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Some remarks on the Ammeter risk process

1 Introduction

The early development of risk theory was, to a large extent, dominated by Swedish scientists. The most prominent pioneers are Filip Lundberg and Harald Cramér. Probably most readers are further familiar with namens as Arfwedson, Esscher, Laurin, Saxén, Segerdahl, and Täcklind. All these pioneers were mainly interested in finding and proving results related to the ruin probability when the claims occur according to a Poisson process. We also mention Ove Lundberg, who 1940 presented a thesis about Markov point processes and, more particularly, mixed Poisson processes.
In the early development of risk theory, the Swiss actuary Hans Ammeter also plays a very important role. In Ammeter (1948) results were presented about the ruin probability, in a model with randomly fluctuating "basic-probabilities" or intensities. Thus the claims occur according to a Cox process. That special Cox process is built up by independent and stochastically identical pieces of mixed Poisson processes. Mathematically the analysis of the Ammeter model is related to the analysis of mixed Poisson process. Due to the war, Ove Lundberg's thesis was not available to Ammeter. However, despite of certain mathematical similarities, the Ammeter model and the mixed Poisson process are different kinds of Cox processes; the Ammeter process is ergodic while the mixed Poisson process is a typical example of a non-ergodic process. For a modern treatment of mixed Poisson processes and – to some extent – of the Ammeter model, see Grandell (1995). The purpose of this paper is to give a modern treatment of the Ammeter model. It is natural to let those (fixed) epochs constituting the border between the underlying mixed Poisson processes, play a fundamental role in the analysis.
In Section 2 we give background for the analysis. The most important part is the simple inequalities in Lemma 1. We will further give a survey of known results when the claims occur according to a renewal and a Poisson process respectively.
Section 3 is devoted to "general" Ammeter models, and Section 4 to the case with infinitely divisible intensities.

2 The risk process

2.1 The general risk process

The usual model of a risk business is based on the following independent objects:

(i) a point process N;
(ii) a sequence $\{Z_k\}_1^\infty$ of independent and identically distributed random variables, having the common distribution function F, with mean value μ, and variance σ^2.

Here N describes the times and $\{Z_k\}$ the costs of the claims. We will here only treat the case with non-negative risk sums, i.e., we assume that $F(0-) = 0$. Notice that, for technical reasons, "zero risk sums" are allowed. The total amount of claims paid by the company in the interval $(0, t]$ is then described by the *claim process*

$$Y(t) = \sum_{k=1}^{N(t)} Z_k\,, \quad \left(\sum_{k=1}^{0} Z_k \stackrel{\text{def}}{=} 0\right).$$

The *risk process*, X, is defined by

$$X(t) = ct - Y(t)\,,$$

where c is a positive real constant.
The *ruin probability* $\Psi(u)$ of a company facing the risk process X and having initial capital u is defined by

$$\Psi(u) = P\{u + X(t) < 0 \text{ for some } t > 0\}\,.$$

Let $\Delta > 0$ be given and put

$$\Psi_\Delta(u) = P\{u + X(k\Delta) < 0 \text{ for some integer } k > 0\}\,.$$

Thus $\Psi_\Delta(u)$ is the probability for the risk process to be ruined at some epoch of the form $t = k\Delta$, $k = 1, 2, \ldots$.
The following simple lemma will be used several times.

Lemma 1. *Let $\Delta > 0$ be given and assume that almost surely no claims occur at the epochs $t = k\Delta$, $k = 0, 1, 2, \ldots$. Then*

$$\Psi(u + c\Delta) \le \Psi_\Delta(u) \le \Psi(u) \le \Psi_\Delta(u - c\Delta) \quad \text{for } u \ge c\Delta\,.$$

Proof. Assume that $u + c\Delta + X(t_0) < 0$ for some $t_0 \in (k_0\Delta, (k_0 + 1)\Delta]$. The premium received by the company in the interval $(t_0, (k_0 + 1)\Delta]$ is less than $c\Delta$. Since $F(0-) = 0$ it therefore follows that

$$u + X((k_0 + 1)\Delta) \leq u + c\Delta + X(t_0) < 0\,.$$

Thus the first inequality follows. The second inequality is trivial, and the third inequality follows from the first one. The restriction $u \geq c\Delta$ is not necessary, but we have introduced it in order to avoid $\Psi_\Delta(u)$ with $u < 0$. ■

The tail behaviour of the claim distribution F is of utmost importance for the ruin probability. Put

$$h(r) \stackrel{\text{def}}{=} \int_0^\infty e^{rz}\, dF(z) - 1\,.$$

Definition 1. *We talk about small claims, or say that F is light-tailed, if there exists $r_\infty > 0$ such that $h(r) \uparrow +\infty$ when $r \uparrow r_\infty$ (we allow for the possibility $r_\infty = +\infty$).* □

The important part of Definition 1 is that $h(r) < \infty$ for some $r > 0$. This means that the tail of F decreases at least exponentially fast, and thus for example the lognormal and the Pareto distributions are excluded.

If $h(r) = \infty$ for all $r > 0$ we talk about *large claims*, or say that F is *heavy-tailed*. A good survey of this case is given by Embrechts and Veraverbeke (1982). In the latter case, we restrict ourselves to claim distributions related to the class $\mathcal{S}$, defined below.

Definition 2. *A distribution G on $[0, \infty)$ belongs to the class $\mathcal{S}$ of subexponential distributions if*

$$\lim_{x \to \infty} \frac{1 - G^{(2)}(x)}{1 - G(x)} = 2\,,$$

where $G^{(2)}(x) = \int_0^x G(x - y)\, dG(y)$. □

All $G \in \mathcal{S}$ have the property, see Embrechts and Veraverbeke (1982, p. 62),

$$\lim_{x \to \infty} \frac{1 - G(x - y)}{1 - G(x)} = 1 \qquad \text{for all } y\,. \tag{1}$$

Much of our discussion will rely on known results in the cases where N is a Poisson process or an ordinary renewal process. Although the Poisson case was the only case known at the time of Ammeter's contribution, we will first give some known results in the renewal case. The use of the tildes and stars in the notations in Sections 2.1 and 2.2 is meant to facilitate the applications in Sections 3 and 4.

2.2 *The ordinary renewal risk process*

The first treatment of the ruin problem when the occurrence of the claims is described by a renewal process is due to Sparre Andersen (1957). In a series of papers Thorin has carried through a systematic study, see for example Thorin (1982).

Let $\widetilde{N}$ be a point process and let S_k denote the time of the kth claim. $\widetilde{N}$ is called an ordinary *renewal process* (with inter-occurrence time distribution K) if the variables S_1, $S_2 - S_1$, $S_3 - S_2$, $\ldots$ are independent and have a common distribution K with mean $1/\widetilde{\alpha}$. Let $\widehat{k}$ denote the Laplace transform, i.e., $\widehat{k}(v) = \int_0^\infty e^{-vs}\, dK(s)$.

The distribution of the costs of the claims is denoted by $\widetilde{F}$ where $\widetilde{F}(0-) = 0$. Thus ruin can only occur at claim epochs. To avoid some technical complications we assume that $\widetilde{F}$ contains an absolutely continuous component. Furthermore $\widetilde{\mu}$, $\widetilde{h}(r)$ and so on have their natural interpretation with respect to $\widetilde{F}$.

The relative *safety loading* $\widetilde{\varrho}$ is defined by

$$\widetilde{\varrho} = \frac{c - \widetilde{\alpha}\widetilde{\mu}}{\widetilde{\alpha}\widetilde{\mu}}\,.$$

The risk process, $\widetilde{X}$, is said to have *positive* safety loading if $\widetilde{\varrho} > 0$.

Consider now the small claim case.

