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Approximation methods for the total claimsize distribution – an algorithmic and graphical presentation

1. Introduction

Traditionally, the distribution of the total claims in a fixed time period (e.g. in one year) has been a central topic in risk theory. Since computers have entered the field, the interest in many traditional approximation methods has faded away whereas more computerintensive methods like recursions, fast Fourier transform and Monte Carlo methods have gained importance. This change of interest retains the field important for researchers. It manifests in the many articles published on this subject recently in insurance mathematics journals.
Most standard textbooks on risk theory contain sections on approximation methods; we refer e.g. to *Beard* et al. (1984), *Gerber* (1979), *Heilmann* (1988), *Hipp* and *Michel* (1990). An earlier review paper is *Kupper* (1971) where also numerical examples can be found. In a sense the present paper can be considered as an update of Kupper's work adjusted to nowadays computer technique.
We consider the classical risk model, where the whole portfolio of a certain insurance business represents the source of risk. The number of claims N in one period is supposed to be Poisson distributed with parameter λ and the claimsizes $(X_k)_{k\in\mathbb{N}}$ are a sequence of iid random variables having common distribution function F with $F(0) = 0$, mean value μ, $\mu_k = E\,X^k$ for $k \geq 2$, and variance σ^2. Furthermore, N and $(X_k)_{k\in\mathbb{N}}$ are independent. Then $S = \sum_{k=1}^{N} X_k$ represents the accumulated claims in one period and has distribution function

$$G(x) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}F^{n*}(x), \quad x \geq 0\,,$$

where F^{n*} denotes the n-fold convolution of F. Notice that F^{n*} is explicitly calculable only for degenerate (deterministic) or exponential claimsizes. In all other cases numerical approximations for G are required, where the convolutions and the infinite sum cause problems.

If not stated otherwise, for ease of notation, we shall always approximate the distribution function $\widetilde{G}$ of the density $\widetilde{g}$ of the standardized random variable

$$\widetilde{S} = \frac{S - \lambda\mu}{\sqrt{\lambda(\sigma^2 + \mu^2)}} \tag{1}$$

The approximation of G is then obtained in the obvious way.
In section 2 we briefly review the different approximation methods and in section 3 we present the graphics together with a discussion.

2. The approximation methods

In the approximations of this section the moments of S determine the constants in certain expansions. We use the following notation

$$\begin{aligned} m_k &= E[S^k], \quad k \in \mathbb{N} \\ \widetilde{m}_k &= E[\widetilde{S}^k], \quad k \in \mathbb{N} \end{aligned}$$

In the Poisson case we have in particular

$$\begin{aligned} m_1 &= \lambda\mu \\ \operatorname{var} S &= \lambda(\sigma^2 + \mu^2) = \lambda\mu_2 \\ \gamma_1 &= \widetilde{m}_3 = \mu_3 / \sqrt{\lambda\mu_2^3} \\ \gamma_2 &= \widetilde{m}_4 - 3 = \mu_4 / (\lambda\mu_2^2) \end{aligned}$$

Furthermore, the normal distribution Φ, its derivatives $\Phi^{(k)}$, $k \in \mathbb{N}$, the normal density ϕ and its derivatives $\phi^{(k)}$, $k \in \mathbb{N}$, play a central role. Obviously, $\phi^{(k)} = \Phi^{(k+1)}$, $k \in \mathbb{N}_0$, holds. Moreover, we need the Hermite polynomials

$$H_k(x) = \phi^{(k)}(x)/\phi(x) = e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k \in \mathbb{N}$$

There exist various recursive formulas for H_k, we shall use the following

$$\begin{aligned} H_{k+1} &= -xH_k(x) + H'_k(x), \quad k \in \mathbb{N} \\ H_0 &= 1 \end{aligned}$$

2.1 The normal approximation and related methods

2.1.1 The normal approximation

The most obvious approximation is to use the CLT which gives for large λ

$$\widetilde{G}(x) \approx \Phi(x)$$

where Φ denotes the standard normal distribution. In practical applications, however, λ is often not large enough and the accuracy of this approximation is not satisfactory if the skewness of the claimsize distribution is large.

2.1.2 The Edgeworth approximation

The Edgeworth approximation can be considered as a refinement of the normal approximation which also takes higher moments into account. The expansion is obtained by means of the moment generating function

$$\widehat{\widetilde{g}}(t) = \int_0^\infty e^{tx} d\widetilde{G}(x)$$

which exists for all $t < t_0 \leq \infty$. For those t we consider the Taylor series

$$\log \widehat{\widetilde{g}}(t) = \sum_{k=0}^\infty \alpha_k t^k \tag{2}$$

where $\alpha_k := \frac{1}{k!} \frac{d^k}{dt^k} \widehat{\widetilde{g}}(t)\Big|_{t=0}$. Due to the standardization we have $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = \frac{1}{2}$. Again by a Taylor expansion we obtain

$$\widehat{\widetilde{g}}(t) = \exp\left(\frac{1}{2}t^2\right) \exp\left(\sum_{k=3}^\infty \alpha_k t^k\right) = \exp\left(\frac{1}{2}t^2\right) \sum_{k=0}^\infty a_k t^k \tag{3}$$

The following result enables us to invert the terms back.

Proposition 1. *For all $k \in \mathbb{N}_0$ we have*

$$t^k e^{t^2/2} = \int_{-\infty}^\infty e^{tu}(-1)^k H_k(u)\phi(u)du$$

where ϕ denotes the standard normal density.

Proof: The result obviously holds for $k = 0$. Then we use induction:

$$\int_{-\infty}^{\infty} e^{tu}(-1)^{k+1}\phi^{(k+1)}(u)du$$

$$= \left[e^{tu}(-1)^{k+1}\phi^{(k)}(u)\right]_{-\infty}^{\infty} + t\int_{-\infty}^{\infty} e^{tu}(-1)^{k}\phi^{(k)}(u)du$$

$$= 0 + t^{k+1}e^{t^2/2}$$ □

We insert this in (3) and obtain (if it is possible to interchange sum and integral)

$$\widehat{\widetilde{g}}(t) = \sum_{k=0}^{\infty} a_k \int_{-\infty}^{\infty} e^{tu}(-1)^k\phi^{(k)}(u)du = \int_{-\infty}^{\infty} e^{tu}\sum_{k=0}^{\infty} a_k(-1)^k\phi^{(k)}(u)du$$

From this we obtain

$$\widetilde{g}(x) = \sum_{k=0}^{\infty} a_k(-1)^k\phi^{(k)}(x) \quad \text{and} \quad \widetilde{G}(x) = \sum_{k=0}^{\infty} a_k(-1)^k\Phi^{(k)}(x) \tag{4}$$

The following result gives the exact values for a_k, $k \in \mathbb{N}_0$.

Proposition 2.

$$a_k = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} H_j(0)\widetilde{m}_{k-j}, \quad for \ k \in \mathbb{N}_0$$

Proof: From equation (3) we obtain by Taylor expansion

$$\sum_{k=0}^{\infty} a_k t^k = e^{-t^2/2}\,\widehat{\widetilde{g}}(t) = \left(\sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(0)\right)\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\widehat{\widetilde{g}}^{(k)}(0)\right)$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{j=0}^{k}\binom{k}{j} H_j(0)\,\widehat{\widetilde{g}}^{(k-j)}(0)$$

Identification of the terms of the same order gives the result. □

Taking the first n terms in (4) gives the Edgeworth approximation of order n for $\widetilde{g}$ and $\widetilde{G}$; i.e. for $\widetilde{g}$ we obtain

$$\begin{aligned}\widetilde{g}(x) \approx \phi(x) &- \frac{1}{6}\widetilde{m}_3\phi^{(3)}(x) \\ &+ \frac{1}{24}(\widetilde{m}_4 - 3)\phi^{(4)}(x) + \frac{1}{120}(\widetilde{m}_5 - 10\widetilde{m}_3)\phi^{(5)}(x) \\ &+ \frac{1}{720}(\widetilde{m}_6 - 15\widetilde{m}_4 + 30)\phi^{(6)}(x) \\ &+ R(x)\,.\end{aligned}$$

Alternatively, *Gerber* (1979) and *Beard* et al. (1984) approximate in (3)

$$\widehat{\widetilde{g}}(t) \approx \exp\left(\frac{1}{2}t^2\right)\exp\left(\sum_{k=3}^{4}\alpha_k t^k\right) = \exp\left(\frac{1}{2}t^2\right)\sum_{k=0}^{\infty} a_k t^k$$

Truncation of the infinite series on the rhs to $\sum_{k=3}^{6} a_k t^k$ yields to different coefficients giving the approximation

$$\begin{aligned}\widetilde{g}(x) \approx \phi(x) &- \frac{1}{6}\gamma_1\phi^{(3)}(x) \\ &+ \frac{1}{24}\gamma_2\phi^{(4)}(x) + \frac{1}{72}\gamma_1^2\phi^{(6)}(x) \\ &+ R(x)\end{aligned} \tag{5}$$

This expansion is traditionally used for the normal power approximation below. The first line of approximation contains terms up to the order $\lambda^{-1/2}$, the second line up to λ^{-1}, and $R(x)$ up to $\lambda^{-3/2}$.

Notice that for heavy-tailed distributions like the Pareto or loggamma the Edgeworth approximation is not possible. Also, in general the series in (4) are divergent and hence the approximation does not necessarily improve by taking terms of higher order. Nevertheless, taking a suitable number of terms, the Edgeworth approximation gives acceptable results in the neighbourhood of the mean as we shall see in section 3.

2.1.3 The normal power approximation

The basic idea of this approximation is to find a function p such that

$$\widetilde{S} \stackrel{d}{=} p(N)$$

where N is a standard normal random variable. Then approximate

$$\widetilde{G}(x) \approx \Phi(p^{\leftarrow}(x)) \tag{6}$$

where $p^{\leftarrow}$ is the inverse (or locally inverse) function of p. A suitable transformation in polynomial form can be obtained using the Edgeworth expansion, e.g. of order 6. We take Gerber's version (5) which yields

$$\widetilde{G}(x) \approx e(x) := \Phi(x) - \frac{1}{6}\gamma_1 \Phi^{(3)}(x) + \frac{1}{24}\gamma_2 \Phi^{(4)}(x) + \frac{1}{72}\gamma_1^2 \Phi^{(6)}(x)$$

The problem reduces then to find a function p such that

$$e(p(x)) = \Phi(x) \tag{7}$$

Set $p(x) = x + \Delta x$, then (7) can be rewritten as

$$0 = \Phi(x) - e(x + \Delta x) =: q(\Delta x)$$

and we have to find the root Δx. We apply Newton's method of second order with starting value 0. The one-step approximation is then

$$\Delta x \approx -\frac{q(0)}{q'(0)} - \frac{1}{2}\frac{q''(0)}{q'(0)}\left(\frac{q(0)}{q'(0)}\right)^2 \tag{8}$$

where

$$\begin{aligned}
q(0) &= \frac{1}{6}\gamma_1 \Phi^{(3)}(x) - \frac{1}{24}\gamma_2 \Phi^{(4)}(x) - \frac{1}{72}\gamma_1^2 \Phi^{(6)}(x) \\
q'(0) &= -\Phi'(x) + \frac{1}{6}\gamma_1 \Phi^{(4)}(x) - \frac{1}{24}\gamma_2 \Phi^{(5)}(x) - \frac{1}{72}\gamma_1^2 \Phi^{(7)}(x) \\
q''(0) &= -\Phi''(x) + \frac{1}{6}\gamma_1 \Phi^{(5)}(x) - \frac{1}{24}\gamma_2 \Phi^{(6)}(x) - \frac{1}{72}\gamma_1^2 \Phi^{(8)}(x)
\end{aligned}$$

Now we use the fact that $\Phi^{(k)} = \phi^{(k-1)} = H_{k-1}\phi$ holds for all $k \in \mathbb{N}$ and approximate the two terms of the righthand side of (8) as follows:

$$-\frac{q(0)}{q'(0)} \approx \frac{\frac{1}{6}\gamma_1(x^2-1) + \frac{1}{24}\gamma_2(x^3-3x) + \frac{1}{72}\gamma_1^2(x^5-10x^3+15x)}{1+\frac{1}{6}\gamma_1(x^3-3x)} \tag{9}$$

$$-\frac{1}{2}\frac{q''(0)}{q'(0)}\left(\frac{q(0)}{q'(0)}\right)^2 \approx \frac{1}{72}\gamma_1^2(x^5-2x^3+x) \tag{10}$$

At first sight the approximations above seem rather arbitrary. They are, however, motivated by the fact that they are asymptotically exact up to the order $O(\lambda^{-3/2})$ for the Poisson claim number N as the parameter $\lambda \to \infty$ [*Kauppi* and *Ojantakanen* (1969)].

