E[(\widetilde{S}-d)_+]$. The last equality means that the desired net stop-loss premiums are simply $1-\pi$ times the corresponding ones obtained by using $\widetilde{S}$ instead of S.

6.2 *Case of general claims*

If negative claim amounts are allowed, the preceding technique can still be applied. The only fundamental difference is that we would condition on the claims/no claims events instead of $S=0$ and $S>0$. The final formulas for $\widetilde{\mu}$, $\widetilde{\sigma}^2$ and $\widetilde{\gamma}$ are the same.

7 Numerical illustrations

The pension fund PK-230 of Held (1982) will be used again, to show the improvements in the approximations resulting from the application of the technique presented in Section 6. We consider only positive amounts at risk, so there are no negative claim amounts.
The probability that there are no claims for this portfolio (under the preceding assumption) is

$$\pi = 0.287247$$

and can be computed directly from the (modified) data. We use the formulas of the previous section to calculate the moments of $\widetilde{S}$:

$$\begin{aligned} E[\widetilde{S}] &= 93\,269.56 \\ \mathrm{Var}[\widetilde{S}] &= 7.380364 \times 10^9 \\ \widetilde{\gamma} &= 1.074408 \times 10^{15} \end{aligned}$$

The coefficient of variation of $\widetilde{S}$ is $\widetilde{\sigma}/\widetilde{\mu} = 0.9211$ and the coefficient of skewness is $\widetilde{\gamma}/\widetilde{\sigma}^3 = 1.6945$. The distribution of $\widetilde{S}$ is still far from normality.

The parameters of the approximating distributions for $\widetilde{F}(\cdot)$ are given in Table 7. The most noticeable changes are the increase in the α parameter. For the non-translated gamma and inverse Gaussian distributions this parameter is now greater than one. In the gamma case that means that the mode is positive (the p.d.f. has a bell shape) while it was zero without the modification.

Table 7 Parameters used in the approximations of $\widetilde{F}(\cdot)$
Pension fund PK-230

	Parameters				
Approximations	μ	σ^2	α	β	x_0
Normal	93 269.56	7.380364×10^9			
Gamma			1.178698	1.26375×10^{-5}	
Translated gamma			1.393012	1.37385×10^{-5}	−8 125.4
Inverse Gaussian			1.178698	1.26375×10^{-5}	
Tr. inverse Gaussian			3.134278	2.06077×10^{-5}	−58 822.8

Table 8 Approximations of the net stop-loss premiums expressed as a percentage of the ≪exact≫ values
(with modification for the mass at zero)
Pension fund PK-230

			% of exact stop-loss premium				
d	$F(d)$	Exact Stop-loss premium	Normal	Gamma	Translated gamma	Inverse Gaussian	Translated inverse Gaussian
280 000	0.97145	2230.10	20.28	107.51	100.31	140.07	99.99
290 000	0.97523	1963.16	16.49	108.09	100.24	145.07	100.40
300 000	0.97843	1729.71	13.25	108.56	100.06	150.21	100.74
360 000	0.98965	814.74	2.76	110.46	97.93	186.07	102.55
370 000	0.99070	715.94	2.05	111.17	97.90	193.91	103.32
380 000	0.99181	628.10	1.50	112.06	98.01	202.49	104.28

Table 8 gives the approximations to net the stop-loss premiums as a percentage of the ≪exact≫ premiums which is shown in column 3. We observe a general improvement of the precision of the approximations. In fact, the relative errors (in absolute value) in Table 8 are all less than the corresponding ones in Table 6. Therefore, all approximations have benefited from the added parameter π and the writing of $F(\cdot)$ as a mixture. As a rule, this technique should be applied whenever the considered portfolio is ≪small≫ since improvements in precision are expected and the computational effort is negligible (the moments have to be computed anyway). Also, it cannot harm even if applied where it is not really necessary.

Conclusions

We have shown that analytical approximations can be useful to quickly provide good approximations of stop-loss premiums. Whenever the third moment of the distribution to be approximated is known, one should use either the translated gamma or inverse Gaussian approximation. If the portfolio is small, one should also ≪remove≫ (partially or not) the probability mass at zero. This technique requires that the probability that there are no claims be known or readily computable. Moreover, this technique, if it can be applied, cannot harm.

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Appendix

If the shape parameter α is «large», numerical problems arise in the inverse Gaussian case (with or without translation). Then the following asymptotic development can be useful to numerically evaluate formulas (43) and (50):

$$1 - \Phi(x) = \frac{\varphi(x)}{x}\left\{1 - \frac{1}{x^2} + \frac{1 \cdot 3}{x^4} - \frac{1 \cdot 3 \cdot 5}{x^6} + \cdots \right. \\ \left. + \frac{(-1)^n \cdot 1 \cdot 3 \cdot \dots \cdot (2n-1)}{x^{2n}}\right\} + R_n(x)$$

where $R_n(x) = (-1)^n \cdot 1 \cdot 3 \cdot \dots \cdot (2n+1) \cdot \int\limits_x^\infty \frac{\varphi(t)}{t^{2n+2}}\, dt.$

From Abramowitz & Stegun (1972).

Summary

This paper presents and compares five analytical formulas for the approximation of stop-loss premiums. Two of them, based on the inverse Gaussian distribution, are not widely known. The authors also suggest a technique which improves the precision of these approximations for ≪small≫ portfolios.

Résumé

L'article présente et compare cinq formules analytiques pour l'approximation de primes stop-loss. Deux d'entre elles, basées sur la distribution gaussienne inverse, sont peu connues. On y suggère aussi une technique permettant d'améliorer la performance de ces approximations lorsque le portefeuille d'assurance est ≪petit≫.

Zusammenfassung

Diese Arbeit erläutert und vergleicht fünf analytische Formeln für die Approximation von Stop-Loss Praemien. Die beiden, welche auf der inversen Gaussverteilung basieren, scheinen neu zu sein. Ferner schlagen die Autoren eine Technik vor, wie die Präzision dieser Approximationen für "kleine" Portefeuilles verbessert werden kann.