The adjustment coefficient or the *Lundberg exponent* R is the positive solution of

$$(\widetilde{h}(r) + 1)\widehat{k}(cr) = 1\,. \tag{2}$$

The following basic results hold, see Grandell (1991):

$$\widetilde{\Psi}(u) = (1 - \widetilde{\mu}R)e^{-Ru} \tag{3}$$

when the claim costs are exponentially distributed with mean $\widetilde{\mu}$, cf. Sparre Andersen (1957, p. 226). We further have:
the *Cramér-Lundberg approximation*, cf. Thorin (1974, p. 94),

$$\lim_{u \to \infty} e^{Ru}\widetilde{\Psi}(u) = \widetilde{C}; \tag{4}$$

the *Lundberg inequality*, cf. Sparre Andersen (1957, p. 224),

$$\widetilde{\Psi}(u) \leq e^{-Ru}\,. \tag{5}$$

Consider now the large claim case. Let $\widetilde{F}_I$ be defined by

$$\widetilde{F}_I(z) = \frac{1}{\widetilde{\mu}} \int_0^z (1 - \widetilde{F}(x))\,dx\,,$$

and assume that $\widetilde{F}_I \in \mathcal{S}$. This includes lognormally and Pareto distributed claims.
Then, cf. Embrechts and Veraverbeke (1982, p. 65),

$$\widetilde{\Psi}(u) \sim \frac{1}{\widetilde{\varrho}}(1 - \widetilde{F}_I(u)) \qquad \text{as} \quad u \to \infty\,. \tag{6}$$

(The sign $\sim$ means that the quotient between the two sides tends to one.)
We will now essentially consider Pareto distributed claims, but we will need a slight generalization to distributions with *regularly varying* tails. Then

$$1 - \widetilde{F}(z) \sim z^{-\delta}L(z) \qquad \text{as} \quad z \to \infty\,, \tag{7}$$

where L is *slowly varying* at infinity, i.e.,

$$L(xz) \sim L(z), \qquad \text{as} \quad z \to \infty\,,$$

for all $x > 0$. Distributions fulfilling (7) are sometimes said to be of the *Pareto type*. We will only consider $\delta > 1$ so that $\widetilde{\mu} < \infty$. It follows from Feller (1971, pp. 279 and 281) that

$$1 - \widetilde{F}_I(z) \sim \frac{1}{\widetilde{\mu}(\delta - 1)} z^{-(\delta-1)}L(z)$$

and

$$1 - \widetilde{F}_I^{(2)}(z) \sim \frac{2}{\widetilde{\mu}(\delta - 1)} z^{-(\delta-1)} L(z) .$$

Thus $\widetilde{F}_I \in \mathcal{S}$ and

$$\widetilde{\Psi}(u) \sim \frac{1}{\widetilde{\varrho}\,\widetilde{\mu}(\delta - 1)} u^{-(\delta-1)} L(u) \qquad \text{as} \quad u \to \infty . \tag{8}$$

In the Poisson case (8) is due to von Bahr (1975).

Example 1. Let $\widetilde{F}$ be a Pareto distribution with $\delta > 1$, i.e.,

$$1 - \widetilde{F}(z) = \left(\frac{z}{a}\right)^{-\delta} \qquad \text{for} \quad z \geq a > 0 .$$

Then (7) holds with $L(z) = a^{\delta}$. Since $\widetilde{\mu} = \frac{\delta a}{\delta - 1}$ we get from (8)

$$\begin{aligned} \widetilde{\Psi}(u) &\sim \frac{a^{\delta}}{\widetilde{\varrho}\,\widetilde{\mu}(\delta - 1)} u^{-(\delta-1)} \\ &= \frac{\widetilde{\alpha} a}{c(\delta - 1) - \widetilde{\alpha}\delta a} \left(\frac{u}{a}\right)^{-(\delta-1)} \qquad \text{as } u \to \infty , \end{aligned} \tag{9}$$

cf. von Bahr (1975). □

2.3 *The Poisson risk process*

Let N^* be a Poisson process with intensity α^*, which – expressed in terms of the renewal process – means that

$$K(t) = 1 - e^{-\alpha^* t} \text{ for } t \geq 0 \quad \text{or} \quad \widehat{k}(v) = \frac{1}{1 + v/\alpha^*} .$$

In the small claim case R is the positive solution of

$$\frac{h^*(r) + 1}{1 + cr/\alpha^*} = 1 \quad \text{or} \quad h^*(r) = \frac{cr}{\alpha^*} . \tag{10}$$

For exponentially distributed claims (4) is reduced to

$$\Psi^*(u) = \frac{1}{1+\varrho^*} e^{-\frac{\varrho^* u}{\mu^*(1+\varrho^*)}} . \tag{11}$$

In the Cramér-Lundberg approximation the constant is explicitly given by

$$C^* = \frac{\varrho^* \mu^*}{h^{*\prime}(R) - c/\alpha^*} . \tag{12}$$

The Poisson versions of (2)–(5) are due to Lundberg (1926) and Cramér (1930).

Let $X^*(t)$ be a Poisson risk process with intensity α and claim distribution F. Assume that $\Psi^*(u)$ is known, or at least "asymptotically" known. Let, as before, $\Psi^*_\Delta(u)$ be the probability that the risk process is ruined at some epoch of the form $t = k\Delta$.

It follows from Lemma 1 that

$$\frac{\Psi^*(u + c\Delta)}{\Psi^*(u)} \le \frac{\Psi^*_\Delta(u)}{\Psi^*(u)} \le 1 , \tag{13}$$

where, of course, the second inequality is trivial.

Let $\widetilde{X}$ be a renewal risk process with inter-occurrence time distribution

$$K(t) = \begin{cases} 0 & \text{for } t < \Delta , \\ 1 & \text{for } t \ge \Delta , \end{cases} \quad \text{or} \quad \widehat{k}(v) = e^{-\Delta v} \tag{14}$$

and claim distribution

$$\widetilde{F}(z) = \sum_{k=0}^{\infty} \frac{(\alpha^* \Delta)^k}{k!} e^{-\alpha^* \Delta} F^{*(k)}(z) , \tag{15}$$

where $F^{*(0)}(z) = \begin{cases} 0 & \text{for } z < 0 , \\ 1 & \text{for } z \ge 0 \end{cases}$ and $F^{*(k)}(z) = \int_0^z F^{*(k-1)}(z-y)\, dF^*(y)$.

Since X^* has stationary and independent increments it follows that $\widetilde{X}$ has ruin probability Ψ^*_Δ, cf. Grandell (1991, p. 67). Thus we have

$$\widetilde{\alpha} = \frac{1}{\Delta} \quad \text{and} \quad \widetilde{\mu} = \alpha^* \Delta \mu^* .$$

In the small claim case we have $\Psi^*(u) \sim C^* e^{-Ru}$, where R and C^* are given by (10) and (12). Consider now $\widetilde{X}$. We have

$$\begin{aligned}\widetilde{h}(r) + 1 &= \sum_{k=0}^{\infty} \frac{(\alpha^* \Delta)^k}{k!} e^{-\alpha^* \Delta} (h^*(r) + 1)^k \\ &= e^{\alpha^* \Delta h^*(r)}, \end{aligned} \tag{16}$$

and thus the Lundberg exponent is given by

$$e^{\alpha^* \Delta h^*(r) - cr\Delta} = 1 \quad \text{or} \quad \alpha^* h^*(r) = cr. \tag{17}$$

Thus X^* and $\widetilde{X}$ have the same R, and it follows from (4) that $\Psi^*_\Delta(u) \sim C_\Delta e^{-Ru}$ for some constant C_Δ. In this case it follows from Cramér (1955, p. 75) that

$$C_\Delta \sim \frac{C^*}{\mu^* \varrho^* R \cdot \alpha^* \Delta} \qquad \text{as } \alpha^* \Delta \to \infty. \tag{18}$$

From Lemma 1 we get $C_\Delta \geq C e^{-Rc\Delta}$, which – in comparison with (18) – merely means that $C_\Delta \geq 0$ for large values of Δ.
The fact that Lemma 1 is rather useless in this case may be the explanation why the underlying simple idea, to our knowledge, has not been used.
In the large claim case the situation is quite different. Assume that $F_I^* \in \mathcal{S}$. From (6) and (1) we then get

$$\lim_{u \to \infty} \frac{\Psi^*(u + c\Delta)}{\Psi^*(u)} = \lim_{u \to \infty} \frac{1 - F_I^*(u + c\Delta)}{1 - F_I^*(u)} = 1,$$

and it follows from (13) that

$$\Psi^*_\Delta(u) \sim \Psi^*(u) \qquad \text{as} \quad u \to \infty. \tag{19}$$

Intuitively, in this case, ruin is caused by a claim so large that the risk process will remain negative until the next epoch of the form $k\Delta$.