We expand $(1+\frac{1}{6}\gamma_1(x^3-3x))^{-1}$ of (9) in a Taylor series and, taking only the terms up to order γ_1^2 into account, we obtain

$$\begin{aligned}-\frac{q(0)}{q'(0)} &\approx \left(\frac{1}{6}\gamma_1(x^2-1) + \frac{1}{24}\gamma_2(x^3-3x) + \frac{1}{72}\gamma_1^2(x^5-10x^3+15x)\right)\\ &\quad\times\left(1-\frac{1}{6}\gamma_1(x^3-3x)+\frac{1}{36}\gamma_1^2(x^3-3x)^2\right)\\ &\approx \frac{1}{6}\gamma_1(x^2-1) + \frac{1}{24}\gamma_2(x^3-3x) + \frac{1}{72}\gamma_1^2(-x^5-2x^3+9x)\end{aligned}$$

This together with (10) gives by (8)

$$\begin{aligned}p(x) &= x + \Delta x\\ &\approx x + \frac{1}{6}\gamma_1(x^2-1) + \frac{1}{24}\gamma_2(x^3-3x) - \frac{1}{36}\gamma_1^2(2x^3-5x)\end{aligned} \tag{11}$$

If we use only the first two terms in (11) we obtain

$$p^{\leftarrow}(x) \approx \sqrt{9/\gamma_1^2 + 6x/\gamma_1 + 1} - 3/\gamma_1$$

and, inserting this in (6), we obtain the normal power approximation of second order

$$\widetilde{G}(x) \approx \Phi\left(\sqrt{9/\gamma_1^2 + 6x/\gamma_1 + 1} - 3/\gamma_1\right)$$

For the normal power approximation of the third order equation (11) has to be inverted; e.g. by Cardano's formula.

2.2 *Approximations using orthogonal polynomials*

2.2.1 The Hilbert space setting

Suppose $I \subset \mathbb{R}$ is an interval and w a positive continuous (weight) function. Furthermore, denote by L^2_w the Hilbert space of all L^2-integrable functions with respect to the measure $w(x)dx$. Then for $f,\, g \in L^2_w$ we define the scalar product

$$\langle f,\, g\rangle := \int_I f(x)\, g(x) w(x) dx$$

and the induced norm

$$\|f\| = \sqrt{\langle f,\, f\rangle}$$

Certain orthogonal polynomials $(\Pi_i)_{i\in\mathbb{N}}$ constitute a basis of L^2_w and every $f \in L^2_w$ has the representation

$$f(x) = \sum_{i=0}^{\infty} A_i \Pi_i(x) w(x)$$

where

$$A_i = \frac{1}{\langle \Pi_i,\, \Pi_i\rangle} \int_I \Pi_i(x)\, f(x) dx\,, \quad i \in \mathbb{N}_0$$

Hence the approximation of order n for $f \in L^2_w$ is given by

$$f(x) \approx f_n(x) = \sum_{i=0}^{n} A_i \Pi_i(x)\, w(x)$$

Different intervals I and different weight functions w yield different approximations. In our case f is the density of a random variable, say Z, and hence

$$A_i = \frac{1}{\langle \Pi_i,\, \Pi_i\rangle} E[\Pi_i(Z)]\,, \quad i \in \mathbb{N}$$

Since Π_i is a polynomial of degree i this implies that for an approximation of order n the n-th moment of Z has to be finite. Furthermore, for I we take the support of Z and for the weight function w a probability density.

2.2.2 The Gamma approximation of Bowers

We approximate the density $\widetilde{g}$ of the standardized random variable

$$\widetilde{S} = \frac{ES}{\operatorname{var} S} S$$

which is in the Poisson case equal to $\widetilde{S} = \frac{\mu}{\mu_2} S$. $\widetilde{S}$ is a non-negative random variable and we choose $I = \mathbb{R}^+$. Furthermore, we choose $w(x) = \Gamma(x;\, 1,\, b) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x}$, $x \geq 0$, and $b = ES/\operatorname{var} S$. Then the Laguerre polynomials

$$\begin{aligned} \Pi_i(x) &= (-1)^i x^{1-b} e^x \frac{d^i}{dx^i} (x^{i+b-1} e^{-x}) \\ &= \Gamma(b+1) \sum_{j=0}^{i} \binom{i}{j} (-1)^{i+j} x^j \frac{1}{\Gamma(b+j)}, \quad i \in \mathbb{N}_0 \end{aligned}$$

constitute a basis of L^2_w. Due to the normalization we obtain

$$A_0 = 1, \quad A_1 = A_2 = 0$$

and

$$A_3 = \frac{\Gamma(b)}{6\Gamma(b+3)} (\widetilde{\mu}_3 - (b+2)(b+1)b)$$

where $\widetilde{\mu}_3 = E[\widetilde{S}^3]$. In the Poisson case $\widetilde{\mu}_3 = \lambda^{5/2} \mu^3 \mu_3 / \mu_2^{3/2}$. Consequently, for $n = 0,\, 1,\, 2$ we obtain a simple Gamma approximation, i.e.

$$\widetilde{g}(x) \approx w(x) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x}, \quad x \geq 0$$

and for $n = 3$ we obtain

$$\begin{aligned} \widetilde{g}(x) &\approx w(x) + A_3 \Pi_3(x) w(x) \\ &= \frac{1}{\Gamma(b)} x^{b-1} e^{-x} + \frac{1}{6} (\widetilde{\mu}_3 - (b+2)(b+1)b) \\ &\quad \times \left\{ \frac{x^3}{\Gamma(b+3)} - \frac{3x^2}{\Gamma(b+2)} + \frac{3x}{\Gamma(b+1)} - \frac{1}{\Gamma(b)} \right\} x^{b-1} e^{-x} \end{aligned}$$

2.2.3 The Gram-Charlier approximation

We approximate the density $\widetilde{g}$ of the standardized random variable

$$\widetilde{S} = \frac{S - ES}{\sqrt{\operatorname{var} S}}$$

which is in the Poisson case given in (1). The support of $\widetilde{S}$ is $\mathbb{R}$, consequently we take $I = \mathbb{R}$. Furthermore, we take $w(x) = \phi(x)$ the standardnormal density. Then the Hermite polynomials

$$\Pi_i(x) = H_i(x)$$

as defined in the beginning of this section constitute a basis of L^2_w. Due to the normalization we obtain

$$A_0 = 1, \quad A_1 = A_2 = 0, \quad A_3 = -\gamma_1/6, \quad A_4 = \gamma_2/24\,.$$

Consequently, for $n = 0, 1, 2$ we obtain a simple normal approximation; i.e.

$$\widetilde{g}(x) \approx w(x) = \phi(x)$$

and for $n = 3$ we obtain

$$\begin{aligned}\widetilde{g}(x) &\approx w(x) + A_3 H_3(x) w(x) \\ &= \phi(x) - \frac{1}{6}\gamma_1 \phi^{(3)}(x)\end{aligned}$$

and for $n = 4$

$$\begin{aligned}\widetilde{g}(x) &\approx w(x) + A_3 H_3(x) w(x) + A_4 H_4(x) w(x) \\ &= \phi(x) - \frac{1}{6}\gamma_1 \phi^{(3)}(x) + \frac{1}{24}\gamma_2 \phi^{(4)}(x)\end{aligned}$$

Notice that for $n \leq 4$ the Gram-Charlier approximation is exactly the Edgeworth approximation of the corresponding order. Only higher order approximations differ.

2.2.4 The Esscher approximation

Most of the preceding approximations are sufficiently precise around the mean but perform poorly in the tails. Exponential tilting shifts the mean to an arbitrary large x-value and hence improves the approximation in the tail considerably. Then the density or distribution function is estimated for this value x. We describe the method in detail: For the random variable S with distribution function G define for $h \in \mathbb{R}$ such that $\widehat{g}(h) < \infty$ the exponentially tilted random variable S_h with distribution function G_h by

$$G_h = \frac{1}{\widehat{g}(h)} \int_0^x e^{hy} dG(y), \quad x \geq 0 \tag{12}$$

G_h is called the *Esscher transform of* G. Then G_h has moment generating function

$$\widehat{g}_h(t) = \widehat{g}(t+h)/\widehat{g}(h) \tag{13}$$

i.e. $\widehat{g}_h$ is essentially a translation of $\widehat{g}$.

Proposition 3. [*Gerber* (1979), Section 4.7]

(a) *Suppose S is compound Poisson with parameter λ and claimsize distribution F, then also S_h is compound Poisson with parameter $\lambda\widehat{f}(h)$ and claimsize distribution F_h.*

(b) *The function $h \mapsto E[S_h]$ is increasing.*

Proof: (a) The moment generating function of S is

$$\widehat{g}(t) = \exp(\lambda(\widehat{f}(t) - 1)$$

and using (13) one obtains

$$\begin{aligned} \widehat{g}_h(t) &= \exp(\lambda(\widehat{f}(t+h) - 1) - \lambda(\widehat{f}(t) - 1)) \\ &= \exp(\lambda\widehat{f}(t)[\widehat{f}(t+h)/\widehat{f}(t) - 1]) \end{aligned}$$

(b) Observe that

$$E[S_h] = \widehat{g}_h'(0) = \widehat{g}'(h)/\widehat{g}(h)$$

and hence

$$\frac{d}{dh}E[S_h] = \left\{\widehat{g}''(h)\widehat{g}(h) - (\widehat{g}'(h))^2\right\}/\widehat{g}^2(h)$$
$$= \widehat{g}_h''(0) - (\widehat{g}_h'(0))^2 = \operatorname{var} S_h > 0 \qquad \square$$

Remark. Denote by $x_0 := \sup\{x;\, G(x) < 1\}$ and by $h_0 := \sup\{h;\, \widehat{g}(h) < \infty\}$. Then in all cases we consider we have

$$\lim_{h \to h_0} ES_h = x_0 \tag{14}$$

For a precise mathematical result and an example where (14) does not hold see *Petrov* (1965).

Then for a given $x \in \operatorname{supp} S$ determine $h \in \mathbb{R}$ such that

$$E[S_h] = x$$

and apply the Edgeworth approximation to G_h or its density g_h. Traditionally, the Edgeworth approximation of third order is taken and one obtains

$$g_h(y) \approx \frac{1}{\sigma(h)}\,\phi\left(\frac{y-x}{\sigma(h)}\right) - \frac{E[(S_h - x)^3]}{6\sigma^4(h)}\,\phi^{(3)}\left(\frac{y-x}{\sigma(h)}\right) \tag{15}$$

where $\sigma^2(h) = \operatorname{var} S_h$. Considering densities in (12) we obtain

$$g(y) = \widehat{g}(h)e^{-hy}g_h(y)$$

and inserting (15) gives an approximation for $g(y)$, which is good as long as y is in the neighbourhood of x. From this one obtains the distribution function

$$G(x) = \widehat{g}(h)\int_{-\infty}^{x} e^{-hy}g_h(y)dy \tag{16}$$

or

$$1 - G(x) = \widehat{g}(h)\int_{x}^{\infty} e^{-hy}g_h(y)dy \tag{17}$$

For numerical reasons for $h < 0\,(x < ES)$ formula (16) is preferable and for $h > 0\,(x > ES)$ formula (17). E.g. substituting (15) in (17) one obtains for $h > 0$

$$1 - G(x) \approx \widehat{g}(h) \int_0^\infty e^{-h(x+\sigma(h)y)} \left(\phi(y) - \frac{E[(S_h - x)^3]}{6\sigma^4(h)} \phi^{(3)}(y) \right) dy \quad (18)$$

Define the so-called Esscher functions as

$$E_k(x) := \int_0^\infty e^{-xy} \phi^{(k)}(y) dy\,, \quad k \in \mathbb{N}_0$$

then (18) can be rewritten as

$$1 - G(x) \approx \widehat{g}(h) e^{-hx} \left(E_0(u) - \frac{E[(S_h - x)^3]}{6\sigma^4(h)} E_3(u) \right)$$

where $u := h\sigma(h)$ and E_0 and E_3 can be rewritten as

$$E_0(x) = [1 - \Phi(x)]/\sqrt{2\pi}\phi(x)$$
$$E_3(x) = \frac{1}{\sqrt{2\pi}} \left(1 - x^2 + x^3 \frac{1 - \Phi(x)}{\phi(x)} \right)$$

These considerations generalize easily to higher order approximations. Obviously, this method is more sophisticated than the previous ones. One important point is that one needs to know or to approximate the moment generating function and the quality of the approximation of $g(x)$ depends on the quality of the approximation of $\widehat{g}(h)$. Moreover, $g(x)$ has to be estimated pointwise. But the great advantage of this method is its accuracy in the tails.

2.3 *The saddlepoint method*

The saddlepoint method has originally been developed by *Daniels* (1954) as an approximation of the density of the mean of n iid observations. The basic idea is an explicit inversion of a Fourier transform which can be obtained in form of an asymptotic expansion in powers of n^{-1}. Its dominant term is called saddlepoint approximation and has the advantage that for an important class of densities the relative error is uniformly n^{-1} over the whole support of the random variable.

The idea of a saddlepoint approximation has since then been applied to many other areas of statistics, see e.g. *Fields* and *Ronchetti* (1990). For compound Poisson and Pólya process the method has been used by *Embrechts* et al. (1985) to approximate tail probabilities $P\big(\sum_{i=1}^{N} X_i > x\big)$, where N is either Poisson or negative binomial distributed.
In the next paragraph 2.3.1 we explain the general idea for the tail probability of the mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and discuss its extensions to random sums in 2.3.2.

