D. Kurzmitteilungen

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An Improved Recursion for the Compound Generalized Poisson Distribution

Abstract Goovaerts and Kaas (1991) gave a two step recursive scheme to evaluate compound generalized Poisson distributions. In this paper, their recursive scheme is improved and more general results are proposed.

1 Introduction

Panjer (1981) introduced a recursive algorithm to evaluate the probability function for a class of compound distributions. The probability function representing the probability for the number of claims is assumed to satisfy a first order recursion. This method does not work directly in the case of compound generalized Poisson distributions (GPD). Goovaerts and Kaas (GK) (1991) for the first time gave an elegant two step recursive scheme to evaluate compound generalized Poisson distributions. They use the fact that GPD is itself a compound Poisson with an associated Borel distribution. They derived a recursive scheme to evaluate a compound Borel distribution for a very particular compounding random variable Z having $P(Z = 0) = 0$ and $P(Z = 1) > 0$. In the second step, they used the Panjer recursion for the compound Poisson to get the final results of compound GPD.
In this paper we give an improved and more generalized recursion scheme to evaluate the compound GPD. The requirement of the conditions $P(Z = 0) = 0$ and $P(Z = 1) > 0$ are totally removed.

2 Improved recursion

To avoid repetition of the derivation, we will use the same definition and notations as described in Goovaerts and Kaas (1991). We use their established results to derive a revised two step recursion. Equations in GK will be referenced by GK(.). Let

$$S = \sum_{i=1}^{N} Z_i$$

where N is assumed to be $GP(\theta, \lambda)$, a generalized Poisson distribution random variable, and Z_i is assumed to be a non-negative integer valued (zero inclusive) random variable. GK considered only positive values ($Z_i \geq 1$) and excluded zero values to avoid possible difficulties. GK claims on page 195 that,

> "*By excluding zero-claims, we avoid problems later on, when we have to compute $P[S = 0]$ to start a recursion.*"

They are right in their recursive scheme, but in our improved recursion Z_i having zero values does not create any problem at all.
Assume Z_i to be a non-negative integer valued random variable. Let h be the smallest value of Z_i taking on a positive probability, and $Z_1, Z_2, \ldots, Z_N$ be independent and identically distributed. Then given the usual assumption of independence of N and Z_i, as in GK(12) we have the pgf of S as

$$G_S(u) = e^{\theta(t-1)} \qquad \text{GK(12)}$$

with

$$te^{-\lambda(t-1)} = G_Z(u) \qquad \text{GK(12)}$$

where $G_Z(u)$ is the pgf of the random variable Z.
Taking log and differentiating we have

$$\frac{t'(u)}{t(u)} - \lambda t'(u) = \frac{G'_Z(u)}{G_Z(u)} \qquad \text{GK(13)}$$

Rearranging it leads to

$$t'(u) = \frac{t(u)}{1 - \lambda t(u)} \frac{G'_Z(u)}{G_Z(u)} \qquad \text{GK(14)}$$

At this point, we rearrange GK(14) as follows

$$ut'(u)\{1-\lambda t(u)\} = t(u)\frac{uG'_Z(u)}{G_Z(u)} \tag{2.1}$$

As in GK(15), let us define the sequences $\{\alpha_n\}$ and $\{r_n\}$ given by

$$t(u) = \sum_{n=0}^{\infty} \alpha_n u^n$$

and

$$\frac{uG'_Z(u)}{G_Z(u)} = \sum_{n=0}^{\infty} r_n u^n \tag{2.2}$$

Notice that we deviated from GK(15) by including α_0 in the sequence. We also excluded the auxiliary sequence $\{\beta_n\}$ since it is not needed at all in our scheme.

The coefficients r_n depend solely on the known probability function of Z. Let $p_n = P[Z_i = n]$, $n = 0, 1, 2, \ldots$; and

$$p_0 = p_1 = \cdots = p_{h-1} = 0\,, \quad p_h > 0\,.$$

So we have

$$G_Z(u) = \sum_{n=h}^{\infty} p_n u^n = u^h \sum_{n=0}^{\infty} p_{n+h} u^n$$

Hence from (2.2), cancelling u^h from both numerator and denominator on the left hand side and transposing the denominator to the right hand side we have

$$\sum_{n=0}^{\infty} (n+h) p_{n+h} u^n = \left\{ \sum_{n=0}^{\infty} p_{n+h} u^n \right\} \left\{ \sum_{n=0}^{\infty} r_n u^n \right\}.$$

Now comparing coefficients of u^n we have

$$(n+h) p_{n+h} = \sum_{j=0}^{n} r_j p_{n+h-j} \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