3 The Ammeter process

We will define the Ammeter process within the framework of Cox processes. Intuitively we shall think of a Cox process N as generated in the following way. First a realization $\alpha(t)$ of a non-negative random process $\lambda = \{\lambda(t);\, t \geq 0\}$ is generated and conditioned upon that realization, N is a non-homogeneous Poisson process with intensity function $\alpha(t)$. The process λ is called the *intensity process*. This intuitive definition of a Cox process suffices for our purposes. A detailed discussion of Cox processes and their impact on risk theory is to be found in Grandell (1991).
A natural measure of the variability of the intensity process is σ_λ^2, defined by

$$\sigma_\lambda^2 \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \operatorname{Var}\left[\int_0^t \lambda(s)\, ds\right].$$

Grandell (1991, p. 123) proposed the approximation R_D of R, given by

$$R_D = \frac{2\varrho\alpha\mu}{\mu^2\sigma_\lambda^2 + \alpha(\sigma^2 + \mu^2)},$$

where D stands for "diffusion".
This approximation must be regarded as based on ad hoc reasoning, although the ideas behind it are due to a diffusion approximation of the risk process, which is reasonable for small values of ϱ.
The first Cox process, other than the Poisson process, used in connection with risk theory was the mixed Poisson process. Below we give the modern definition.

Definition 3. *Let Λ be a non-negative random variable with distribution U and mean α. The Cox process obtained by letting*

$$\lambda(\cdot) = \Lambda, \quad \text{a.s.},$$

is called a mixed Poisson process. □

For a mixed Poisson process we have $\sigma_\lambda^2 = \lim_{t\to\infty} \frac{1}{t} t^2 \operatorname{Var}[\Lambda] = \infty$.

Definition 4. *Let $\Delta > 0$ be fixed and let $\{L_k;\, k = 0, 1, \ldots\}$ be a sequence of non-negative, independent, and identically distributed random variables with distribution U and mean α. The Cox process obtained by letting*

$$\lambda(t) = L_k \qquad \text{for } k\Delta \leq t < (k+1)\Delta,$$

is called an Ammeter process. □

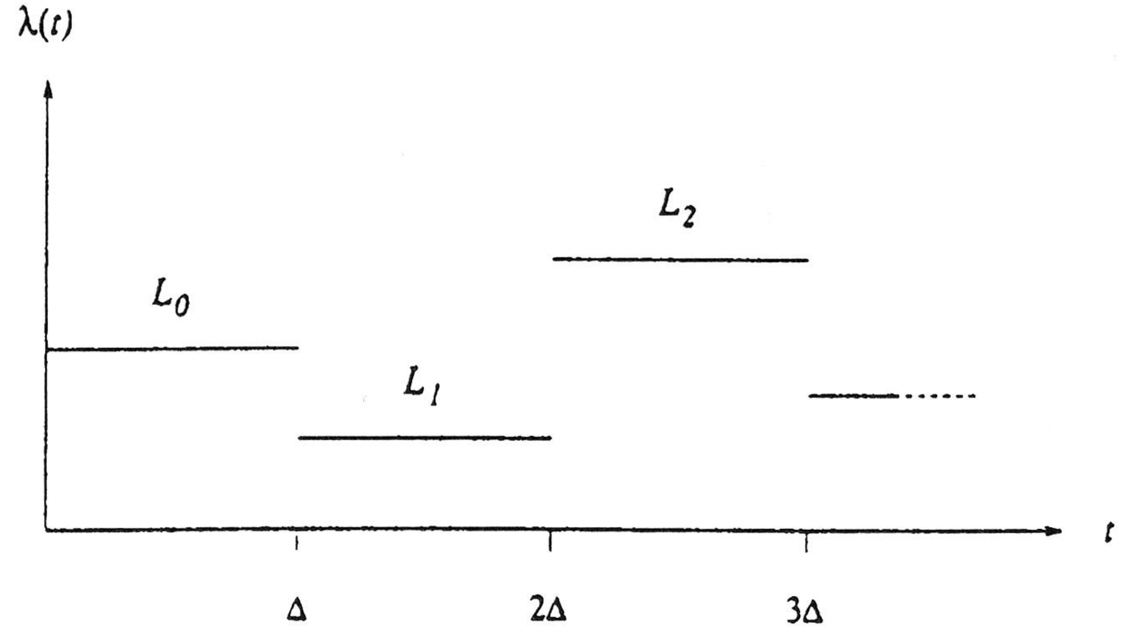

Figure 1. Illustration of the intensity process in the Ammeter case.

The Cox process with this intensity is called an Ammeter process, since it is essentially the model considered by Ammeter (1948). The Ammeter process is technically related to the mixed Poisson process since it can be looked upon as built up by a sequence of independent mixed Poisson processes. However, besides from that relation, it is very different from a mixed Poisson process. For an Ammeter process we have

$$\sigma_\lambda^2 = \lim_{k\to\infty} \frac{1}{k\Delta} \operatorname{Var}\left[\sum_{j=0}^{k-1} \Delta L_j\right] = \Delta\sigma_L^2 ,$$

where $\sigma_L^2 \stackrel{\text{def}}{=} \operatorname{Var}[L_k]$. Thus we have

$$R_D = \frac{2\varrho\alpha\mu}{\Delta\mu^2\sigma_L^2 + \alpha(\sigma^2 + \mu^2)} .$$

For exponentially distributed claims, R_D reduces to

$$R_D = \frac{\varrho}{\mu\left(1 + \frac{\Delta\sigma_L^2}{2\alpha}\right)} .$$

Let now N be an Ammeter process, F the distribution of the costs of the claims, $Y(\cdot)$ the corresponding claim process, and $X(t) = ct - Y(t)$ the risk process. Let $\widehat{u}$ denote the Laplace transform of U, i.e.,

$$\widehat{u}(v) = \int_0^\infty e^{-v\ell}\, dU(\ell)\,.$$

As usual $\Psi(u)$ denotes the ruin probability.
A fundamental property in the analysis of Ammeter processes is that $\{\widetilde{Z}_k\}$, defined by

$$\widetilde{Z}_k \stackrel{\text{def}}{=} Y((k+1)\Delta) - Y(k\Delta), \quad k = 0, 1, \ldots$$

is a sequence of independent and identically distributed random variables. Let $\widetilde{F}$ denote their common distribution function. Then we have, cf. (15),

$$\widetilde{F}(z) = \int_0^\infty \sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta} F^{(k)}(z)\, dU(\ell) \quad \text{and} \quad \widetilde{\mu} = \alpha\,\Delta\mu\,. \tag{20}$$

Let $\widetilde{X}$ be a renewal risk process with inter-occurrence time distribution K given by (14) and claim distribution $\widetilde{F}$ given by (20). Obviously the sequences $\{X(k\Delta)\}_{k=0}^\infty$ and $\{\widetilde{X}(k\Delta)\}_{k=0}^\infty$ have the same distribution. Since in a renewal model ruin can only occur at claim epochs, it follows that $\widetilde{X}$ has ruin probability $\Psi_\Delta(u)$. Thus the situation is somewhat contrary to the Poisson case, since here it is reasonable to regard $\Psi_\Delta(u)$ as known. Form Lemma 1 we now get, cf. (13),

$$1 \le \frac{\Psi(u)}{\Psi_\Delta(u)} \le \frac{\Psi_\Delta(u - c\Delta)}{\Psi_\Delta(u)} \quad \text{for } u \ge c\Delta\,. \tag{21}$$

Consider the small claim case and assume that U is light-tailed, i.e., that $\widehat{u}$ fulfills the same condition as h. Then we have, cf. (16),

$$\widetilde{h}(r) + 1 = \int_0^\infty e^{\ell\Delta h(r)}\, dU(\ell) = \widehat{u}(-\Delta h(r))\,.$$

This implies that $\widetilde{F}$ is light-tailed.