2.3.1 Saddlepoint expansions for tail probabilities

We follow *Jensen* (1988). Suppose we want to approximate the distribution tail $G_n(x) = P\big(\frac{1}{n}\sum_{k=1}^{n} X_k > x\big)$ where $(X_k)_{k\in\mathbb{N}}$ are iid with common distribution function F and Fourier-Laplace transform

$$\varphi(h+iu) = E\exp((h+iu)X_1)$$

defined for $h < \tau \leq \infty$. Then we shall use the following inversion formula [*Widder* (1941), Theorem 7.6.b, p.70]

$$\overline{F}(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\varphi(h+iu)}{h+iu}\exp(-(h+iu)x)du \tag{19}$$

which holds for $0 < h < \tau$ if $\varphi(h+iu)/(h+iu)$ is integrable. Furthermore, we assume that $\varphi(h+iu) \to 0$ for $|u| \to \infty$. By a change of variables we obtain for G_n:

$$\overline{G}_n(x) = \frac{\varphi^n(h)\exp(-nhx)}{\sqrt{n}\,h\sigma(h)}\,\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\varphi_h^n\left(\frac{u}{\sqrt{n}}\right)\frac{du}{1+iu/(\sqrt{n}\,h\sigma(h))} \tag{20}$$

where

$$\varphi_h(u) = \frac{\varphi(h+iu/\sigma(h))}{\varphi(h)}\exp(-iux/\sigma(h)) \tag{21}$$

Notice that $\varphi(h+it)/\varphi(h)$ is the characteristic function of an exponentially tilted random variable as defined in (13). Again h is chosen such that x is equal to the mean, i.e. h is the solution of $(d/dh)\ln\varphi(h) = x$. Furthermore, we have introduced a new scale $\sigma(h)$, which we choose to be equal to the standard deviation of the exponentially tilted random variable, i.e. $\sigma^2(h) = (d^2/dh^2)\ln\varphi(h)$. The point h is the saddlepoint and the saddlepoint expansion is now obtained by expanding the cumulant generating function $\ln\varphi_h(u)$; we denote by $\rho_k(h)$ its cumulants; i.e. $\rho_k(h) = \kappa^{(k)}(h)/(\kappa''(h))^{k/2}$ with $\kappa(h) = \ln\varphi(h)$. Then a Taylor expansion gives

$$\begin{aligned}
\varphi_h^n\left(\frac{u}{\sqrt{n}}\right) &= \exp\left\{n\left(\ln\varphi\left(h+\frac{iu}{\sqrt{n}\sigma(h)}\right) - \ln\varphi(h) - \frac{iux}{\sqrt{n}\sigma(h)}\right)\right\} \\
&= \exp\left\{n\left(\ln\varphi(h) + (ln\varphi)'(h)\frac{iu}{\sqrt{n}\sigma(h)}\right.\right. \\
&\quad \left.\left. + (\ln\varphi)''(h)\frac{(iu)^2}{2n\sigma^2(h)} + \cdots - \ln\varphi(h) - \frac{iux}{\sqrt{n}\sigma(h)}\right)\right\} \\
&= \exp\left\{-\frac{u^2}{2} + \sum_{k=3}^{\infty}\frac{\rho_k(h)}{k!\sqrt{n^{k-2}}}(iu)^k\right\}
\end{aligned}$$

We expand $\exp\left\{\sum_{k=3}^{\infty}\frac{\rho_k(h)}{k!\sqrt{n^{k-2}}}(iu)^k\right\}$ and consider all terms up to order $O(\frac{1}{n})$; this gives

$$\varphi_h^n\left(\frac{u}{\sqrt{n}}\right) \approx e^{-u^2/2}\left\{1 + \frac{\rho_3(h)}{6\sqrt{n}}(iu)^3 + \frac{\rho_4(h)}{24n}(iu)^4 + \frac{\rho_3^2(h)}{36n}(iu)^6\right\}$$

If we set now

$$B_k(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2}\frac{(iu)^k du}{1+iu/z}$$

then

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_h^n\left(\frac{u}{\sqrt{n}}\right)\frac{du}{1+iu/(\sqrt{n}\,h\sigma(h))} \\
&\quad = B_0(z) - \frac{\rho_3(s)}{6\sqrt{n}}B_3(z) + \frac{\rho_4(s)}{24n}B_4(z) + \frac{\rho_3^2(s)}{72n}B_6(z) + R(z) \qquad (22)
\end{aligned}$$

where $z = \sqrt{n}\,h\sigma(h)$, and expressions for $B_k(z)$ can be found e.g. in *Abramovitz* and *Stegun* (1970), pp. 297–330:

$$B_0(z) = \sqrt{2\pi}\,ze^{z^2/2}(1-\Phi(z)) \qquad B_3(z) = z^3B_0(z) - (z^3 - z)$$

$$B_4(z) = z\,B_3(z) \qquad B_6(z) = z^3B_3(z) - 3z^2$$

2.3.2 Saddlepoint expansions for a compound Poisson sum

The situation we are interested in is the approximation of $\overline{G}(x) = P\Big(\sum_{k=1}^{N} X_k > x\Big)$ where N is Poisson with mean λ, where λ is fixed, and we consider the limiting behaviour of $\overline{G}(x)$ as $x \to \infty$ [*Embrechts* et al. (1985)]. We denote by

$$\varphi_X(h+iu) = E\exp((h+iu)X_1)$$

the Fourier-Laplace transform of X_1, then the Poisson sum S has Fourier-Laplace transform

$$\varphi_S(h+iu) = \exp(-\lambda(1-\varphi_X(h+iu)))$$

Now assume φ_X satisfies, as required for fixed n in section 2.3.1, that $\varphi_X(h+iu) \to 0$ as $|u| \to \infty$, then

$$\varphi_S(h+iu) \to e^{-\lambda}, \quad |u| \to \infty$$

This effect which arises from the discreteness of N in 0 has to be removed and it can be done by defining $\widetilde{N}$ by $P(\widetilde{N} = k) = p_k/(1-p_0)$, $k \in \mathbb{N}$, and $\widetilde{S} = \sum_{k=1}^{\widetilde{N}} X_k$. This implies, that

$$P(S > x) = (1-p_0)P(\widetilde{S} > x) \tag{23}$$

and with $P(\widetilde{N} = 0) = 0$ we obtain

$$\varphi_{\widetilde{S}}(h+iu) = \sum_{k=1}^{\infty} \varphi_X^k(h+iu)P(\widetilde{N} = k) = \frac{1}{1-p_0}(\varphi_S(h+iu) - p_0)$$

which tends to 0 as $|u| \to \infty$.
We apply the inversion formula (19) and obtain together with (23)

$$\overline{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\varphi_S(h+iu) - p_0)}{h+iu} \exp(-(h+iu)x)du$$

for $h < \tau \leq \infty$ where $(\varphi_S(h+iu) - p_0)/(h+iu)$ can be shown to be integrable if φ_X satisfies this condition. The saddlepoint h is chosen such that $(d/dh)\ln\varphi_S(h) = \lambda\varphi'_X(h) = x$ and the normalizing scale as $\sigma^2(h) = (d^2/dh^2)\ln\varphi_S(h) = \lambda\varphi''_X(h)$, i.e. the exponentially tilted random variable S_h has been standardized. Then we obtain

$$\overline{G}(x) = \frac{(\varphi_S(h) - p_0)\exp(-hx)}{h\,\sigma(h)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_1(u)\frac{du}{1 + iu/h\,\sigma(h)}$$

where

$$\varphi_1(u) = \frac{\varphi_S(h + iu/\sigma(h)) - p_0}{\varphi_S(h) - p_0}\exp(-ixu/\sigma(h))$$

Write

$$\varphi_1(u) = \left(\frac{\varphi_S(h + iu/\sigma(h))}{\varphi_S(h)} - \frac{p_0}{\varphi_S(h)}\right)\frac{1}{1 - p_0/\varphi_S(h)}$$

ignore p_0/φ_S and set

$$\varphi_2(u) = \frac{\varphi_S(h + iu/\sigma(h))}{\varphi_S(h)}\exp(-ixu/\sigma(h))$$

Notice that $\varphi_2 = \varphi_{h,S}$ of (21), and if we approximate

$$\overline{G}(x) \approx \frac{(\varphi_S(h) - p_0)\exp(-hx)}{h\,\sigma(h)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_2(u)\frac{du}{1 + iu/h\,\sigma(h)}$$

then we obtain for the integral of the righthandside the same expansion as for the integral in (22) with $n = 1$, with

$$\frac{\rho_k(h)}{\sqrt{n^{k-2}}} = \frac{\beta_k(h)}{\sqrt{\lambda^{k-2}}} = \frac{\varphi_X^{(k)}(h)}{\sqrt{(\varphi''_X(h))^k\lambda^{k-2}}}$$

If $\beta_3(h) \to 0$ and $\beta_5(h)/\beta_4(h) \to 0$ as $h \to \tau$ then for a large class of distributions F the expansions of (22) are of order $O(\beta_3(h)/\sqrt{\lambda})$ and $O(\beta_5(h)/\lambda^{3/2})$, respectively, as $y \to \infty$.
A short version of the saddlepoint approximation can be obtained replacing $B_k(z)$ in (22) by their asymptotic equivalents for $z \to \infty$.

$$B_0(z) \to 1\,, \quad B_3(z) \sim 3/z \to 0\,, \quad B_4(z) \to 3\,, \quad B_6(z) \to 15$$

Consequently,

$$\overline{G}(x) \approx \frac{(\varphi_s(h) - p_0)\exp(-hx)}{h\,\sigma(h)} \frac{1}{\sqrt{2\pi}} \left\{ 1 + \frac{1}{24\lambda}(3\beta_4(h) - 5\beta_3^2(h)) \right\}$$

The advantage of the saddlepoint method is that for a large class of distributions the error is uniformly bounded [see *Embrechts* et al. (1986)].

2.4 *Discrete methods*

The methods we discuss in this section start with an arithmetic claimsize distribution; i.e. there exists some $d > 0$ such that

$$f(j) = P(Y = jd)\,, \quad j \in \mathbb{N}_0$$

If Y has an absolutely continuous distribution function F, then a discretisation procedure is applied as e.g.

$$f(0) = F(0)\,, \qquad f(1) = F\left(\frac{3d}{2}\right) - F(0)$$

$$f(j) = F\left(\frac{2j+1}{2}d\right) - F\left(\frac{2j-1}{2}d\right) \quad \text{for} \quad j \geq 2$$

[see *Feilmeier* and *Bertram* (1987), p.48]. Notice that the approximation methods of this section all work on a finite support. Furthermore, the number of calculations increases as well with a finer discretisation (smaller d) as with a greater support (for heavy tailed distributions).

We also want to emphasize the fact, that for all approximation methods introduced in sections 2.1.1–2.2.3 only λ and the first few moments of the claimsize distribution have to be known (or estimated); for the Esscher and saddlepoint approximation a transform has to be calculated. For the discrete approximations in this section the whole claimsize distribution has to be known.

2.4.1 The Panjer approximation

We derive the recursion formula for N Poisson distributed, i.e.

$$p_k = P(N = k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for} \quad k \in \mathbb{N}_0$$

Notice that p_k satisfies the following recurrence relation

$$p_0 = e^{-\lambda}, \quad p_k = \frac{\lambda}{k} p_{k-1}, \quad k \in \mathbb{N}$$

Furthermore, the distribution of the total claim amount S is

$$g(j) = P(S = jd) = \sum_{k=0}^{\infty} p_k \, f^{k*}(j), \quad j \in \mathbb{N}_0$$

For instance we have

$$g(0) = e^{-\lambda} \sum_{k=0}^{\infty} \lambda^k f^k(0)/k! = e^{\lambda(f(0)-1)} \tag{24}$$

Now the moment generating function satisfies for $t < \tau \leq \infty$

$$\widehat{g}(t) = E\, e^{tS} = \sum_{k=0}^{\infty} p_k \, \widehat{f}^k(t)$$

Taking the derivative yields

$$\begin{aligned}
\widehat{g}\,'(t) &= \sum_{k=1}^{\infty} k p_k \, \widehat{f}^{k-1}(t) \, \widehat{f}'(t) \\
&= \lambda \widehat{f}'(t) \sum_{k=1}^{\infty} p_{k-1} \, \widehat{f}^{k-1}(t) \\
&= \lambda \widehat{f}'(t) \, \widehat{g}(t) \, .
\end{aligned}$$

Since $\widehat{g}(t) = \sum_{j=0}^{\infty} g(j)\, e^{tjd}$ and $\widehat{f}(t) = \sum_{j=0}^{\infty} f(j)\, e^{tjd}$, we obtain

$$\sum_{j=1}^{\infty} jd\, g(j)\, e^{tjd} = \lambda \sum_{j=1}^{\infty} jd\, f(j)\, e^{tjd} \cdot \sum_{l=0}^{\infty} g(l)\, e^{tld}$$

We compare the coefficients of e^{tjd} and obtain for $j \in \mathbb{N}$

$$jg(j) = \lambda \sum_{l=1}^{j} l\, f(l)\, g(j-l)$$

which gives together with (24) the *Panjer recursion formula*:

$$g(0) = e^{\lambda(f(0)-1)}$$
$$g(j) = \frac{\lambda}{j} \sum_{l=1}^{j} l\, f(l)\, g(j-l)\,, \quad j \in \mathbb{N}$$

Notice that the important point is the recurrence relation for p_k, $k \in \mathbb{N}_0$; the method works for all p_k satisfying

$$p_k = \left(a + \frac{b}{k}\right) p_{k-1}\,, \quad k \in \mathbb{N}$$

for some a, $b \in \mathbb{R}$.

