

**Zeitschrift:** Mitteilungen / Schweizerische Vereinigung der  
Versicherungsmathematiker = Bulletin / Association Suisse des  
Actuaires = Bulletin / Swiss Association of Actuaries

**Herausgeber:** Schweizerische Vereinigung der Versicherungsmathematiker

**Band:** - (1995)

**Heft:** 1

**Rubrik:** Kurzmitteilungen

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 20.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## D. Kurzmitteilungen

A.H. SHARIF and H.H. PANJER

### An Improved Recursion for the Compound Generalized Poisson Distribution

**Abstract** Goovaerts and Kaas (1991) gave a two step recursive scheme to evaluate compound generalized Poisson distributions. In this paper, their recursive scheme is improved and more general results are proposed.

#### 1 Introduction

Panjer (1981) introduced a recursive algorithm to evaluate the probability function for a class of compound distributions. The probability function representing the probability for the number of claims is assumed to satisfy a first order recursion. This method does not work directly in the case of compound generalized Poisson distributions (GPD). Goovaerts and Kaas (GK) (1991) for the first time gave an elegant two step recursive scheme to evaluate compound generalized Poisson distributions. They use the fact that GPD is itself a compound Poisson with an associated Borel distribution. They derived a recursive scheme to evaluate a compound Borel distribution for a very particular compounding random variable  $Z$  having  $P(Z = 0) = 0$  and  $P(Z = 1) > 0$ . In the second step, they used the Panjer recursion for the compound Poisson to get the final results of compound GPD.

In this paper we give an improved and more generalized recursion scheme to evaluate the compound GPD. The requirement of the conditions  $P(Z = 0) = 0$  and  $P(Z = 1) > 0$  are totally removed.

#### 2 Improved recursion

To avoid repetition of the derivation, we will use the same definition and notations as described in Goovaerts and Kaas (1991). We use their established

results to derive a revised two step recursion. Equations in GK will be referenced by GK(.). Let

$$S = \sum_{i=1}^N Z_i$$

where  $N$  is assumed to be  $GP(\theta, \lambda)$ , a generalized Poisson distribution random variable, and  $Z_i$  is assumed to be a non-negative integer valued (zero inclusive) random variable. GK considered only positive values ( $Z_i \geq 1$ ) and excluded zero values to avoid possible difficulties. GK claims on page 195 that,

*“By excluding zero-claims, we avoid problems later on, when we have to compute  $P[S = 0]$  to start a recursion.”*

They are right in their recursive scheme, but in our improved recursion  $Z_i$  having zero values does not create any problem at all.

Assume  $Z_i$  to be a non-negative integer valued random variable. Let  $h$  be the smallest value of  $Z_i$  taking on a positive probability, and  $Z_1, Z_2, \dots, Z_N$  be independent and identically distributed. Then given the usual assumption of independence of  $N$  and  $Z_i$ , as in GK(12) we have the pgf of  $S$  as

$$G_S(u) = e^{\theta(t-1)} \quad \text{GK(12)}$$

with

$$te^{-\lambda(t-1)} = G_Z(u) \quad \text{GK(12)}$$

where  $G_Z(u)$  is the pgf of the random variable  $Z$ .

Taking log and differentiating we have

$$\frac{t'(u)}{t(u)} - \lambda t'(u) = \frac{G'_Z(u)}{G_Z(u)} \quad \text{GK(13)}$$

Rearranging it leads to

$$t'(u) = \frac{t(u)}{1 - \lambda t(u)} \frac{G'_Z(u)}{G_Z(u)} \quad \text{GK(14)}$$

At this point, we rearrange GK(14) as follows

$$ut'(u)\{1 - \lambda t(u)\} = t(u) \frac{uG'_Z(u)}{G_Z(u)} \quad (2.1)$$

As in GK(15), let us define the sequences  $\{\alpha_n\}$  and  $\{r_n\}$  given by

$$t(u) = \sum_{n=0}^{\infty} \alpha_n u^n$$

and

$$\frac{uG'_Z(u)}{G_Z(u)} = \sum_{n=0}^{\infty} r_n u^n \quad (2.2)$$

Notice that we deviated from GK(15) by including  $\alpha_0$  in the sequence. We also excluded the auxiliary sequence  $\{\beta_n\}$  since it is not needed at all in our scheme.

The coefficients  $r_n$  depend solely on the known probability function of  $Z$ . Let  $p_n = P[Z_i = n]$ ,  $n = 0, 1, 2, \dots$ ; and

$$p_0 = p_1 = \dots = p_{h-1} = 0, \quad p_h > 0.$$

So we have

$$G_Z(u) = \sum_{n=h}^{\infty} p_n u^n = u^h \sum_{n=0}^{\infty} p_{n+h} u^n$$

Hence from (2.2), cancelling  $u^h$  from both numerator and denominator on the left hand side and transposing the denominator to the right hand side we have

$$\sum_{n=0}^{\infty} (n+h) p_{n+h} u^n = \left\{ \sum_{n=0}^{\infty} p_{n+h} u^n \right\} \left\{ \sum_{n=0}^{\infty} r_n u^n \right\}.$$

Now comparing coefficients of  $u^n$  we have

$$(n+h)p_{n+h} = \sum_{j=0}^n r_j p_{n+h-j} \quad \text{for } n = 0, 1, 2, \dots$$

Then the sequence  $\{r_n\}$  can be evaluated recursively as follows:

$$r_n = \frac{1}{p_h} \left\{ (n+h)p_{n+h} - \sum_{j=0}^{n-1} r_j p_{n+h-j} \right\} \quad \text{for } n = 0, 1, 2, \dots \quad (2.3)$$

Note that  $r_0 = h$  and GK(19) is a particular case of our (2.3) for  $h = 1$ .