Since $(\widetilde{h}(r)+1)\widehat{k}(cr) = \widehat{u}(-\Delta h(r))e^{-cr\Delta}$ it follows that R is the positive solution of

$$\widehat{u}(-\Delta h(r))e^{-cr\Delta} = 1\,. \tag{22}$$

A Lundberg inequality now readily follows.

Theorem 1. *$\Psi(u) \le Ce^{-Ru}$, for some $C < \infty$.*

Proof. The theorem is an immediate consequence of (21) and (5):

$$\Psi(u) \le \Psi_{\Delta}(u - c\Delta) \le e^{-R(u-c\Delta)} = e^{cR\Delta}e^{-Ru}\,.$$

Since $e^{cR\Delta}e^{-Ru} \ge 1$ for $u < c\Delta$, the theorem holds for all values of u. ■

From the proof of Theorem 1 it is seen that we can choose $C = e^{cR\Delta}$. Although we do not at all claim that this is the smallest possible value of C we must in general, cf. (5), accept that $C > 1$.
Put, cf. (4), $C_{\Delta} \overset{\text{def}}{=} \lim_{u\to\infty} e^{Ru}\Psi_{\Delta}(u)$. We do conjecture that also the Cramér-Lundberg approximation holds for $\Psi(u)$. This conjecture is based on Asmussen (1989), Asmussen and Rolski (1994), and Grigelionis (1993). In any case we may define

$$\underline{C} \overset{\text{def}}{=} \liminf_{u\to\infty} e^{Ru}\Psi(u) \quad \text{and} \quad \overline{C} \overset{\text{def}}{=} \limsup_{u\to\infty} e^{Ru}\Psi(u)\,,$$

and it follows from (21) that

$$C_{\Delta} \le \underline{C} \le \overline{C} \le e^{cR\Delta}C_{\Delta}\,. \tag{23}$$

Recall from Section 2.3 that a similar use of Lemma 1 in the Poisson case turned out to yield little information for large values of Δ. The reason was that $R\Delta$ became large for large values of Δ.
Here the situation is – at least sometimes – quite different, since Δ is involved in the definition of X and thus also in R.

Proposition 1. *Assume that there exists a positive solution r_0 of*

$$\widehat{u}(-r/\alpha)e^{-(1+\varrho)r} = 1\,. \tag{24}$$

Then we have

$$e^{cR\Delta} \le e^{(1+\varrho)r_0}\,.$$

Proof. Put $R_0 = \mu\alpha\Delta R$. Noticing that $cR\Delta = (1+\varrho)R_0$ it follows from (22) that R_0 is the positive solution of

$$\widehat{u}\left(-\Delta h\left(\frac{r}{\mu\alpha\Delta}\right)\right)e^{-(1+\varrho)r} = 1\,.$$

Since $\widehat{u}(-v)$ is increasing in v and since $h(r) \geq r\mu$ it follows that

$$\widehat{u}\left(-\Delta h\left(\frac{r}{\mu\alpha\Delta}\right)\right) \geq \widehat{u}(-r/\alpha)\,. \tag{25}$$

Thus we have

$$1 = \widehat{u}\left(-\Delta h\left(\frac{R_0}{\mu\alpha\Delta}\right)\right)e^{-(1+\varrho)R_0} \geq \widehat{u}(-R_0/\alpha)e^{-(1+\varrho)R_0}\,,$$

which implies

$$\widehat{u}(-R_0/\alpha)e^{-(1+\varrho)R_0} \leq \widehat{u}(-r_0/\alpha)e^{-(1+\varrho)r_0}\,.$$

Since $\widehat{u}(-r/\alpha)e^{-(1+\varrho)r}$ is convex in r, it follows that $R_0 \leq r_0$ and the proposition is proved. ■

Remark 1. Suppose now that Δ is large. In "kind" cases we have $\Delta h(r/(\mu\alpha\Delta)) \approx r/\alpha$ and thus we have approximate equality in (25). Therefore the bound in Proposition 1 ought to be the best possible bound holding for all Δ. □

Consider

$$U(\ell) = \begin{cases} 0 & \text{for } \ell < \alpha\,, \\ 1 & \text{for } \ell \geq \alpha\,, \end{cases} \qquad \text{or} \qquad \widehat{u}(v) = e^{-\alpha v}\,,$$

which corresponds to the Poisson case. Then (24) reduces to $-\varrho r = 0$, which obviously has no positive solution.

Example 2. The simplest non-trivial example, to which we will return, is probably when L_k is exponentially distributed. Then $\widehat{u}(v) = 1/(1+\alpha v)$ and (24) reduces to

$$\frac{1}{1-r}e^{-(1+\varrho)r} = 1\,. \tag{26}$$

Some values of r_0 and $e^{(1+\varrho)r_0}$ are given in Table 1.

Table 1. Values of r_0 and $e^{(1+\varrho)r_0}$ for exponentially distributed intensity.

ϱ	r_0	$e^{(1+\varrho)r_0}$
5 %	0.0937	1.1034
10 %	0.1761	1.2138
15 %	0.2490	1.3316
20 %	0.3137	1.4571
25 %	0.3714	1.5908
30 %	0.4230	1.7330

It is seen from (24), and illustrated in Table 1, that r_0 increases with increasing values of ϱ. This is quite natural, since the higher safety loading the quicker the risk process "recovers" after ruin.

Example 3. A natural extension of the case considered in Example 2 is when L_k is Γ-distributed. Then

$$u(\ell) \stackrel{\text{def}}{=} U'(\ell) = \frac{\beta^\gamma}{\Gamma(\gamma)} \ell^{\gamma-1} e^{-\beta\ell}, \quad \text{for } \ell \geq 0,$$

where γ is called the shape parameter and β the scale parameter. In this case we say that the intensity is $\Gamma(\gamma, \beta)$. We have

$$\alpha = \frac{\gamma}{\beta}, \quad \sigma_L^2 \stackrel{\text{def}}{=} \text{Var}[L_k] = \frac{\gamma}{\beta^2}, \quad \text{and} \quad \hat{u}(v) = \left(1 + \frac{v}{\beta}\right)^{-\gamma},$$

and (24) reduces to

$$\left(1 - \frac{r}{\gamma}\right)^{-\gamma} e^{-(1+\varrho)r} = 1 \quad \text{or} \quad \frac{1}{1 - r/\gamma} e^{-(1+\varrho)r/\gamma} = 1 . \tag{27}$$

□

Consider now the case where $\widetilde{F}$ is heavy-tailed, or more precisely that $\widetilde{F}_I \in \mathcal{S}$. Then (6) applies to $\Psi_\Delta(u)$. From (1) and (21) its follows that $\Psi(u) \sim \Psi_\Delta(u)$, cf. (19), and thus

$$\Psi(u) \sim \frac{1}{\varrho}(1 - \widetilde{F}_I(u)) \quad \text{as } u \to \infty . \tag{28}$$

Assume now, for some $\delta > 1$ and some slowly varying function L, that

$$1 - U(\ell) \sim \omega \ell^{-\delta} L(\ell) \quad \text{and} \quad 1 - F(z) \sim \phi z^{-\delta} L(z)\,, \tag{29}$$

where $\omega \geq 0$ and $\phi \geq 0$. If $\omega = 0$, then (29) means that $1 - U(\ell) = o(\ell^{-\delta} L(\ell))$, i.e., that $\lim_{\ell \to \infty} \frac{1-U(\ell)}{\ell^{-\delta} L(\ell)} = 0$, and similarly if $\phi = 0$.

Proposition 2. *Assume that* (29) *holds and that at least one of the constants* ω *or* ϕ *are strictly positive. Then*

$$\Psi(u) \sim \frac{\omega(\mu\Delta)^{\delta} + \phi\alpha\Delta}{\varrho\alpha\Delta\mu(\delta - 1)} u^{-(\delta-1)} L(u) \quad \text{as } u \to \infty\,. \tag{30}$$

In the proof of Proposition 2 we will need the following Lemma.

Lemma 2. $1 - U(\ell) = o(\ell^{-\delta} L(\ell))$ *implies* $P\{N(\Delta) > n\} = o(n^{-\delta} L(n))$ *as* $n \to \infty$.