2.4.2 The fast Fourier transform method

Using the fact, that for the characteristic functions of f and g the following relation holds

$$\varphi_S(t) = \exp[\lambda(\varphi_Y(t) - 1)]$$

we proceed according to the following diagram

$$f \longrightarrow \varphi_Y \xrightarrow{\exp[\lambda(\varphi_Y(t)-1)]} \varphi_S \longrightarrow g$$

This means we have to determine the characteristic function and its inverse. The method we use can be explained as follows: For a given vector $\underline{a} = (a_0, \cdots, a_{n-1})^T$, the vector $\underline{b} = (b_0, \cdots, b_{n-1})^T$ with

$$\underline{b} = W \underline{a}\,, \quad \text{where} \quad W = \left(e^{kj \cdot 2\pi i/n}\right)_{k,\, j=0,\, \cdots,\, n-1}$$

is the Fourier transform of $\underline{a}$. Now we have for $\overline{W} = \left(e^{-kj \cdot 2\pi i/n}\right)_{k,\, j=0,\, \ldots,\, n-1}$ that

$$W \cdot \overline{W} = nI$$

where I is the unit matrix, which provides the inversion procedure. Thus we obtain

$$\underline{b} = W\underline{a} \stackrel{\text{def}}{=} FFT^{+}(\underline{a})$$
$$\underline{a} = \frac{1}{n}\overline{W}\underline{b} \stackrel{\text{def}}{=} \frac{1}{n}\, FFT^{-}(\underline{b})$$

Now consider the characteristic function φ_Y of the distribution $f(k) = P(Y = kd)$, $k \in \mathbb{N}_0$. If we compute $\varphi_Y(t_j)$ for $t_j = \frac{2\pi j}{nd}$, $j \in \mathbb{N}_0$, we obtain

$$\varphi_Y(t_j) = \sum_{k=0}^{\infty} f(k) e^{ikd\cdot 2\pi j/nd} = \sum_{k=0}^{\infty} f(k) e^{jk\frac{2\pi i}{n}}$$

If we set

$$\widetilde{f}(k) = \sum_{l=0}^{\infty} f(k+ln)\,, \quad k = 0, 1, \ldots, n-1$$

we obtain by periodicity

$$\varphi_Y(t_j) = \sum_{k=0}^{n-1} \widetilde{f}(k)\, e^{jk\frac{2\pi i}{n}}\,.$$

We define the vectors $\underline{\varphi}_Y = (\varphi_Y(t_0), \ldots, \varphi_Y(t_{n-1}))^T$ and $\underline{\widetilde{f}} = (\widetilde{f}(0), \ldots, \widetilde{f}(n-1))^T$. Then we may rewrite the last formula as

$$\underline{\varphi}_Y = FFT^{+}(\underline{\widetilde{f}})$$

Applying this result to the compound Poisson distribution we obtain with the corresponding notation

$$\underline{\varphi}_S = FFT^{+}(\underline{\widetilde{g}}) = \exp[\lambda(FFT^{+}(\underline{\widetilde{f}}) - 1)]$$

and solved for $\underline{\widetilde{g}}$

$$\underline{\widetilde{g}} = \frac{1}{n} FFT^{-}(\exp[\lambda(FFT^{+}(\underline{\widetilde{f}}) - 1)])$$

This way to calculate $\underline{\widetilde{g}}$ from the given values $\underline{\widetilde{f}}$, is called the fast Fourier transform method, even it is not yet fast. To justify its name, let us return e.g. to the transformation FFT^+. The complexity of the algorithm can be represented by the number of complex multiplications which is of the order $O(n^2)$, because we need n^2 multiplications to compute the product $W\underline{a}$. If n is even, the algorithm can be accelerated. For $n = 2m$, $m \in \mathbb{N}$, let w_n be the n-th unit root $w_n = e^{2\pi i/n}$. We compute b_{2k}:

$$\begin{aligned} b_{2k} &= \sum_{j=0}^{n-1} a_j\, w_n^{2kj} = \sum_{j=1}^{m-1} \left(a_j\, w_n^{2kj} + a_{j+m}\, w_n^{2k(j+m)} \right) \\ &= \sum_{j=1}^{m-1} (a_j + a_{j+m})\, w_m^{kj} \end{aligned}$$

since $w_n^{2km} = w_n^{kn} = 1$ and $w_n^{2kj} = \left(w_n^2\right)^{kj} = w_m^{kj}$. In the same way, with $w_n^m = -1$ we get:

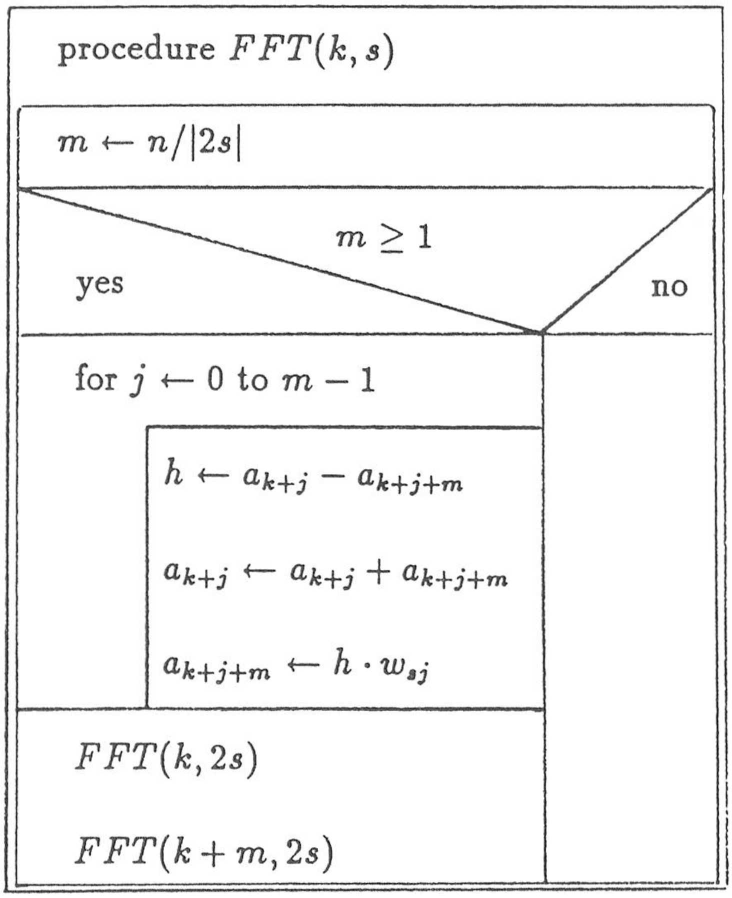
$$\begin{aligned} b_{2k+1} &= \sum_{j=0}^{n-1} a_j\, w_n^{(2k+1)j} = \sum_{j=1}^{m-1} \left(a_j\, w_n^{(2k+1)j} + a_{j+m}\, w_n^{(2k+1)(j+m)} \right) \\ &= \sum_{j=1}^{m-1} (a_j - a_{j+m})\, w_n^j\, w_m^{kj} \end{aligned}$$

If we denote $a_j^* = a_j + a_{j+m}$ and $a_j^{**} = (a_j - a_{j+m})w_n^j$, $0 \leq j \leq m-1$, we may replace one Fourier transformation of order n by two of the order $m = \frac{n}{2}$:

$$b_{2k} = \sum_{j=0}^{m-1} a_j^* w_m^{jk}\,, \qquad b_{2k+1} = \sum_{j=0}^{m-1} a_j^{**} w_m^{jk}$$

A considerable reduction of the number of calculations can be achieved by choosing $n = 2^l$. One can prove, that in this case the complexity of the algorithm is of the order $O(n \cdot l) = O(n \cdot \log_2 n)$ [see *Bühlmann* (1984)]. Obviously the same is true for the transformation FFT^-.

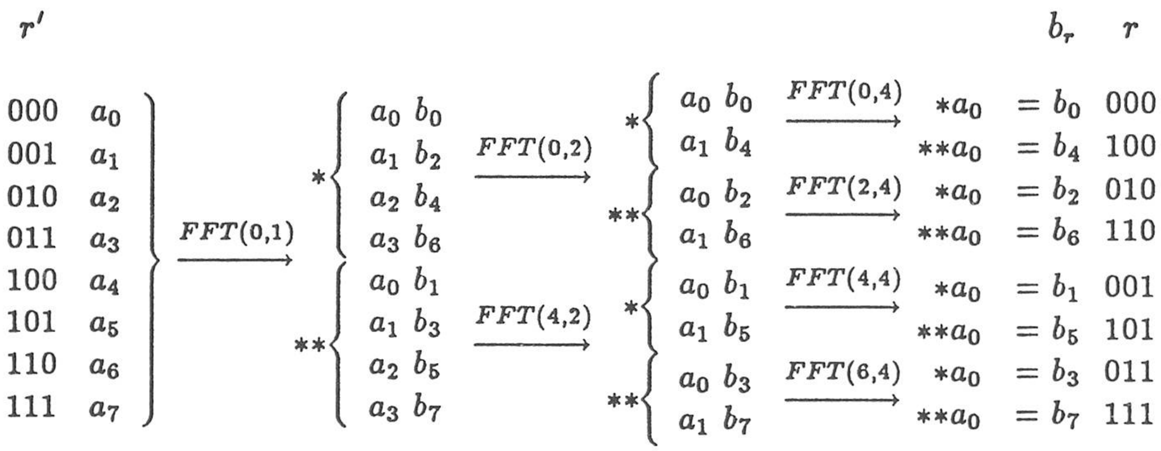
If $n = 2^l$, the fast Fourier transformation can be programmed recursively according to the following Nassi-Shneiderman diagram



Herein k represents the index of the first element and $n/|s|$ the order of the fast Fourier transformation. s is also needed to pick the correct unit root $w_{sj} \stackrel{\text{def}}{=} w_n^{sj}$. If we define $w_l = e^{2l\pi i/n}\big|_{-n/2<l<n/2}$ we get

$$FFT^+(\underline{a}) \longleftrightarrow FFT(0, 1)$$
$$FFT^-(\underline{a}) \longleftrightarrow FFT(0, -1)$$

To illustrate, how the procedure FFT works, we give an example with $n = 8$ (i.e. $l = 3$):



The b-rows show, which elements b_k are computed with the help of the corresponding a-values. After the completed transformation, the element with index r is at a different place, say r'. The two numbers r and r' are related in the following way: If we reverse the digits in the binary representation of r (with l positions) we get a binary representation of r' [for a proof see *Schwarz* (1977)]. To get the right order of the elements of the Fourier transform, we have to renumber them, after finishing the procedure FFT.

In the sequel we shall abbreviate fast Fourier transform by FFT. For a recent summary of the FFT method in insurance mathematics see *Embrechts* et al. (1992).

2.4.3 The Monte-Carlo method

Monte-Carlo simulation is nowadays a well established tool in almost all fields of applied mathematics. Random number generators are installed on every computer, generating standard uniform random numbers. From these one can derive random numbers of any distribution. In our case we have to generate random numbers for the claimsize distribution F and also for the Poisson distribution.

The simplest method to generate random numbers from a distribution F is by means of the generalized inverse

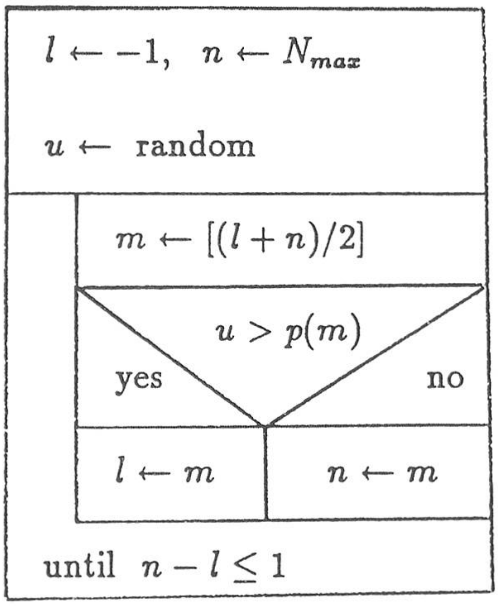
$$F^{\leftarrow}(u) = \inf \{x; F(x) \geq u\}$$

if it can be calculated explicitly. This is easy e.g. for the exponential distribution.

For uniform distributed U

$$X = -\frac{1}{a} \ln(1 - U)$$

is exponentially distributed with parameter a.
To generate Poisson distributed random numbers n, we create an array p with $p(k) = P[N \le k] = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$ for $0 \le k \le N_{\max}$. The following binary search algorithm returns n.