Having evaluated the sequence  $\{r_n\}$ , we are in a final stage of evaluating the desired sequence  $\{\alpha_n\}$ , the compound Borel probability distribution. Comparing the coefficients of  $u^n$  in (2.1) we have

$$n\alpha_n - \lambda \sum_{j=0}^n j\alpha_j\alpha_{n-j} = \sum_{j=0}^n \alpha_j r_{n-j} \quad (2.4)$$

Hence rearranging we have the recursion in its most general form:

$$\alpha_n = \frac{1}{n-h-\lambda\alpha_0 n} \left\{ \sum_{j=0}^{n-1} \alpha_j (\lambda j \alpha_{n-j} + r_{n-j}) \right\} \\ \text{for } n = h+1, h+2, \dots \quad (2.5)$$

Note that the first term in the first brace on the right hand side becomes zero for  $j = 0$ . Now the initial values  $\alpha_0, \alpha_1, \dots, \alpha_h$  will be determined from the definition of  $t(u)$ . Since by definition, as in GK(12)

$$te^{-\lambda(t-1)} = G_Z(u)$$

we have

$$\left\{ \sum_{n=0}^{\infty} \alpha_n u^n \right\} e^{-\lambda(\sum_{n=0}^{\infty} \alpha_n u^n)} = \left\{ \sum_{n=h}^{\infty} p_n u^n \right\} e^{-\lambda}.$$

Now comparing the coefficients of  $u^n$  we have for  $h > 0$

$$\alpha_0 = \alpha_1 = \dots = \alpha_{h-1} = 0 \quad \text{and} \quad \alpha_h = p_h e^{-\lambda}$$

For  $h = 0$ , the value of  $\alpha_0$  is given by the implicit relation

$$\alpha_0 e^{-\lambda\alpha_0} = p_0 e^{-\lambda}.$$

Even though  $\alpha_0$  does not have an explicit expression in  $p_0$  and  $\lambda$ , it has a unique value given by the above relation where uniqueness is guaranteed by the GPD parametric restriction namely  $\lambda < 1$ . Therefore (2.5) can be rewritten as

$$\alpha_n = \frac{1}{n - h - \lambda\alpha_0 n} \left\{ \sum_{j=h}^{n-1} \alpha_j (\lambda j \alpha_{n-j} + r_{n-j}) \right\} \\ \text{for } n = h + 1, h + 2, \dots \quad (2.6)$$

For  $h = 1$ , this simplifies to

$$\alpha_n = \frac{1}{n - 1} \left\{ \sum_{j=1}^{n-1} \alpha_j (\lambda j \alpha_{n-j} + r_{n-j}) \right\} \\ \text{for } n = 2, 3, \dots \quad (2.7)$$

with  $\alpha_0 = 0$  and  $\alpha_1 = p_1 e^{-\lambda}$ .

Note that GK(23) is a particular case of our above results for  $h = 1$ . The recursive evaluation of  $\alpha_n$  by GK(20)–(22) is enhanced by our recursion (2.7) which is elegant and computationally more efficient. A generalized version is given by (2.6) above.

Finally, the compound GPD is evaluated in the second step by application of Panjer's recursion for compound Poisson as in GK(24) with the starting value  $P(S = 0) = G_S(0) = e^{\theta(\alpha_0 - 1)}$ . For  $h = 1$ ,  $P(S = 0) = P(N = 0) = e^{-\theta}$  as was pointed out in GK.

### 3 Conclusions

Our recursive scheme (2.7) is more elegant and computationally more efficient as compared to the scheme in GK(20)–(22). Also our recursion in (2.6) is a generalized result in the sense that  $h$  could be any non-negative (including zero) integer while in GK it was fixed at  $h = 1$ . Our approach could be easily extended to some other Lagrange distributions to evaluate the compound distribution, namely the generalized negative binomial distribution and the Borel-Tanner distribution.

The generalized Poisson probability model was originally developed in the field of reliability specially in queueing theory. It had its application in

finding the distribution of busy period in a queueing model. The generalized Poisson model has also been used in another statistical research area namely biostatistics. Consul (1990) used it to model the distribution of injuries in auto-accidents. Because of the peculiarity of being embedded in a Lagrange expansion, it has not been very popular. We believe that our efficient algorithm might entice more actuaries to use the generalized Poisson model.

## References

- Consul, P.C. (1990). A model for distribution of injuries in auto-accidents. *Bulletin of the Swiss Association of Actuaries*, 1990, 1:161 – 168.
- Goovaerts, M.J. and Kaas, R. (1991). Evaluating compound generalized Poisson distributions recursively. *ASTIN Bulletin*, 21(2):193 – 198.
- Panjer, H. (1981). Recursive evaluation of a family of compound distributions. *ASTIN Bulletin*, 12:22 – 26.

A.H. Sharif and H.H. Panjer  
Department of Statistics and Actuarial Science  
University of Waterloo, Waterloo  
Ontario, Canada N2L 3G1.