Proof. By partial integration, properties of the $\Gamma(n+1, \Delta)$-distribution, and that $n! > n^n e^{-n}$, we get, for $a < 1$,

$$\begin{aligned}
P\{N(\Delta) > n\} &= \int_0^\infty \frac{\Delta^{n+1} \ell^n}{n!} e^{-\ell\Delta} (1 - U(\ell))\, d\ell \\
&\leq \int_0^{an/\Delta} \frac{\Delta^{n+1} \ell^n}{n!} e^{-\ell\Delta}\, d\ell \\
&\quad + \int_{an/\Delta}^\infty \frac{\Delta^{n+1} \ell^n}{n!} e^{-\ell\Delta}\, d\ell \cdot (1 - U(an/\Delta)) \\
&\leq \frac{an}{\Delta} \frac{\Delta (an)^n}{n!} e^{-an} + (1 - U(an/\Delta)) \\
&\leq \frac{(an)^{n+1} e^{-an}}{n^n e^{-n}} + (1 - U(an/\Delta)) \\
&\leq na^n e^n + (1 - U(an/\Delta))\,.
\end{aligned}$$

Choose $a < 1/e$. Then

$$\lim_{n\to\infty} \frac{P\{N(\Delta) > n\}}{n^{-\delta}L(n)} \leq \lim_{n\to\infty} \frac{n(ae)^n + (1 - U(an/\Delta))}{n^{-\delta}L(n)}$$
$$= (a/\Delta)^{-\delta} \lim_{n\to\infty} \frac{(1 - U(an/\Delta))}{(an/\Delta)^{-\delta}L(an/\Delta)} = 0 . \quad \blacksquare$$

Proof of Proposition 2. It follows from Willmot (1990), after a slight modification, cf. Grandell (1995), that $1 - U(\ell) \sim \omega\ell^{-\delta}L(\ell)$ implies $P\{N(\Delta) > n\} \sim \omega\Delta^{\delta}n^{-\delta}L(n)$ for $\omega > 0$. For $\omega = 0$ Lemma 2 applies.
From Stam (1973, p. 311) it follows that

$$1 - \widetilde{F}(z) \sim (\omega\mu^{\delta}\Delta^{\delta} + \phi\alpha\Delta)z^{-\delta}L(z) .$$

Since $\widetilde{\mu} = \alpha\Delta\mu$ the proposition now follows exactly as (8) followed from (7). $\blacksquare$

The case where both ω and ϕ are positive seems somewhat artificial, since it means that U and F have almost exactly the same tail behaviour. If we let one of those constants be zero, it is no restriction to let the other be one.
Let us first consider the case $\phi = 0$ and $\omega = 1$. This means that the heavy-tail behaviour of $\widetilde{F}$ is caused by the variation of the intensity. It does not imply that F is necessarily light-tailed, but it is less heavy-tailed than U. Then it follows from (30) that

$$\Psi(u) \sim \frac{(\mu\Delta)^{\delta-1}}{\varrho\alpha(\delta-1)}u^{-(\delta-1)}L(u) \quad \text{as } u \to \infty .$$

Now consider the case $\phi = 1$ and $\omega = 0$. This means that F is heavy-tailed, and we can really talk about large claims. Then it follows from (30) that

$$\Psi(u) \sim \frac{1}{\varrho\mu(\delta-1)}u^{-(\delta-1)}L(u) \quad \text{as } u \to \infty .$$

In this case, cf. (8), there is no essential influence of the intensity. This observation is in agreement with Asmussen et al. (1994).
We will now consider more in detail the case where F is heavy-tailed and U is light-tailed. It turns out that we will need both that $F \in \mathcal{S}$ as well as that

$F_I \in \mathcal{S}$. The relation between these two conditions is not at all trivial. In order to cope with these requirements, we will rely on the class $\mathcal{S}^*$, defined below.

Definition 5. *A distribution F on $[0,\infty)$ belongs to the class $\mathcal{S}^*$ if it has finite expectation μ and*

$$\lim_{x\to\infty} \int_0^x \frac{1-F(x-y)}{1-F(x)}(1-F(y))\,dy = 2\mu\,. \qquad \square$$

From Klüppelberg (1989) it follows that the lognormal and the Pareto distribution belong to $\mathcal{S}^*$.
The following proposition was proposed by Klüppelberg (1994).

Proposition 3. *Let $F \in \mathcal{S}^*$ and U be light-tailed. Then*

$$\Psi(u) \sim \frac{1}{\varrho}(1-F_I(u)) \quad \text{as } u \to \infty\,.$$

Proof. We will first show that F and $\widetilde{F}$ are tail-equivalent, i.e., that

$$\lim_{x\to\infty} \frac{1-\widetilde{F}(z)}{1-F(z)} = \text{const.} \in (0,\infty)\,,$$

since it will imply (Klüppelberg 1988, p. 134) that $\widetilde{F} \in \mathcal{S}^*$. From (20) we get

$$\begin{aligned} 1-\widetilde{F}(z) &= \int_0^\infty \left(1-\sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta} F^{(k)}(z)\right) dU(\ell) \\ &= \int_0^\infty \sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta}(1-F^{(k)}(z))\,dU(\ell)\,, \end{aligned}$$

and thus

$$\frac{1-\widetilde{F}(z)}{1-F(z)} = \int_0^\infty \sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta} \frac{1-F^{(k)}(z)}{1-F(z)}\,dU(\ell)\,.$$

Since $F \in \mathcal{S}^*$ implies (Klüppelberg 1988, p. 135) $F \in \mathcal{S}$, we have, see Athreya and Ney (1972, p. 148),

$$\lim_{z\to\infty} \frac{1 - F^{(k)}(z)}{1 - F(z)} = k, \quad \text{for } k = 0, 1, 2, \ldots$$

Further, see Athreya and Ney (1972, p. 149), given any $\varepsilon > 0$ there exists a $D < \infty$ such that

$$\frac{1 - F^{(k)}(z)}{1 - F(z)} \le D(1+\varepsilon)^k \quad \text{for all } k \text{ and } z\,.$$

Since U is light-tailed, this implies, for ε small enough, that

$$\begin{aligned}
&\int_0^\infty \sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta} \frac{1 - F^{(k)}(z)}{1 - F(z)}\, dU(\ell) \\
&\le \int_0^\infty \sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta} D(1+\varepsilon)^k \, dU(\ell) \\
&= D \cdot \int_0^\infty e^{\varepsilon\Delta\ell}\, dU(\ell) < \infty\,.
\end{aligned}$$

By dominated convergence we get

$$\lim_{z\to\infty} \frac{1 - \widetilde{F}(z)}{1 - F(z)} = \int_0^\infty \sum_{k=0}^\infty \frac{(\ell\Delta)^k}{k!} e^{-\ell\Delta} k \, dU(\ell) = \alpha\Delta\,,$$

which implies (Klüppelberg 1988, pp. 134 and 135) that $\widetilde{F}_I \in \mathcal{S}$. Thus (6) yields $\Psi(u) \sim \frac{1}{\varrho}(1 - \widetilde{F}_I(u))$. By L'Hospital's rule

$$\begin{aligned}
\lim_{u\to\infty} \frac{1 - \widetilde{F}_I(u)}{1 - F_I(u)} &= \lim_{u\to\infty} \frac{\frac{1}{\alpha\Delta\mu}\int_u^\infty (1 - \widetilde{F}(x))\, dx}{\frac{1}{\mu}\int_u^\infty (1 - F(x))\, dx} \\
&= \frac{1}{\alpha\Delta} \lim_{u\to\infty} \frac{1 - \widetilde{F}_I(u)}{1 - F_I(u)} = 1
\end{aligned}$$

follows, and the proposition is proved. ■

Remark 2. We will now consider the relation between Proposition 2, in the case $\omega = 0$ and $\phi = 1$, and Proposition 3. Let F be of the Pareto type, i.e., $1 - F(z) \sim z^{-\delta} L(z)$.
Although it follows from Klüppelberg (1989) that $F \in \mathcal{S}^*$, we will give a direct proof, since it will give some insight in the properties of $\mathcal{S}^*$. Following Klüppelberg (1988, p. 135) we get

$$\begin{aligned}\int_0^x \frac{1-F(x-y)}{1-F(x)}(1-F(y))\,dy &= 2\int_0^{x/2} \frac{1-F(x-y)}{1-F(x)}(1-F(y))\,dy \\ &\le 2\frac{1-F(x/2)}{1-F(x)}\int_0^{x/2} 1-F(y)\,dy \\ &\sim 2\cdot 2^{\delta}\mu < \infty\,.\end{aligned}$$