Herein, "random" generates standard uniform random numbers and $[x]$ denotes the integer part of x.
For more sophisticated algorithms see e.g. *Morgan* (1984) or *Ripley* (1987).
To simulate values $s_1, s_2, \ldots, s_m$, $m \in \mathbb{N}$, of the total claimsize S we first simulate Poisson random numbers $n_1, n_2, \ldots, n_m$, and then n_k random numbers y_{ki}, $i = 1, \ldots, n_k$, $k = 1, \ldots, m$, according to the claimsize distribution F. Then

$$s_k = \sum_{i=1}^{n_k} y_{ki}\,, \quad k = 1, \ldots, m\,,$$

are simulated values of S. An approximation for the total claimsize distribution

G is now given by

$$G(x) \approx \frac{1}{m} \sum_{k=1}^{m} I_{(s_k \le x)}\,, \qquad x \in \mathbb{R}\,.$$

We did not include any figures of the outcome in section 3, the reason beeing that the method can be made arbitrarily exact. Nevertheless, the computertime needed to gain comparable exactness as for other discrete methods is considerably higher. A more detailed discussion of the pros and cons of this method can be found in *Feilmeier* and *Bertram* (1987).

3. Examples

This section is devoted to a graphical presentation of the approximation of the tail $\overline{G}(x) = 1 - G(x)$ of the total claimsize distribution by the methods described in the previous section. As one can imagine it was not easy to choose an appropriate sample of the graphics: On the one hand we wanted to show some typical pictures, but also on the other hand show how wrong things can go. The graphics are collected at the end of the section. The underlining explanations to the figures are in the same order as the approximating curves; i.e. top to bottom.
We restrict ourselves to the Poisson model, so we have to specify the Poisson parameter λ and the claimsize distribution. The methods are of particular interest for small samples and hence we decided to take always $\lambda = 10$ and to choose the parameters of the claimsize distributions appropriately; the expected total claim amount ES ranges for our examples from 10 to 27.
For the two discrete methods, the Panjer approximation and the FFT method, the error is uniformly bounded and can be made arbitrarily small by taking a sufficiently small discretisation parameter d. Furthermore, both methods show virtually the same curve. So we only show here the FFT approximation and we consider it as a reference curve for the quality of the other approximation methods.

3.1 Exponential claimsizes

The one exception where we did not take $\lambda = 10$ is for exponentially distributed claimsizes with density

$$f(x) = a\,e^{-ax}\,, \quad x \ge 0,\, a > 0\,.$$

It is well-known that in this case it is possible to calculate explicitly the distribution function G of the total claim amount which is given e.g. in *Heilmann* (1988)

$$G(x) = 1 - e^{-(\lambda + ax)} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} \frac{(ax)^j}{j!}, \quad x \geq 0$$

So we had hoped for an excellent reference curve for all our approximation methods, namely the true distribution. Unfortunately, the analytical representation as an infinite series caused numerical problems and we did not obtain sufficient exactness for $\lambda = 10$. On the other hand for $\lambda = 1$ and a standard exponential claimsize distribution we obtained a very accurate tail $1 - G$ and we present it in Figure 1 and 2 together with some approximation results. We also calculated the relative error $|\overline{G}(x) - \overline{H}(x)|/\overline{G}(x)$ for different approximations $\overline{H}$. Whereas for the FFT approximation the error remains bounded for the normal power approximation it is increasing and becomes rather large in the tail.

3.2 Gamma distributed claimsizes

The gamma density is given by

$$f(x) = \frac{1}{b\,\Gamma(a)} \left(\frac{x}{b}\right)^{a-1} e^{-x/b}, \quad x \geq 0,\ a, b > 0\,.$$

In our example we took $a = 5$ and $b = 1/5$ which implies with $\lambda = 10$ that $ES = 10$. All methods performed rather well around the mean whereas in the extreme tails the approximations by orthogonal polynomials perform rather poorly as can be seen in Figure 3 where the range from 20 to 30 has been chosen. Since the moment generating function is explicitly given, approximations like Esscher and saddlepoint are easily computed and almost coincide with the FFT curve. Our reference curve, the FFT approximation, has been calculated with $d = 0.02$ and $n = 2^{12} = 4096$.

3.3 Weibull distributed claimsizes

The Weibull density is given by

$$f(x) = \frac{a}{b} \left(\frac{x}{b}\right)^{a-1} e^{-(x/b)^a}, \quad x \geq 0,\ a, b > 0$$

For our examples we took two different parameter sets such that the tail decreases for one set of parameters faster than exponential and for the other slower.

For $a = 2$ and $b = 3$ we obtain quite reasonable approximations also for the far end tail (Figure 4). For the FFT approximation we took $d = 0.05$ and $n = 4096$. The situation is different in the subexponential case, here we took $a = 0.5$ and $b = 0.75$. Figure 5 shows approximations in the range $10 - 60$. Here the Gram-Charlier approximation shows a rather strange behaviour between 25 and 45.

Figure 6 shows that in the far tail, which is noticably heavier than in Figure 4, the approximations are further apart from each other. For the FFT approximation we took $d = 0.03$ and $n = 8192$.

3.4 Lognormal claimsize

The claimsizes have density

$$f(x) = \frac{1}{\sqrt{2\pi}\,xb} \exp\left\{ -\frac{(\ln x - a)^2}{2b^2} \right\}, \quad x \geq 0,\ a > 0,\ b > 1$$

Notice that the moment generating function is infinite for all positive arguments, hence approximation methods like Esscher and saddlepoint are no longer possible. All moments are finite, but grow very fast for certain parameter values which can cause serious trouble. Edgeworth expansions of different order make this very obvious.

For $a = 1$ and $b = 1/5$ the FFT approximation and the Edgeworth expansions of orders 3, 4, 5 and 6 seem to amalgamate into one curve (Figure 7). Even in the tails they are very close (Figure 8).

This is not very surprising by the following table which shows the first six moments of S and the coefficients of the Edgeworth expansion.

EX	EX^2	EX^3	EX^4	EX^5	EX^6
2.77319	8.00447	24.0468	75.1886	244.692	828.818

a_3	a_4	a_5	a_6
-0.055963	0.004889	-0.0003557	0.001588

Moments and Edgeworth coefficients for lognormal claimsizes with $(a, b) = \left(1, \frac{1}{5}\right)$

Table 1

The graphics change gradually when one decreases a and increases b gradually. The moments grow faster and for $a = 1/5$ and $b = 1$ the disaster is shown in Figures 9–12. The plain line is obtained by FFT approximation with $d = 0.05$ and $n = 4096$. Figure 13 shows 9–12 together.
The first six moments of S and the coefficients of the Edgeworth expansion are given in the following table.

EX	EX^2	EX^3	EX^4	EX^5	EX^6
2.01375	11.0232	164.002	6634.24	729416	$2.17 \cdot 10^8$

a_3	a_4	a_5	a_6
-0.236206	0.227492	-0.476461	2.28838

Moments and Edgeworth coefficients for lognormal claimsizes with $(a,\, b) = \left(\frac{1}{5},\, 1\right)$
Table 2

3.5 *Pareto distributed claimsizes*

The Pareto density is given by

$$f(x) = \frac{b}{a}\left(\frac{a}{a+x}\right)^{b+1}, \quad x \geq 0,\ a,\, b > 0\,.$$

Also here the moment generating function has its singular point in 0. We took the parameters $a = 6$ and $b = 5$ which guarantees the existence of 4 finite moments. Some approximations are shown in Figure 14.

3.6 *Monte Carlo simulation*

As mentioned in section 2 also simulation methods provide useful approximations. Indeed they give good results but as ad hoc methods they proved to be slow. Also the many textbooks on simulation methods show clearly that there is a lot more to say to this subject than we are prepared to do in this paper.
Nevertheless, we show some examples in Figures 15–18. As Poisson parameter we took $\lambda = 10$ and simulated the Poisson variable N by the binary search algorithm described in section 2.4.3. Then we simulated 1000 total claims, where we used Marsaglia's polar method to simulate normal random numbers which

we transformed into lognormal ones. The other distributions we simulated by inversion.
It should be noted, that the heavier the tails are the less points determine the tails. Therefore particular methods have been developed to simulate distribution tails [see e.g. *Keller* and *Klüppelberg* (1991)].

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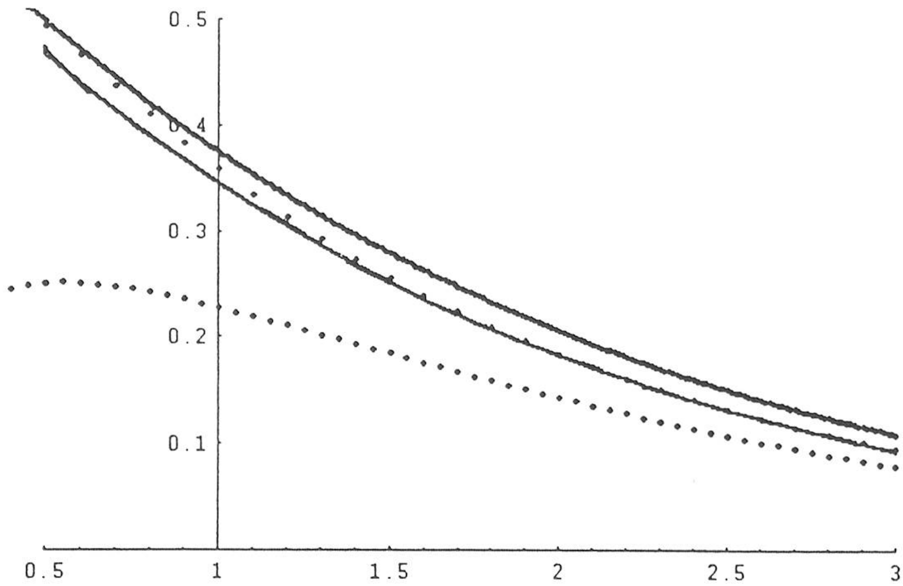
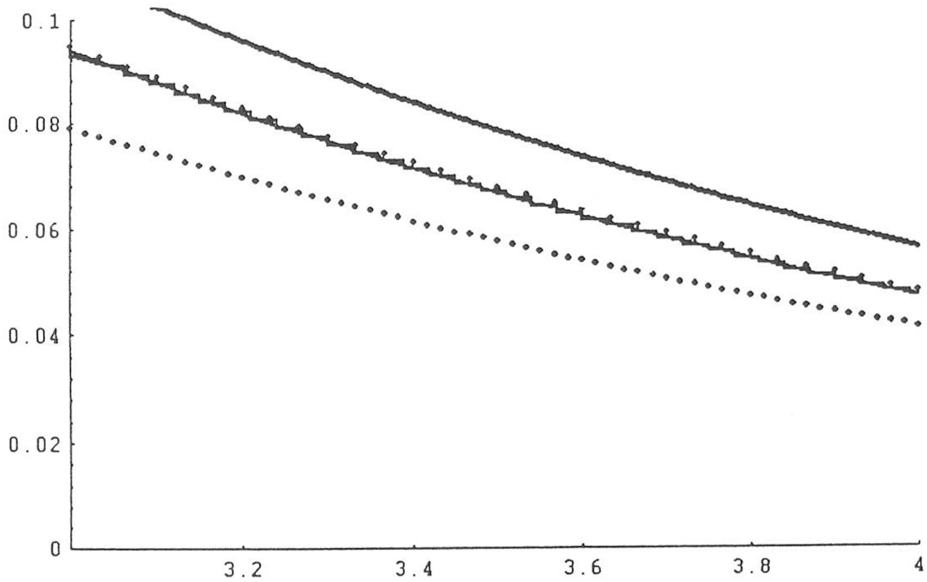
Figure 1 Exponential distribution



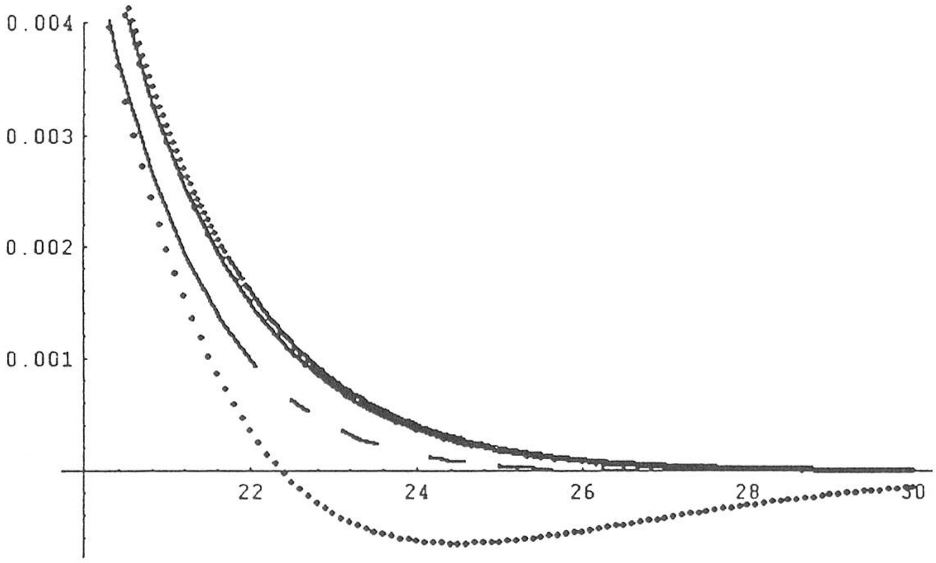
______________ Normal power approximation of 2nd order
. . . . Esscher approximation of 3rd order
______________ Exact distribution and FFT approximation
. . . . Saddlepoint approximation of 3rd order

