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D. Kurzmitteilungen

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An Improved Elementary Upper Bound for the Variance of a Stop-Loss Risk

We present a simple upper bound for the variance of a stop-loss risk. Besides the priority, the mean and the variance of the risk, it depends only on the expected value of the stop-loss risk. Knowing only the net premium of a stop-loss contract, it is thus possible to calculate an estimate of the security loading if one applies for example the standard deviation principle. In case the priority is not too small, our upper bound is an improvement of the upper bound by *Kremer* (1990).

Let X denote a random variable with finite mean μ and variance σ^2. The following notations will be used:

$SL(d) = E[(X-d)_+]$	: the expected value of the stop-loss risk to the priority d
$SL^C(d) = E[(d-X)_+] = d - \mu + SL(d)$	: The "complement" of the net stop-loss premium
$\mathrm{Var}(d) = \mathrm{Var}[(X-d)_+]$	: the variance of the stop-loss risk

Theorem. The variance of the stop-loss random variable $(X-d)_+$ satisfies the inequality:

$$\mathrm{Var}(d) \le \sigma^2 - 2SL(d)\,SL^C(d). \tag{1}$$

Proof. The identity $X = (X-d)_+ - (d-X)_+ + d$ implies the relationship

$$\begin{aligned} \mathrm{Cov}[X,(X-d)_+] &= \mathrm{Var}[(X-d)_+] - \mathrm{Cov}[(X-d)_+,(d-X)_+] \\ &= \mathrm{Var}[(X-d)_+] + E[(X-d)_+]\cdot E[(d-X)_+]. \end{aligned} \tag{2}$$

Therefore one has

$$\begin{aligned} 0 &\le \mathrm{Var}[X-(X-d)_+] \\ &= \mathrm{Var}[X] + \mathrm{Var}[(X-d)_+] - 2\,\mathrm{Cov}[X,(X-d)_+] \\ &= \mathrm{Var}[X] - \mathrm{Var}[(X-d)_+] - 2E[(X-d)_+]\cdot E[(d-X)_+], \end{aligned}$$

from which the result follows.

Quite recently Kremer (1990) has obtained, applying the Cauchy-Schwarz inequality, the less simple upper bound

$$\mathrm{Var}(d) \leq \sigma^2 + (d-\mu)^2 - SL(d)^2 - \frac{1}{F(d)} \cdot SL^C(d), \tag{3}$$

which depends also on the probability $F(d) = Pr(X \leq d)$. Provided d is not too small, we show that actually the inequality (1) is an improvement to this upper bound. Let us write $B_1(d)$, $B_2(d)$ for the right-hand sides of (1) and (3) respectively.

Corollary. Assume that the distribution function $F(x)$ is integrable and that $\lim_{x\to-\infty} xF(x) = 0$. If d_0 is the solution of the fixed point equation

$$F(d_0) \cdot \int_{-\infty}^{d_0} F(x)\,dx = 1, \tag{4}$$

then one has

$$B_1(d) \leq B_2(d) \quad \text{if and only if} \quad d \geq d_0. \tag{5}$$

Proof. A calculation shows that

$$B_2(d) - B_1(d) = SL^C(d) \cdot \left(SL^C(d) - \frac{1}{F(d)} \right).$$

It follows that $B_2(d) \geq B_1(d)$ if and only if $F(d)\,SL^C(d) \geq 1$. Through partial integration one gets

$$\begin{aligned} SL^C(d) &= \int_{-\infty}^{d} (d-x)f(x)\,dx \\ &= dF(d) - \int_{-\infty}^{d} xf(x)\,dx \\ &= dF(d) - xF(x)\Big|_{-\infty}^{d} + \int_{-\infty}^{d} F(x)\,dx \\ &= \int_{-\infty}^{d} F(x)\,dx\,, \end{aligned}$$

and the result is shown.

In practice the decision rule (5) is often not very useful since the distribution function $F(x)$ is in most cases not known. An approximate decision rule, depending only on the probability $F(d)$, is obtained as follows. The inequality

$$\begin{aligned} F(d)\, SL^C(d) &= F(d) \cdot \int_{-\infty}^{d} (d-x)\, dF(x) \\ &\geq F(d) \cdot (dF(d) - \mu) \end{aligned}$$

shows that if

$$d \geq \frac{1+\mu F(d)}{F(d)^2}\,, \tag{6}$$

one has necessarily $B_1(d) \leq B_2(d)$. In this case the upper bound (1) improves on the bound (3) even in the sense of Kremer (1990).

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Reference

Kremer, E. (1990): An elementary upper bound for the loading of a stop-loss cover. Scandinavian Actuarial Journal.