Thus $F \in \mathcal{S}^*$ follows by dominated convergence. Notice that a careless interchange of limits and the integration in the first term leads to a wrong result.
Thus Proposition 3 is a strict generalization of Proposition 2 when U is light-tailed. The really nice thing with Proposition 2 is that F and U contribute to the ruin probability in a rather symmetric way. It is therefore tempting to consider F light-tailed and $U \in \mathcal{S}$ or $\mathcal{S}^*$. However, both $\mathcal{S}$ and $\mathcal{S}^*$ are defined in terms of convolution properties, while there seems not to be any natural convolution of U with itself involved. Therefore such an approach seems to be difficult. □

4 Infinitely divisible Ammeter processes

We will now restrict ourselves to small claims. A major problem in this case is that $\widetilde{C}$, occurring in (4), is not given in an explicit form and therefore (23) is not so easy to use. In the infinitely divisible case the situation is nicer, since then the explicit form given in (12) can be used. This is, in fact, the essential observation in Ammeter's original approach.

Definition 6. *A random variable L with distribution U is called infinitely divisible, ID, if for each n there exists a random variable $L^{(n)}$ such that L has the same distribution as the sum of n independent copies of $L^{(n)}$.* □

An Ammeter process is ID if its structure distribution U is ID. This holds in a "point process" sense, but we will only need that the mixed Poisson distributed random variable $N(\Delta)$ is ID. It is well-known, see for example Feller (1968, p. 290), that this implies that $N(\Delta)$ is compound Poisson distributed, i.e., that

$$N(\Delta) = \xi_1 + \xi_2 + \cdots + \xi_{\overline{N}}\,, \qquad (N(\Delta) = 0 \text{ if } \overline{N} = 0)\,, \tag{31}$$

where $\xi_1, \xi_2, \ldots$ are independent and identically distributed discrete variables, $\overline{N}$ is Poisson distributed with mean $\overline{\alpha}$, and $\overline{N}$ is independent of the ξ_is. In order to make the representation unique we assume that $P\{\xi_i = 0\} = 0$.
For any discrete random variable N, its generating function $G_N(s)$ is given by $G_N(s) \stackrel{\text{def}}{=} E[s^N]$. Then (31) can be written in the form

$$\widehat{u}(\Delta(1-s)) = e^{\overline{\alpha}(G_\xi(s)-1)}, \quad \text{for } |s| \le 1\,, \tag{32}$$

where $\widehat{u}(\Delta(1-s))$ follows from the representation of $N(\Delta)$ as a mixed Poisson distribution, cf. Grandell (1995), and $e^{\overline{\alpha}(G_\xi(s)-1)}$ from the representation as a compound Poisson distribution. Since we assumed that $P\{\xi_i = 0\} = 0$, we have $G_\xi(0) = 0$. Thus we get

$$\widehat{u}(\Delta) = e^{-\overline{\alpha}} \quad \text{or} \quad \overline{\alpha} = -\log(\widehat{u}(\Delta)) \tag{33}$$

and

$$\overline{\alpha}(G_\xi(s) - 1) = \log(\widehat{u}(\Delta(1-s)))$$

or

$$G_\xi(s) = 1 - \frac{\log(\widehat{u}(\Delta(1-s)))}{\log(\widehat{u}(\Delta))}\,.$$

We will introduce an *associate* Poisson risk process X^*, with the same premium c as in X. The position of the claims is described by a Poisson process N^* with intensity

$$\alpha^* = \frac{\overline{\alpha}}{\Delta} = -\frac{\log(\widehat{u}(\Delta))}{\Delta}\,.$$

The claim distribution F^* is given by

$$F^*(z) = \sum_{k=1}^{\infty} P\{\xi_i = k\} F^{(k)}(z)\,,$$

and thus

$$h^*(r) + 1 = \sum_{k=1}^{\infty} P\{\xi_i = k\}(h(r)+1)^k = G_\xi(h(r)+1)$$

or

$$h^*(r) = -\frac{\log(\widehat{u}(-\Delta h(r)))}{\log(\widehat{u}(\Delta))}\,. \tag{34}$$

Notice that both α^* and F^* depend on Δ.
Although the processes X and X^* are different, the sequences $\{X(k\Delta)\}_{k=0}^{\infty}$ and $\{X^*(k\Delta)\}_{k=0}^{\infty}$ have the same distribution. Thus

$$\Psi_\Delta(u) = \Psi_\Delta^*(u)\,. \tag{35}$$

Since N^* is a Poisson process (12) applies, i.e.,

$$\Psi^*(u) \sim C^* e^{-Ru}, \quad \text{where} \quad C^* = \frac{\varrho^*\alpha^*\mu^*}{\alpha^* h^{*\prime}(R) - c}\,. \tag{36}$$

The Lundberg exponent in (36) is the positive solution of

$$cr = \alpha^* h^*(r) = \frac{\log(\widehat{u}(-\Delta h(r)))}{\Delta} \quad \text{or} \quad \widehat{u}(-\Delta h(r))e^{-cr\Delta} = 1\,.$$

Thus we have the same R for X and X^*, cf. (17) and (22), which is "as it shall be", cf. (35). Further $\alpha^*\mu^* = \alpha\mu$ which follows from the construction of X^* or from $\alpha^*\mu^* = \alpha^*\mu G'_\xi$ (1) and

$$\alpha^* h^{*\prime}(r) = -\frac{\widehat{u}'(-\Delta h(r))h'(r)}{\widehat{u}(-\Delta h(r))}\,.$$

From (22) it follows that $\widehat{u}(-\Delta h(R)) = e^{cR\Delta}$, and we get

$$C^* = \frac{\varrho\alpha\mu}{-\widehat{u}'(-\Delta h(R))h'(R)e^{-cR\Delta} - c}\,, \tag{37}$$

where it may be noticed that $\widehat{u}'(v) < 0$ for all v.

It follows from Lemma 1 that

$$\begin{aligned}\Psi^*(u+c\Delta) \le \Psi^*_\Delta(u) = \Psi_\Delta(u) \le \Psi(u) \le \Psi_\Delta(u-c\Delta) \\ = \Psi^*_\Delta(u-c\Delta) \le \Psi^*(u-c\Delta)\end{aligned}$$

for $u \ge c\Delta$ and that, cf. (23),

$$e^{-cR\Delta}C^* \le C^*_\Delta \le \underline{C} \le \overline{C} \le e^{cR\Delta}C^*_\Delta \le e^{cR\Delta}C^*\,, \tag{38}$$

or, using Proposition 1,

$$e^{-(1+\varrho)r_0}C^* \le C^*_\Delta \le \underline{C} \le \overline{C} \le e^{(1+\varrho)r_0}C^*_\Delta \le e^{(1+\varrho)r_0}C^*\,. \tag{39}$$

Naturally we are better off if we can calculate $C_\Delta = C^*_\Delta$, but (38) and (39) give bounds which only require C^*. We know only one case where C_Δ can be simply calculated, and will consider that case in the continuation of Example 2.
We can now apply the diffusion approximation to X^*, i.e., we consider $R^*_D = 2\varrho\mu^*/(\sigma^{*^2} + \mu^{*^2})$. By differentiation of (34) we find – not very surprisingly – that $R^*_D = R_D$. Then it follows, cf. Grandell (1991, p. 17), that

$$R \le \frac{2\varrho\alpha\mu}{\Delta\mu^2\sigma^2_L + \alpha(\sigma^2+\mu^2)}\,. \tag{40}$$