Figure 2 Exponential distribution



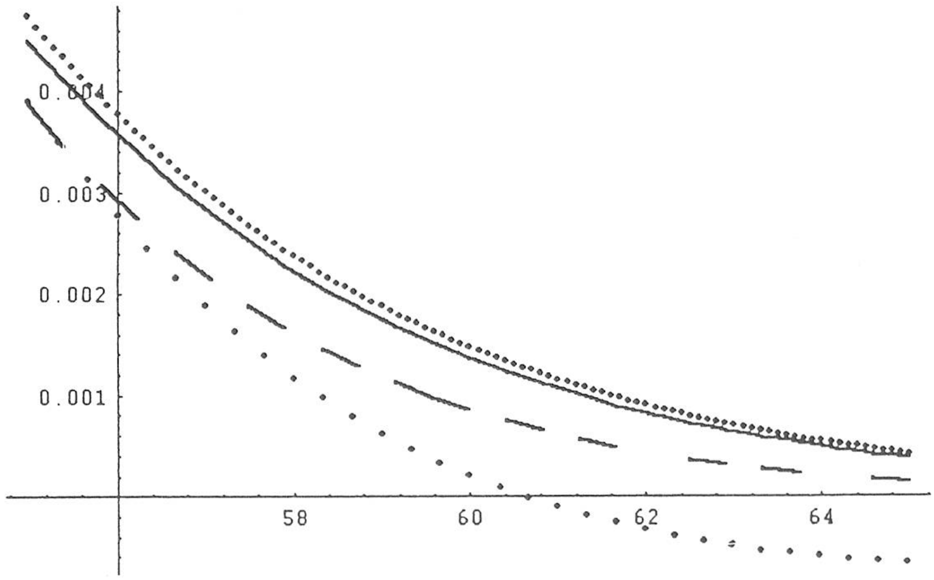
______________ Normal power approximation of 2nd order
. . . . Esscher approximation of 3rd order
______________ Exact distribution and FFT approximation
. . . . Saddlepoint approximation of 3rd order

Figure 3 Gamma distribution



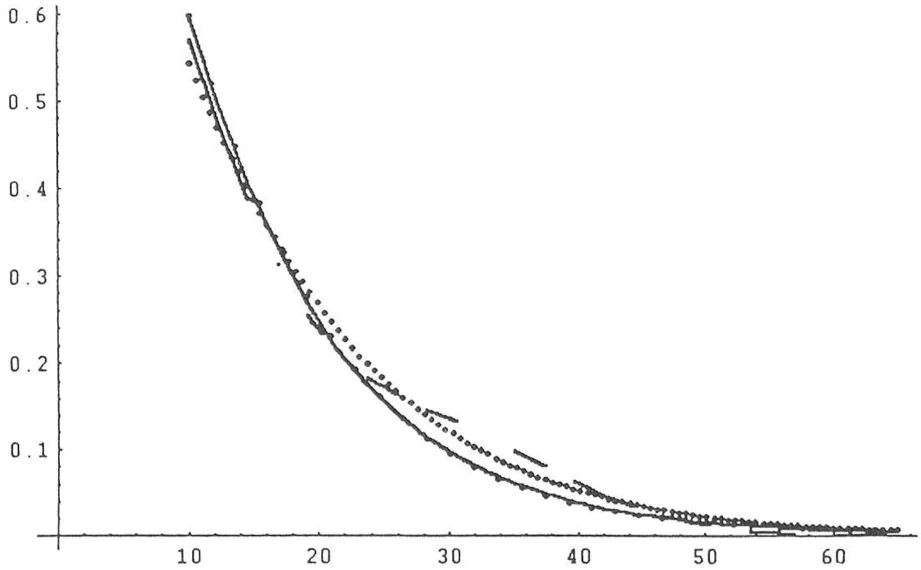
. . . . Normal power approximation of 2nd order
———————— FFT approximation
. . . . Esscher and saddlepoint approximation of 3rd order
\- - - - - - - - - Gram-Charlier approximation of 3rd order
. . . . Gamma approximation of Bowers of 3rd order

Figure 4 Weibull distribution ($a = 2$, $b = 3$)



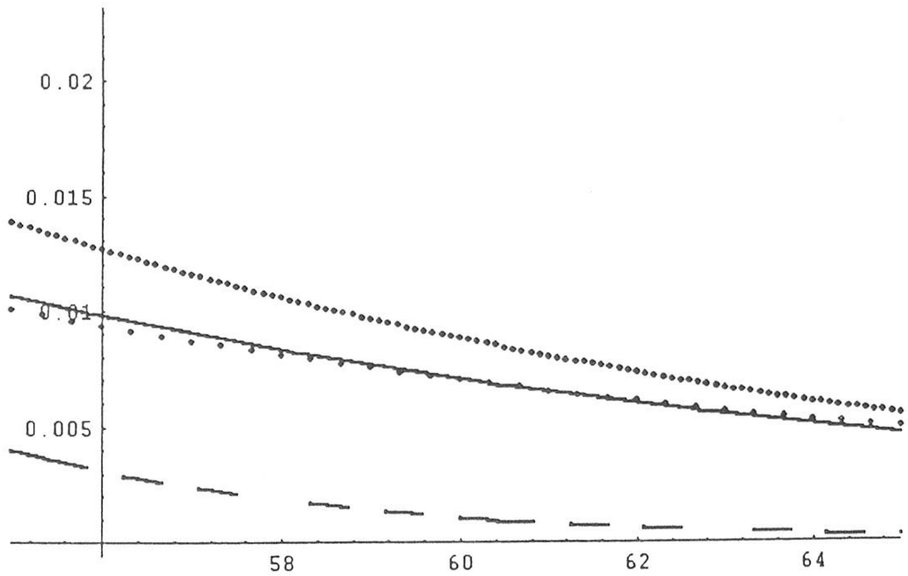
. . . . Normal power approximation of 2nd order
———————— FFT approximation
\- - - - - - - - - Gram-Charlier approximation of 3rd order
. . . . Gamma approximation of Bowers of 3rd order

Figure 5 Weibull distribution ($a = 0.5$, $b = 0.75$)



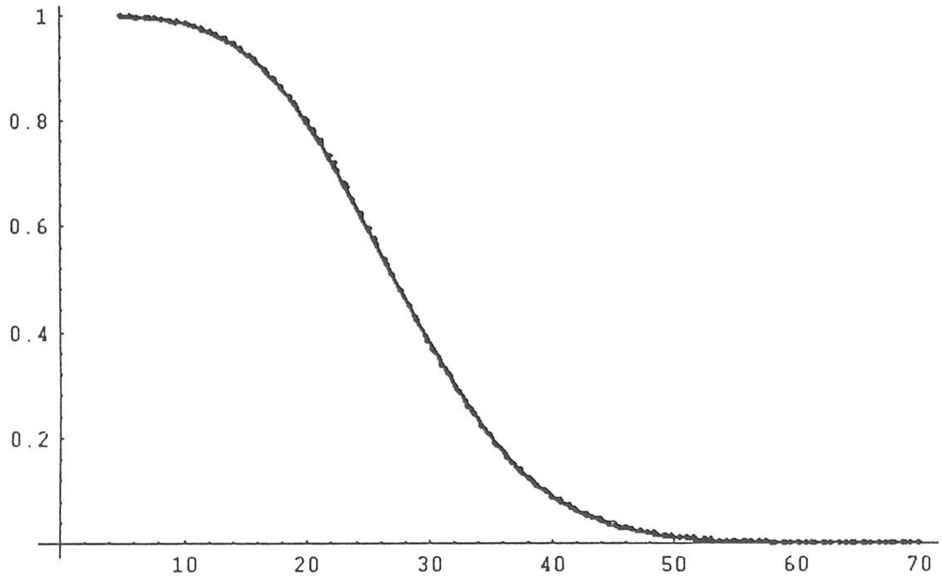
- - - - - - - - - Gram-Charlier approximation of 3rd order
. . . . Normal power approximation of 2nd order
——————— FFT approximation
. . . . Gamma approximation of Bowers of 3rd order

Figure 6 Weibull distribution ($a = 0.5$, $b = 0.75$)



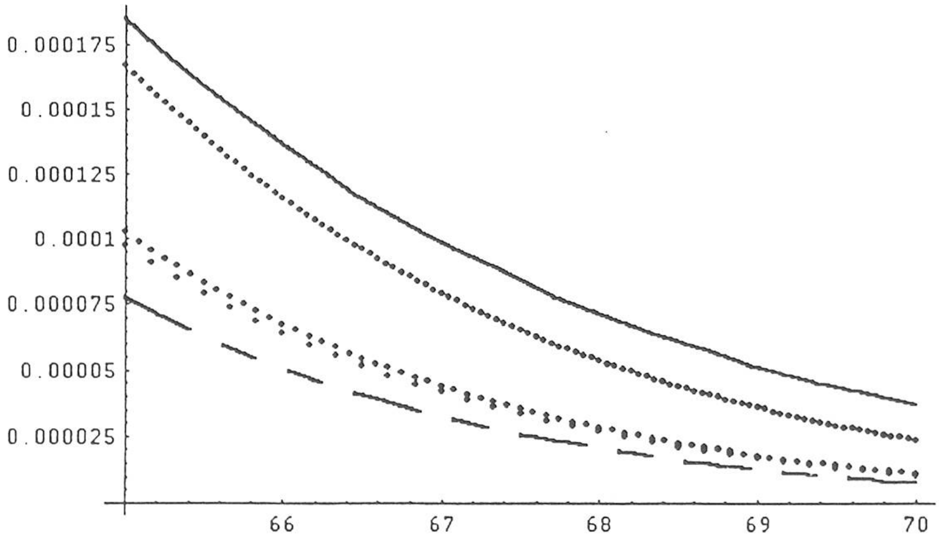
. . . . Normal power approximation of 2nd order
——————— FFT approximation
. . . . Gamma approximation of Bowers of 3rd order
- - - - - - - - - Gram-Charlier approximation of 3rd order

Figure 7 Lognormal distribution ($a = 1$, $b = 1/5$)



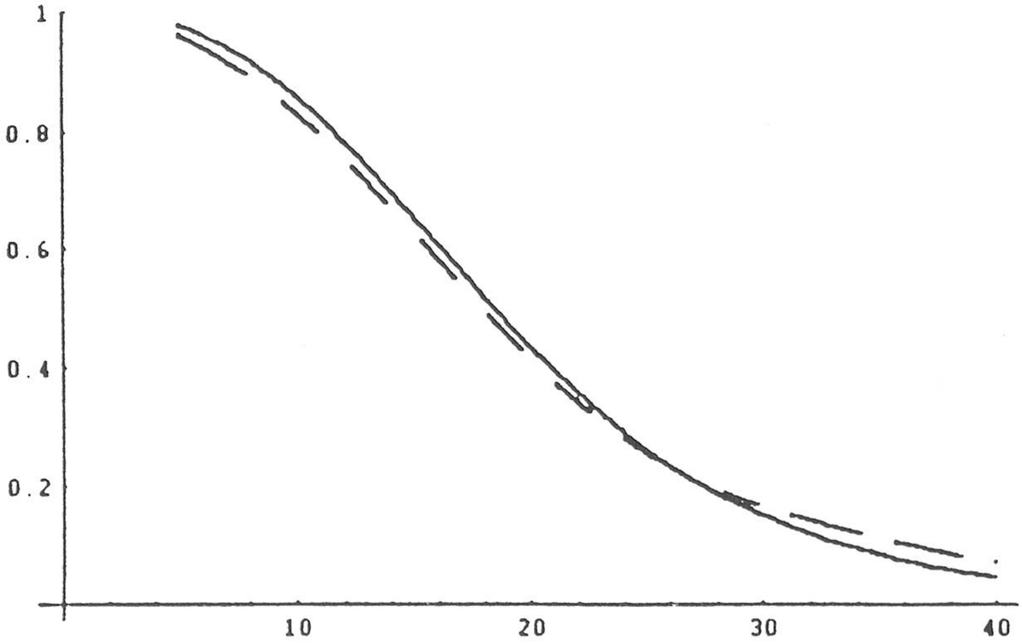
Edgeworth approximations of order 3, 4, 5, 6 and FFT approximation

Figure 8 Lognormal distribution ($a = 1$, $b = 1/5$)