Then the sequence $\{r_n\}$ can be evaluated recursively as follows:

$$r_n = \frac{1}{p_h}\left\{(n+h)p_{n+h} - \sum_{j=0}^{n-1} r_j p_{n+h-j}\right\} \qquad \text{for} \quad n = 0, 1, 2, \ldots (2.3)$$

Note that $r_0 = h$ and GK(19) is a particular case of our (2.3) for $h = 1$. Having evaluated the sequence $\{r_n\}$, we are in a final stage of evaluating the desired sequence $\{\alpha_n\}$, the compound Borel probability distribution. Comparing the coefficients of u^n in (2.1) we have

$$n\alpha_n - \lambda \sum_{j=0}^{n} j\alpha_j\alpha_{n-j} = \sum_{j=0}^{n} \alpha_j r_{n-j} \tag{2.4}$$

Hence rearranging we have the recursion in its most general form:

$$\alpha_n = \frac{1}{n-h-\lambda\alpha_0 n}\left\{\sum_{j=0}^{n-1} \alpha_j(\lambda j\alpha_{n-j} + r_{n-j})\right\}$$
$$\text{for } n = h+1, h+2, \ldots \tag{2.5}$$

Note that the first term in the first brace on the right hand side becomes zero for $j = 0$. Now the initial values $\alpha_0, \alpha_1, \ldots, \alpha_h$ will be determined from the definition of $t(u)$. Since by definition, as in GK(12)

$$te^{-\lambda(t-1)} = G_Z(u)$$

we have

$$\left\{\sum_{n=0}^{\infty} \alpha_n u^n\right\} e^{-\lambda(\sum_{n=0}^{\infty} \alpha_n u^n)} = \left\{\sum_{n=h}^{\infty} p_n u^n\right\} e^{-\lambda}.$$

Now comparing the coefficients of u^n we have for $h > 0$

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{h-1} = 0 \qquad \text{and} \qquad \alpha_h = p_h e^{-\lambda}$$

For $h = 0$, the value of α_0 is given by the implicit relation

$$\alpha_0 e^{-\lambda\alpha_0} = p_0 e^{-\lambda}.$$

Even though α_0 does not have an explicit expression in p_0 and λ, it has a unique value given by the above relation where uniqueness is guaranteed by the GPD parametric restriction namely $\lambda < 1$. Therefore (2.5) can be rewritten as

$$\alpha_n = \frac{1}{n - h - \lambda\alpha_0 n}\left\{\sum_{j=h}^{n-1} \alpha_j(\lambda j \alpha_{n-j} + r_{n-j})\right\}$$
$$\text{for } n = h+1, h+2, \ldots \tag{2.6}$$

For $h = 1$, this simplifies to

$$\alpha_n = \frac{1}{n-1}\left\{\sum_{j=1}^{n-1} \alpha_j(\lambda j \alpha_{n-j} + r_{n-j})\right\}$$
$$\text{for } n = 2, 3, \ldots \tag{2.7}$$

with $\alpha_0 = 0$ and $\alpha_1 = p_1 e^{-\lambda}$.
Note that GK(23) is a particular case of our above results for $h = 1$. The recursive evaluation of α_n by GK(20) – (22) is enhanced by our recursion (2.7) which is elegant and computationally more efficient. A generalized version is given by (2.6) above.
Finally, the compound GPD is evaluated in the second step by application of Panjer's recursion for compound Poisson as in GK(24) with the starting value $P(S = 0) = G_S(0) = e^{\theta(\alpha_0 - 1)}$. For $h = 1$, $P(S = 0) = P(N = 0) = e^{-\theta}$ as was pointed out in GK.

3 Conclusions

Our recursive scheme (2.7) is more elegant and computationally more efficient as compared to the scheme in GK(20) – (22). Also our recursion in (2.6) is a generalized result in the sense that h could be any non-negative (including zero) integer while in GK it was fixed at $h = 1$. Our approach could be easily extended to some other Lagrange distributions to evaluate the compound distribution, namely the generalized negative binomial distribution and the Borel-Tanner distribution.
The generalized Poisson probability model was originally developed in the field of reliability specially in queueing theory. It had its application in

finding the distribution of busy period in a queueing model. The generalized Poisson model has also been used in another statistical research area namely biostatistics. Consul (1990) used it to model the distribution of injuries in auto-accidents. Because of the peculiarity of being embedded in a Lagrange expansion, it has not been very popular. We believe that our efficient algorithm might entice more actuaries to use the generalized Poisson model.

References

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