Remark 3. We will, by a counterexample, show that (40) does not hold in general. Consider the case

$$\begin{aligned}P\{Z_k = 1\} = 1 \;&\Rightarrow\; h(r) = e^r - 1 \\ P\{L_k = 0\} = P\{L_k = 2\} = \frac{1}{2} \;&\Rightarrow\; \widehat{u}(v) = \frac{1}{2}(1+e^{-2v})\,,\end{aligned}$$

where, of course, L_k is *not* infinitely divisible. We have $\mu = 1, \sigma^2 = 0, \alpha = 1$, and $\sigma^2_L = 1$, and thus

$$R_D = \frac{2\varrho}{\Delta + 1}\,.$$

Let

$$f(r,\Delta) \stackrel{\text{def}}{=} \frac{1}{2}(1+\exp\{2\Delta(e^r-1)\})e^{-(1+\varrho)r\Delta}\,.$$

From (22) it follows that R is the positive solution of $f(r, \Delta) = 1$. If we can show that $f(R_D, \Delta) < 1$, it follows that $R_D < R$, which would be a contradiction to (40). For $\Delta \to \infty$ we have $R_D \sim 2\varrho/\Delta$ and thus

$$f(R_D, \Delta) \sim \frac{1}{2}\left(1 + \exp\left\{2\Delta \cdot \frac{2\varrho}{\Delta}\right\}\right)e^{-2(1+\varrho)\varrho} = \frac{1}{2}(1 + e^{4\varrho})e^{-2(1+\varrho)\varrho} .$$

Choosing, for example, $\varrho = 1$, we get

$$f(R_D, \Delta) \sim \frac{1}{2}(1 + e^4)e^{-4} = \frac{1}{2}(1 + e^{-4}) < 1 .$$

Thus (40) does not hold in general. (Exact computations show that $f(R_D, \Delta)$ < 1 for $\Delta \geq 4.502$.) □

From the point of view of the diffusion approximation, the inequality (40) is in the "wrong" direction. For our purpose it is in the "right" direction, since it means that

$$cR\Delta \leq cR_D\Delta = \frac{2\varrho(1+\varrho)\alpha^2\Delta\mu^2}{\Delta\mu^2\sigma_L^2 + \alpha(\sigma^2 + \mu^2)} \leq \frac{2\varrho(1+\varrho)\alpha^2}{\sigma_L^2} . \qquad (41)$$

Formula (41) indicates that (38) is useful when σ_L is large. This is quite natural, since then the (random) intensity is probably large in an interval $[k\Delta, (k+1)\Delta)$ where ruin occurs and it is rather probable that the risk process remains negative until the end of that interval. If, on the other hand, σ_L is small, we are "locally" close to the Poisson case, where we know that (38) is quite useless. Notice that this does not necessarily imply that the risk process itself is close to the Poisson case, since the natural measure of the variability is $\Delta\sigma_L^2$.
In view of Remark 1 and (41), the following proposition is not very surprising.

Proposition 4. *Assume that* L_k *is infinitely divisible. Then* $r_0 \leq 2\varrho\alpha^2/\sigma_L^2$, *where* r_0 *is given by* (24).

Proof. By the assumption of infinite divisibility, it follows from (32) and (33) that

$$\widehat{u}(v) = \exp\left\{-\log(\widehat{u}(\Delta))\left(G_\xi\left(1 - \frac{v}{\Delta}\right) - 1\right)\right\},$$

where G_ξ is a generating function, depending on Δ, with $G_\xi(0) = 0$. Notice that $\log(\widehat{u}(\Delta)) \le 0$. Thus r_0 is the positive solution of

$$-\log(\widehat{u}(\Delta))\left(G_\xi\left(1+\frac{r}{\Delta\alpha}\right)-1\right)-(1+\varrho)r=0\,,$$

independently of the choice of Δ.
Let $p_j(\Delta) = P\{\xi_k = j\}$, so that

$$G_\xi(s) = p_1(\Delta)s + p_2(\Delta)s^2 + \cdots \ge p_1(\Delta)s + p_2(\Delta)s^2 \quad \text{for } s \ge 0\,.$$

By similar arguments as in the proof of Proposition 1 it follows that $r_0 \le \widetilde{r}(\Delta)$, where $\widetilde{r}(\Delta)$ is the positive solution of

$$-\log(\widehat{u}(\Delta))\left(\left(1+\frac{r}{\Delta\alpha}\right)p_1(\Delta)+\left(1+\frac{r}{\Delta\alpha}\right)^2 p_2(\Delta)-1\right)-(1+\varrho)r$$
$$=0\,. \tag{42}$$

Using (32) again, we get

$$G_\xi(s) = 1 - \frac{\log \widehat{u}(\Delta(1-s))}{\log \widehat{u}(\Delta)}\,,$$

which, by routine differentiation, yields

$$p_1(\Delta) = G'_\xi(0) = \frac{\Delta\widehat{u}'(\Delta)}{\widehat{u}(\Delta)\log(\widehat{u}(\Delta))} \sim \frac{\Delta\alpha}{-\log(\widehat{u}(\Delta))}$$

and

$$p_2(\Delta) = \frac{1}{2}G''_\xi(0) = -\frac{\Delta^2\widehat{u}''(\Delta)}{2\widehat{u}(\Delta)\log(\widehat{u}(\Delta))} + \frac{\Delta^2\widehat{u}'(\Delta)^2}{2\widehat{u}(\Delta)^2\log(\widehat{u}(\Delta))}$$
$$\sim \frac{\Delta^2\sigma_L^2}{-2\log(\widehat{u}(\Delta))}\,,$$

as $\Delta \to 0$. Putting this into (42), it follows that $\widetilde{r}(0) \stackrel{\text{def}}{=} \lim_{\Delta\to 0}\widetilde{r}(\Delta)$ is the solution of

$$1 + \frac{r\sigma_L^2}{2\alpha^2} - (1+\varrho) = 0\,.$$

Thus, $\widetilde{r}(0) = 2\varrho\alpha^2/\sigma_L^2$, and the proposition follows. ∎

By considering the same case as in Remark 3, it is seen that Proposition 4 does not hold in general.

Example 3 continued. The most important example of an infinitely divisible structure distribution is certainly when U is $\Gamma(\gamma,\beta)$. Recall that $\alpha = \gamma/\beta$. We choose to parameterize with α and γ, which gives $\sigma_L^2 = \alpha^2/\gamma$. Then R is the positive solution of

$$\left(1 - \frac{\Delta\alpha h(r)}{\gamma}\right)^{-\gamma} e^{-cr\Delta} = 1 \quad \text{or} \quad h(r) = \frac{\gamma}{\Delta\alpha}\left(1 - e^{-cr\Delta/\gamma}\right), \tag{43}$$

which is eq. (35) in Ammeter (1948, p. 196).
We have $\widehat{u}'(v) = -\frac{\gamma}{\beta}(1+\frac{v}{\beta})^{-\gamma-1} = -\alpha(1+\frac{\alpha v}{\gamma})^{-\gamma-1}$, and thus

$$\begin{aligned}\widehat{u}'(-\Delta h(R)) &= -\alpha\left(1 - \frac{\Delta\alpha h(R)}{\gamma}\right)^{-\gamma-1} \\ &= -\alpha e^{-cR\Delta(-\gamma-1)/\gamma} = -\alpha e^{cR\Delta} e^{cR\Delta/\gamma}.\end{aligned}$$

Thus (37) reduces to

$$C^* = \frac{\varrho\mu\alpha e^{-cR\Delta/\gamma}}{\alpha h'(R) - ce^{-cR\Delta/\gamma}}. \tag{44}$$

Now we simplify further and let the claims be exponentially distributed with mean μ. Then $h(r) = \mu r/(1-\mu r)$ and $h'(r) = \mu/(1-\mu r)^2$ and (44) leads to

$$C^* = \frac{\varrho(1-\mu R)^2 e^{-cR\Delta/\gamma}}{1-(1+\varrho)(1-\mu R)^2 e^{-cR\Delta/\gamma}}$$

or, equivalently,

$$C^* = \frac{\varrho(1-\mu R)(1-(1+\Delta\alpha/\gamma)\mu R)}{1-(1+\varrho)(1-\mu R)(1-(1+\Delta\alpha/\gamma)\mu R)}. \tag{45}$$

Ammeter (1948, p. 196) considered this case for $\mu = 1$, $\alpha = 1$, $\Delta = 1000$, $\gamma = 100$, and $u = 1000$. Further he compared with the Poisson case. We will consider the same case, but restrict ourselves to $\varrho = 10\,\%$. Notice that $\sigma_\lambda^2 = 1000 \cdot 0.01 = 10$.