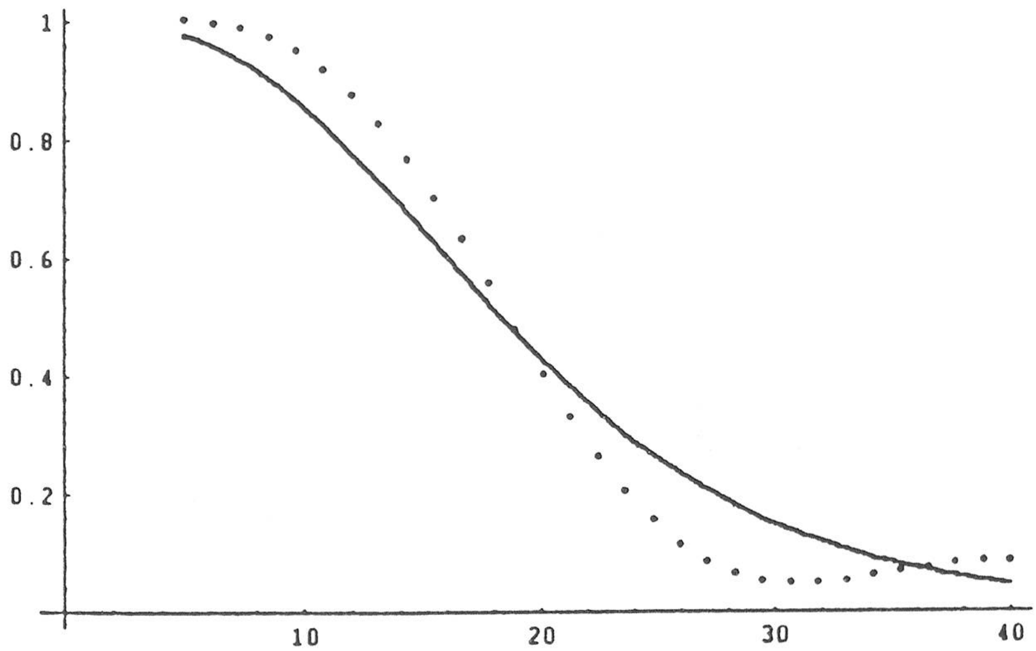
______________	FFT approximation
. . . .	Edgeworth approximation of 6th order
. . . .	Edgeworth approximation of 5th order
. . . .	Edgeworth approximation of 4th order
_ _ _ _ _ _ _ _ _	Edgeworth approximation of 3th order

Figure 9 Lognormal distribution ($a = 1/5$, $b = 1$)



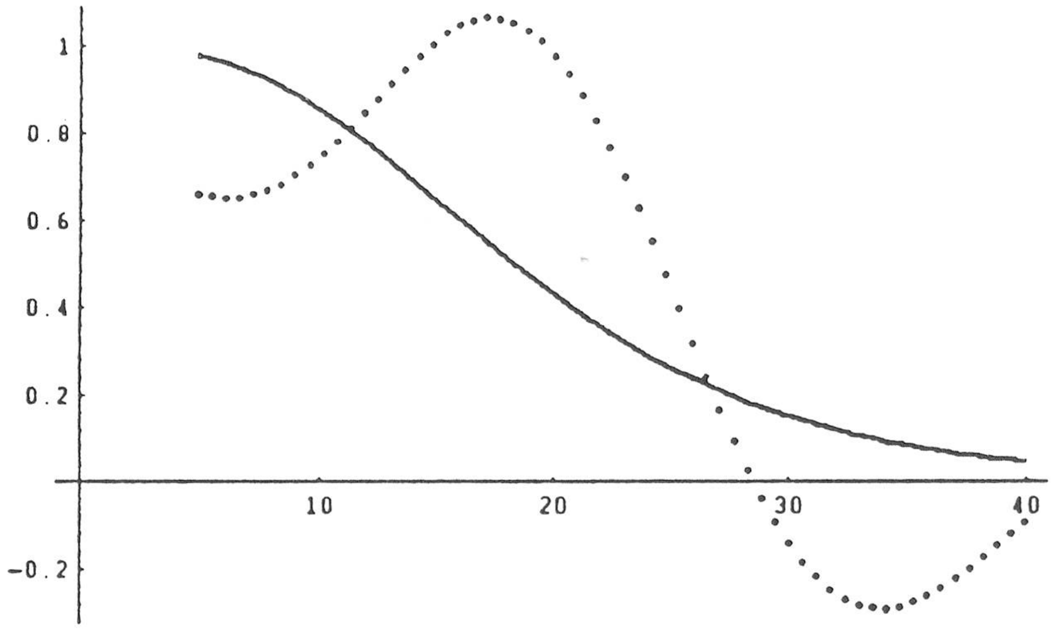
Edgeworth approximation of order 3

Figure 10 Lognormal distribution ($a = 1/5$, $b = 1$)



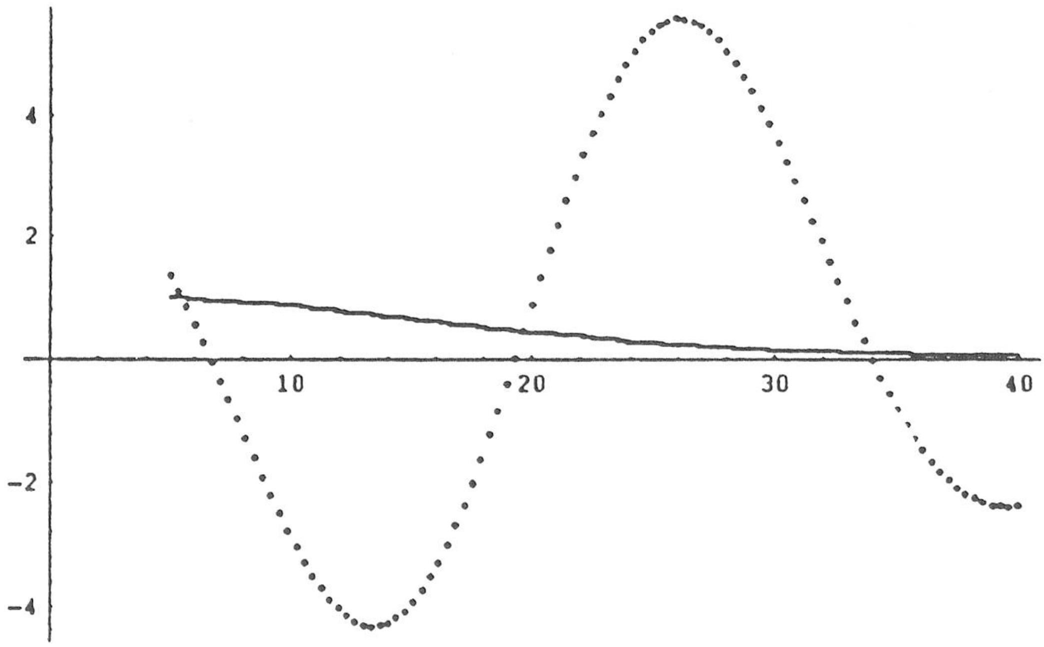
Edgeworth approximation of order 4

Figure 11 Lognormal distribution ($a = 1/5$, $b = 1$)



Edgeworth approximation of order 5

Figure 12 Lognormal distribution ($a = 1/5$, $b = 1$)



Edgeworth approximation of order 6

Figure 13 Lognormal distribution ($a = 1/5$, $b = 1$)

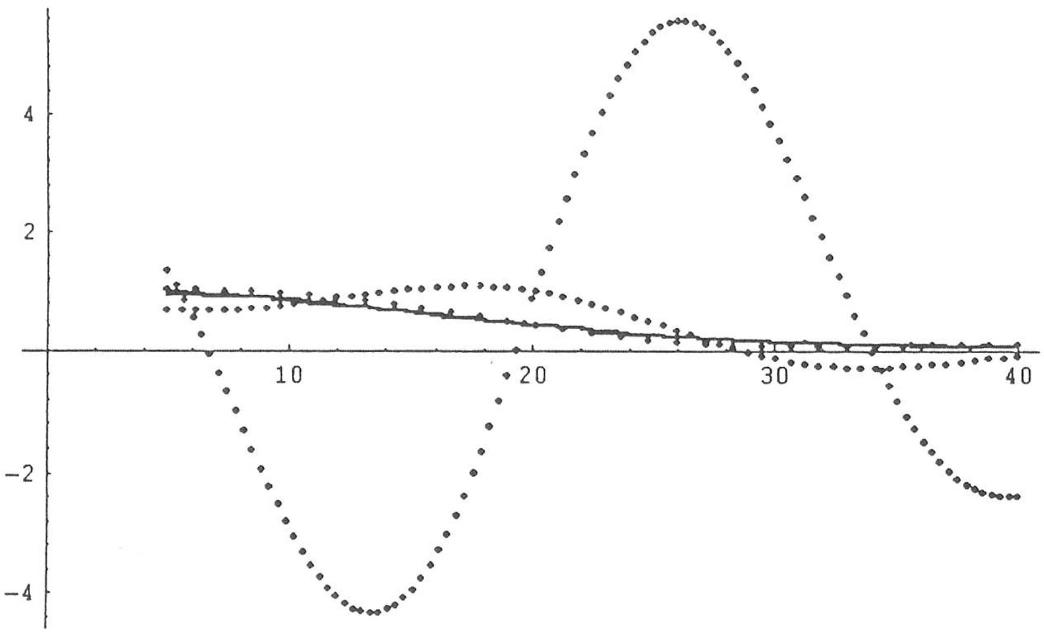
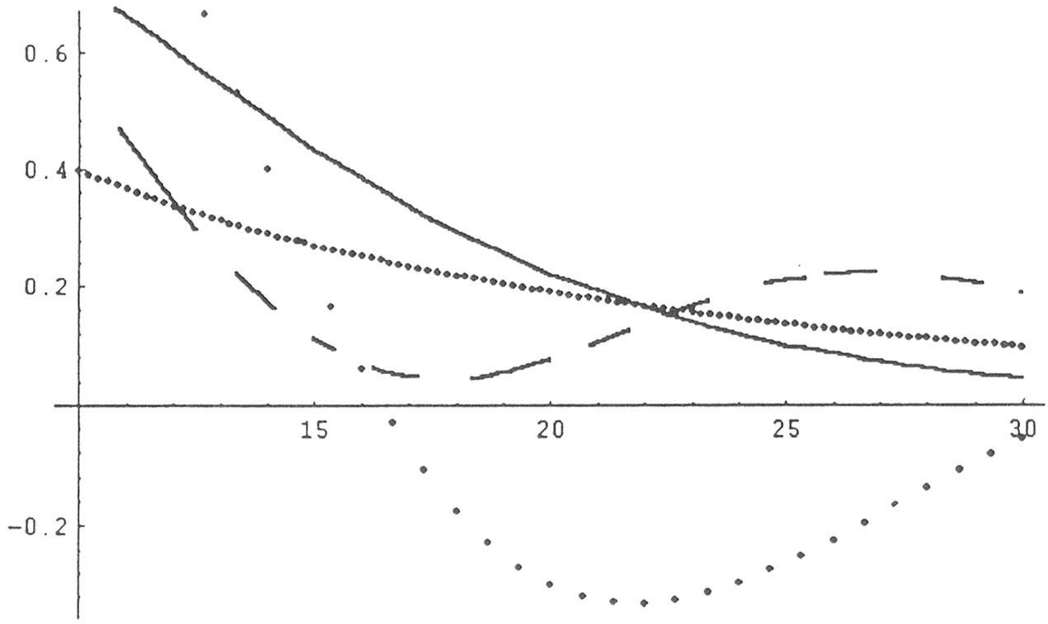


Figure 14 Pareto distribution ($a = 6$, $b = 5$)



————————	FFT approximation
. . . .	Normal power approximation of 2nd order
- - - - - - - - -	Gram Charlier approximation of 3rd order
. . . .	Bowers approximation of 3rd order

Figure 15 Standard exponential distribution

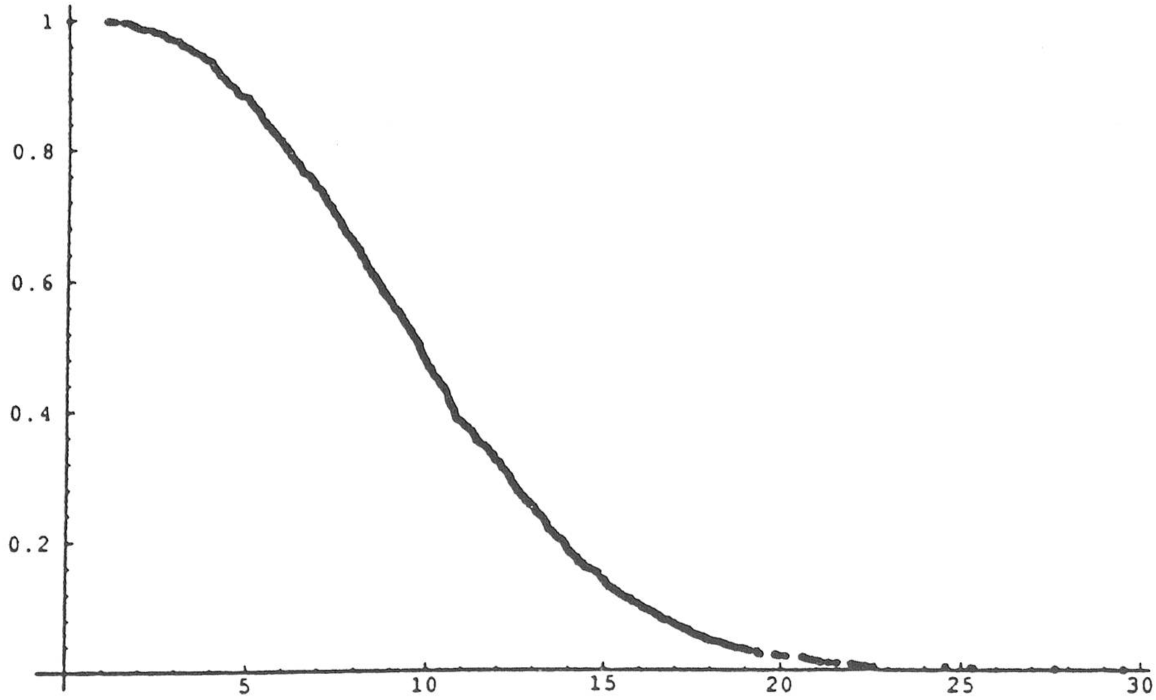


Figure 16 Standard exponential distribution

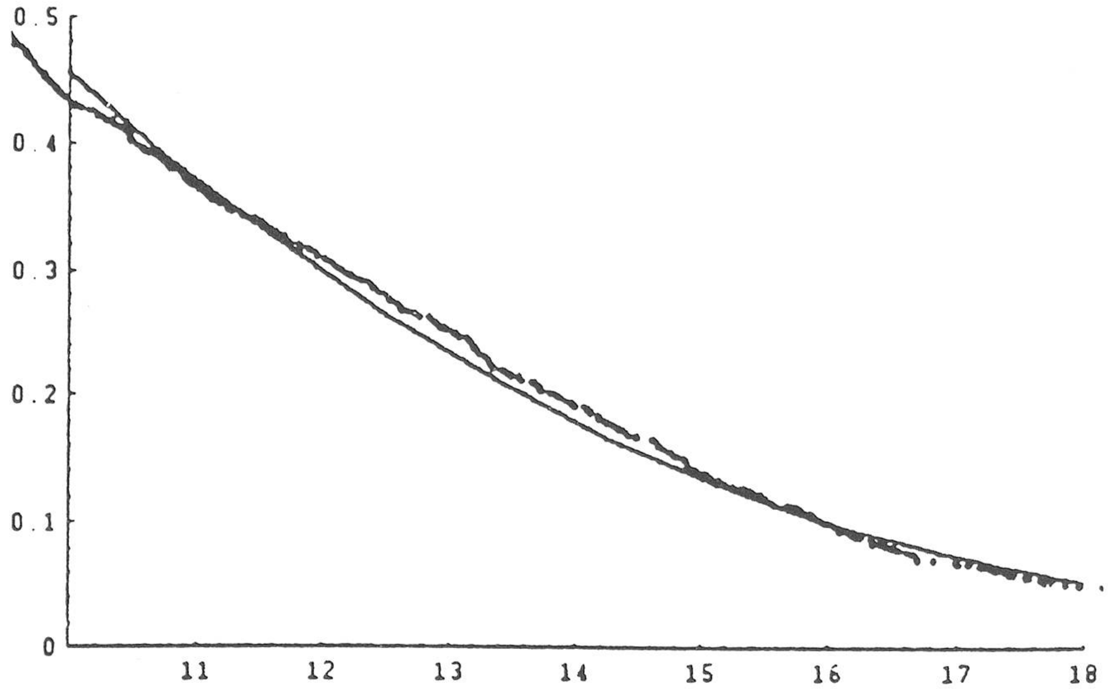


Figure 17 Lognormal distribution $(a = 1/5,\ b = 1)$

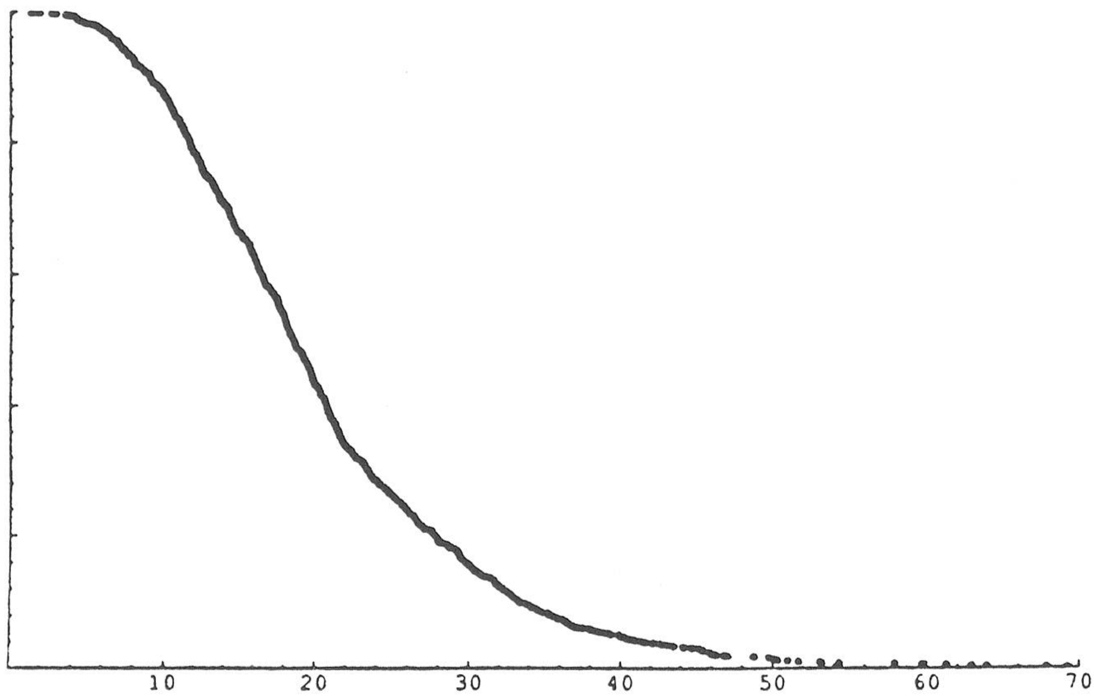
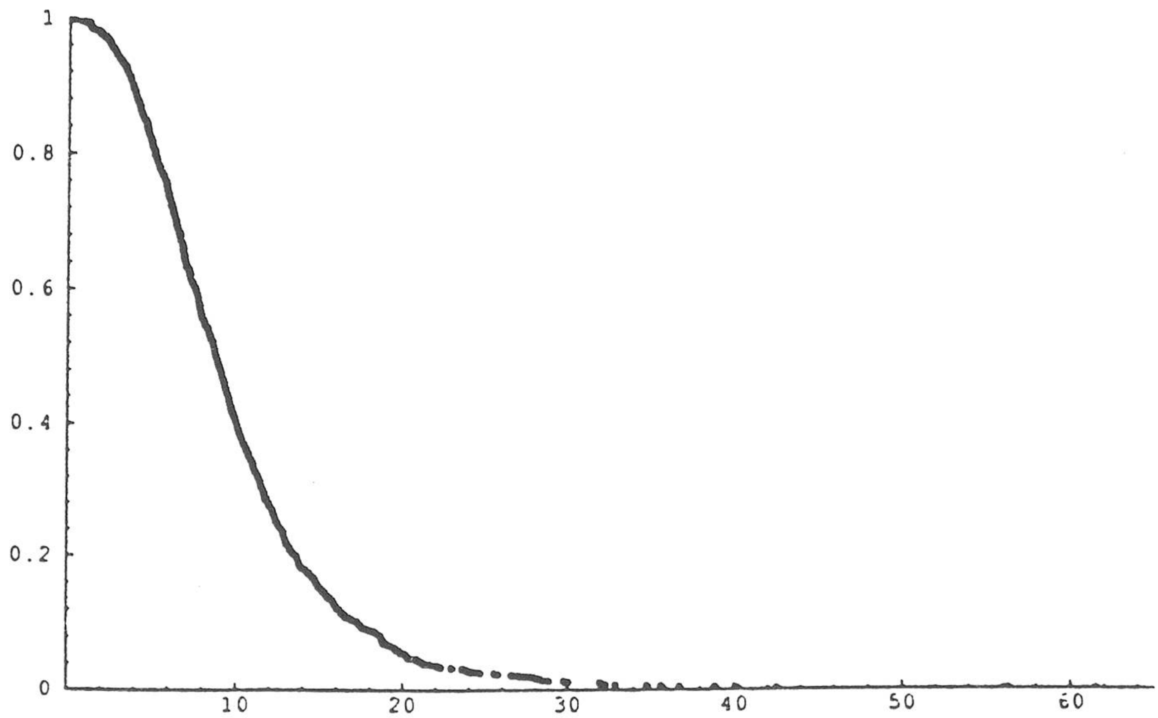


Figure 18 Pareto distribution $(a = 2,\ b = 3)$



References

Abramovitz, M., Stegun, I.A.: Handbook of mathematical functions, Dover books on advanced mathematics, (Ninth printing 1970).

Beard, R.E., Pentikäinen, T., Pesonen, R.: Risk Theory, 3rd Edition, Chapman and Hall, London (1984).

Bühlmann, H.: Numerical Evaluation of the compound Poisson Distribution: Recursion or Fast Fourier Transform? Scand. Actuarial J. (1984), pp. 116–126.

Daniels, H.E.: Saddlepoint approximations in statistics, Ann. Math. Statist. **25** (1954), pp. 631–650.

Embrechts, P., Jensen, J.L., Maejima, M., Teugels, J.L.: Approximations for compound Poisson and Pólya Processes, Adv. Appl. Prob. **17** (1985), pp. 623–637.

Embrechts, P., Grübel, R., Pitts, S.M.: Some applications of the fast Fourier transform algorithm in insurance mathematics, Statistica Neerlandica, to appear (1992).

Feilmeier, M., Bertram, J.: Anwendung numerischer Methoden in der Risikotheorie, Verlag Versicherungswirtschaft e.V., Karlsruhe (1987).

Fields, C., Ronchetti, E.: A small sample asymptotics, Lecture Notes - Monograph Series, Institute of Math. Statistics **13** (1990).

Gerber, H.U.: An Introduction to Mathematical Risk Theory, Huebner Foundation Monographs **8**, distributed by Richard D. Irwin Inc., Homewood Illinois (1979).

Heilmann, W.R.: Fundamentals of Risk Theory, Verlag Versicherungswirtschaft e.V., Karlsruhe (1988).

Hipp, C., Michel, R.: Risikotheorie: stochastische Modelle und statistische Methoden, Verlag Versicherungswirtschaft e.V., Karlsruhe (1990).

Jensen, J.L.: Uniform saddlepoint approximations, Adv. Appl. Prob. **20** (1988), pp. 622–634.

Kauppi, L., Ojantakanen, P.: Approximations of the generalized Poisson function, ASTIN Bulletin **5** (1969), pp. 213–266.

Keller, B., Klüppelberg, C.: Statistical estimation of large claim distributions, Mitteilungen der SVVM (1991), pp. 203–216.

Kupper, J.: Methoden zur Berechnung der Verteilungsfunktion des Totalschadens, Mitteilungen der SVVM (1971), pp. 279–315.

Morgan, B.J.T.: Elements of Simulation, Chapman and Hall, London (1984).

Petrov, V.V.: On the probabilities of large deviations for the sums of independent random variables, Th. Prob. Appl. **10** (1965), pp. 287–298.

Ripley, B.D.: Stochastic Simulation, Wiley, New York (1987).

Schwarz, H.R.: Elementare Darstellung der schnellen Fouriertransformation, Computing **18** (1977), pp. 107–116.

Widder, D.V.: The Laplace Transform, Princeton University Press, Princeton, NJ (1941).

Summary

The present paper is an outcome of a seminar in insurance mathematics at ETH Zürich in SS 92. The aim was to introduce the students to the traditional algorithms using modern computer tools. Since graphical representations are more intuitive than rows of numbers the students were encouraged to use programs offering graphical aids as e.g. MATHEMATICA. The resulting graphics were most interesting and we hope that this opinion is shared by our readers.

Zusammenfassung

Die vorliegende Arbeit ist das Ergebnis eines Seminars für Versicherungsmathematik an der ETH Zürich im Sommersemester 1992. Ziel des Seminars war es, die Studenten mit den traditionellen Algorithmen bekanntzumachen, wobei zeitgemässe Computermöglichkeiten einbezogen werden sollten. Da graphische Darstellungen intuitiver sind als Zahlenreihen, wurden die Studenten ermutigt, Programmpakete wie MATHEMATICA zu verwenden, die gute graphische Möglichkeiten bieten. Wir fanden die präsentierten Graphiken höchst interessant und hoffen sehr, dass unsere Leser diese Meinung teilen.

Résumé

Le présent article est le produit d'un séminaire de mathématiques d'assurance qui a eu lieu à l'EPF de Zürich durant le semestre d'été 92. Le but était de présenter aux étudiants les algorithmes classiques en utilisant les outils modernes de l'informatique. Les représentations graphiques donnent une meilleure intuition que les tableaux de valeurs numériques et les étudiants ont été encouragés à utiliser des logiciels permettant des représentations graphiques tels que MATHEMATICA. Les résultats graphiques sont particulièrement intéressants et nous espérons que cette opinion sera partagée par nos lecteurs.