From Table 1 and (27) we get $r_0 = 17.6134$. This yields $e^{(1+\varrho)r_0} \approx 2.6 \cdot 10^8$, and thus (39) is useless in this case. Solving (43) we get $R = 0.01482$ which yields $e^{cR\Delta} \approx 1.2 \cdot 10^7$, and thus also (38) is useless.
Let us therefore consider the ruin probability for the related Poisson process N^*. Recall that the only thing we now can say about the Ammeter process is that $\Psi_\Delta(u) \leq \Psi^*(u)$. From (36) and (45) we get

$$\Psi^*(u) \sim \frac{0.1(1-R)(1-11R)e^{-Ru}}{1-1.1(1-R)(1-11R)} = 0.8875e^{-0.01482u},$$

and thus $\Psi^*(1000) \approx 3.3 \cdot 10^{-7}$. In Table 4 given by Ammeter (1948, p. 196), values of $\Psi_{1000}(1000)$ are given. Ammeter uses a relation, related to (18), between $\Psi_\Delta(u)$ and $\Psi^*(u)$. To the best of our understanding, that relation is not correct. We will return to this in Remark 4 below.
In the Poisson case, or when $\sigma_L^2 = 0$, we have by (11) and (18), $R = 0.09091$, $\Psi(1000) = 3.0 \cdot 10^{-40}$ and $\Psi_{1000}(1000) \approx 3.3 \cdot 10^{-41}$. Thus we are far from that case. □

Example 2 continued. Let now both U and F be exponential distributions, with means α and μ respectively. Then it follows from (45), with $\gamma = 1$, that

$$C^* = \frac{\varrho(1-\mu R)(1-(1+\Delta\alpha)\mu R)}{1-(1+\varrho)(1-\mu R)(1-(1+\Delta\alpha)\mu R)}.$$

In Grandell (1995) it is shown that

$$\Psi_\Delta(u) = (1-(1+\alpha\Delta)\mu R)e^{-Ru}, \tag{46}$$

and it follows that

$$C_\Delta = (1-(1+\alpha\Delta)\mu R).$$

The main idea in the proof of (46) is to use the fact that $\widetilde{Z}_k$ is exponentially distributed given that $\widetilde{Z}_k > 0$. By "looking" at the risk process at those epochs $k\Delta$ where $\widetilde{Z}_k > 0$, (46) follows from (2). This implies that we consider a different inter-occurrence time distribution than given by (14).
In this case R is the positive solution of

$$\alpha\mu\Delta r = (1-\mu r)(1-e^{(1+\varrho)\alpha\mu\Delta r}),$$

and it is easy to realize, cf. (26), that $\alpha\mu\Delta R \to r_0$ as $\Delta \to \infty$. Thus

$$cR\Delta \to (1+\varrho)r_0\,, \quad C_\Delta \to 1 - r_0 \quad \text{and} \quad C^* \to \frac{\varrho(1-r_0)}{1-(1+\varrho)(1-r_0)}\,, \tag{47}$$

as $\Delta \to \infty$. Further, cf. (41), since $\sigma_L = \alpha$

$$cR_D\Delta \to 2\varrho(1+\varrho) \quad \text{as} \quad \Delta \to \infty\,. \tag{48}$$

As a matter of curiosity we may notice that $C_\Delta e^{cR\Delta} \to 1$ as $\Delta \to \infty$.
In Table 2 we consider the case $\alpha = \mu = 1$. The case "$\Delta = \infty$" refers to the limits (47) and (48), i.e., "$R \cdot \infty$" $= r_0$.

Table 2. Illustration of bounds for exponentially distributed intensity and claims.

ϱ	Δ	$R \cdot \Delta$	$e^{-cR\Delta}C^*$	C_Δ	$e^{cR\Delta}C_\Delta$	$e^{cR\Delta}C^*$	$e^{cR\Delta}$	$e^{cR_D\Delta}$
10 %	1	0.060383	0.8472	0.8792	0.9396	0.9676	1.0687	1.0761
10 %	10	0.148163	0.7540	0.8370	0.9852	1.0446	1.1770	1.2012
10 %	100	0.172884	0.7275	0.8254	0.9983	1.0642	1.2095	1.2407
10 %	1000	0.175804	0.7244	0.8240	0.9998	1.0664	1.2133	1.2455
10 %	∞	0.176134	0.7240	0.8239	1.0000	1.0667	1.2138	1.2461
20 %	1	0.110300	0.7237	0.7794	0.8897	0.9430	1.1415	1.1735
20 %	10	0.266005	0.5762	0.7074	0.9734	1.0909	1.3760	1.4918
20 %	100	0.308213	0.5387	0.6887	0.9969	1.1287	1.4475	1.6009
20 %	1000	0.313142	0.5344	0.6865	0.9997	1.1331	1.4561	1.6145
20 %	∞	0.313698	0.5339	0.6863	1.0000	1.1335	1.4571	1.6161
30 %	1	0.152175	0.6224	0.6957	0.8478	0.9244	1.2188	1.2969
30 %	10	0.361152	0.4453	0.6027	0.9639	1.1388	1.5992	1.9155
30 %	100	0.415920	0.4048	0.5799	0.9958	1.1937	1.7172	2.1484
30 %	1000	0.422255	0.4003	0.5773	0.9996	1.2000	1.7314	2.1781
30 %	∞	0.422970	0.3998	0.5770	1.0000	1.2007	1.7330	2.1815

The intention of presenting Table 2 is to illustrate first (38) and second (40) and (41). Certainly this example suits the methods used very well, and hence far reaching conclusions ought not to be drawn. It does, however, seem as if the unique possibility to calculate C_Δ does not improve the bounds drastically. Further it seems that the diffusion approximation R_D works reasonably well even for ϱ as large as 30 %. The critical "parameter" seems to be σ_L, and we believe that the above conclusions hold reasonably generally for $\sigma_L \approx 1$. □

Remark 4. In the end of Example 3, we mentioned a relation between $\Psi_\Delta(u)$ and $\Psi^*(u)$, used by Ammeter. Suppose that $\mu = \alpha = 1$. Ammeter (1948, p. 195) states, as if it was a well-known fact, that

$$C_\Delta^*/C^* = 1/(1 + \varrho R\Delta) \tag{49}$$

in the Poisson case. In Laurin (1930, p. 111), which is one of Ammeter's general references to risk theory, (49) is mentioned after the comment "We shall only give the final result which is suggested by Lundberg's discussion on this subject:" If we compare with (18), it follows that (49) holds as an approximation, for large values of Δ. Therefore we believe (49) was motivated by a heuristic argument.
In the case where both U and F are exponential distributions it follows from Example 2 that

$$C_\Delta^*/C^* = \frac{1 - (1+\varrho)(1-R)(1-(1+\Delta)R)}{\varrho(1-R)}, \tag{50}$$

which is not in agreement with (49). Neither (50) is in agreement with (49) nor (18) for large values of Δ. This is, however, no contradiction, since C^* does depend on Δ.
This is most certainly not to be regarded as a severe criticism of Ammeter's approach. In fact, Ammeter proposed e^{-Ru} as an approximation of $\Psi_\Delta(u)$, which means – see (5) – that he was "on the safe side". □

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Zusammenfassung

Wir untersuchen verschiedene Fragen im Zusammenhang mit der Ruinwahrscheinlichkeit, falls die Schäden gemäss einem Ammeter-Prozess eintreten. Der Ammeter-Prozess ist ein sehr spezieller Cox-Prozess, der aus unabhängigen gemischten Poisson-Prozessen aufgebaut wird.

Summary

We consider certain questions related to the ruin probability, when the claims occur according to an Ammeter process. The Ammeter process is a very special Cox process, built up by independent mixed Poisson processes.

Résumé

Nous traitons de la probabilité de ruine lorsque les sinistres surviennent selon un processus d'Ammeter. Ce dernier est un cas très particulier des processus de Cox, à savoir celui généré par des processus de Poisson mixtes et indépendants.