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Three Actuarial Applications of Decision Trees

1 Introduction

The central focus of Operations Research (OR) is the use of quantitative methods in decision-making. One of the disciplines of OR, Decision Analysis, provides a rational methodology for decision-making under uncertainty. Decision trees have often proved to be especially convenient for decision problems such that choices must be made at different times, over an extended duration. A tree representation decomposes complex decision problems into their component, and more manageable, parts. It attempts to describe and quantify the relative advantages of alternative policies. It enables managers to explore the worth of acquiring experimental data, however imperfect, to reduce uncertainties. In addition, the pictorial presentation of all decisions and payoffs makes decision trees very powerful management communications tools.
The methodology of decision tree analysis is well known [see for instance *Brown et al.* (1974), *Keeney* (1982), or *Raiffa* (1968)]. It involves structuring a decision problem in terms of a decision flow diagram (the decision tree), assessing utilities or preferences for consequences, and expressing uncertainties about the outcome of events as probabilities. The tree is first constructed, with a basic guideline that the flow of events should be chronological from the base of the tree to its extremities. Any choice of acts is shown as a fork, with a separate branch for each act. Random events are also represented by branches in separate forks. Traditionally, squares are used to represent decision nodes, and circles for chance nodes. Conditional probabilities are then assigned to each branch, using Bayes Theorem to revise prior beliefs into posterior probabilities, whenever experimental results are available. Payoffs, that can be profits, costs, assets, utilities, or any numerical measure that enables a ranking of outcomes, are computed for each path of the tree. The search for the optimal decision is a backward induction process, that involves the computation of an expected value at each chance node, and the selection of the best alternative at each decision node (branch pruning), until the optimal policy is found. A sensitivity analysis is then usually performed, varying the value of critical probabilities or payoffs across their range.
While Decision Analysis is widely used in many industries, very few actuarial applications have been published. Surveys of Decision Analysis, such as *Corner*

and Kirkwood (1991), mention very few, if any, insurance articles. Surveys of applications of OR in insurance [such as *Haelhing von Lanzenauer/Wright* (1991), *Jewell* (1974), and *Shapiro* (1986)] in fact describe very few real-life applications. Actuaries seem to be disenchanted by OR. The Institute of Actuaries does not include it in its course of study. The Society of Actuaries recently downgraded its OR exam, making it an elective test worth only 15 credits (450 credits are required for Fellowship). The Casualty Actuarial Society simply eliminated OR from its exam curriculum (while keeping numerical analysis!), to make room for finance.

The present paper is an attempt to reverse this trend. It presents three actuarial applications of Decision Analysis. Each has already been published, but is further developed by the present author. Section 2 is a cost-benefit analysis of laws mandating the installation of smoke detectors in residential households. The decision tree has only one decision node (to require detectors or not) versus 19 chance nodes. It shows how a fairly complex decision problem can be broken down into basic events, so that published information can be used to estimate probabilities. Two different payoff functions, property damage and lives saved, lead to the conclusion that mandating the use of smoke detectors in all homes is one of the most efficient public policies.

Section 3 evaluates three different mortgage loans, a fixed-rate and two adjustable-rate, to help decision makers select the loan that best fits their attitude towards risk. Since the number of different interest rate scenarios during the time horizon of four years is close to five million, an extensive use of simulation is required. A mean-variance approach, and an expected exponential utility analysis, are the payoff functions used to select a mortgage.

Section 4 models the decision process of a life insurance underwriter, who has to decide whether to request a medical examination of an applicant. A routine and an extended medical test, both imperfect, are available. They are used to compute posterior probabilities that the applicant is a substandard risk, and to devise the optimal medical strategy as a function of the size of the policy.

2 Benefits of Smoke Detectors Laws

In 1989, The United States experienced $688,000$ structural fires, including $513,500$ fires in residential properties. They lead to $4,335$ civilian deaths (down from $6,015$ in 1978) and over 7.5 billions of dollars of property damage. The average property loss per structural fire amounted to $\$10,927$ (*Karter,* 1990).

In an attempt to reduce losses, 39 states have enacted some type of law mandating the use of smoke detectors (SD) (*LeCoque/Harris,* 1990). An estimated 82 % of U.S. households have now installed SD (*Hall,* 1989). It is a well-documented fact that fires in houses with SD have lower death rates, but the precise impact of SD laws still has to be assessed. Moreover, lower death rates may reflect the characteristics of those who purchase SD rather than the effects of SD themselves. The following model, first developed by *Jensen/Tome/Darby* (1989), provides a quantitative evaluation of the effects of SD laws.

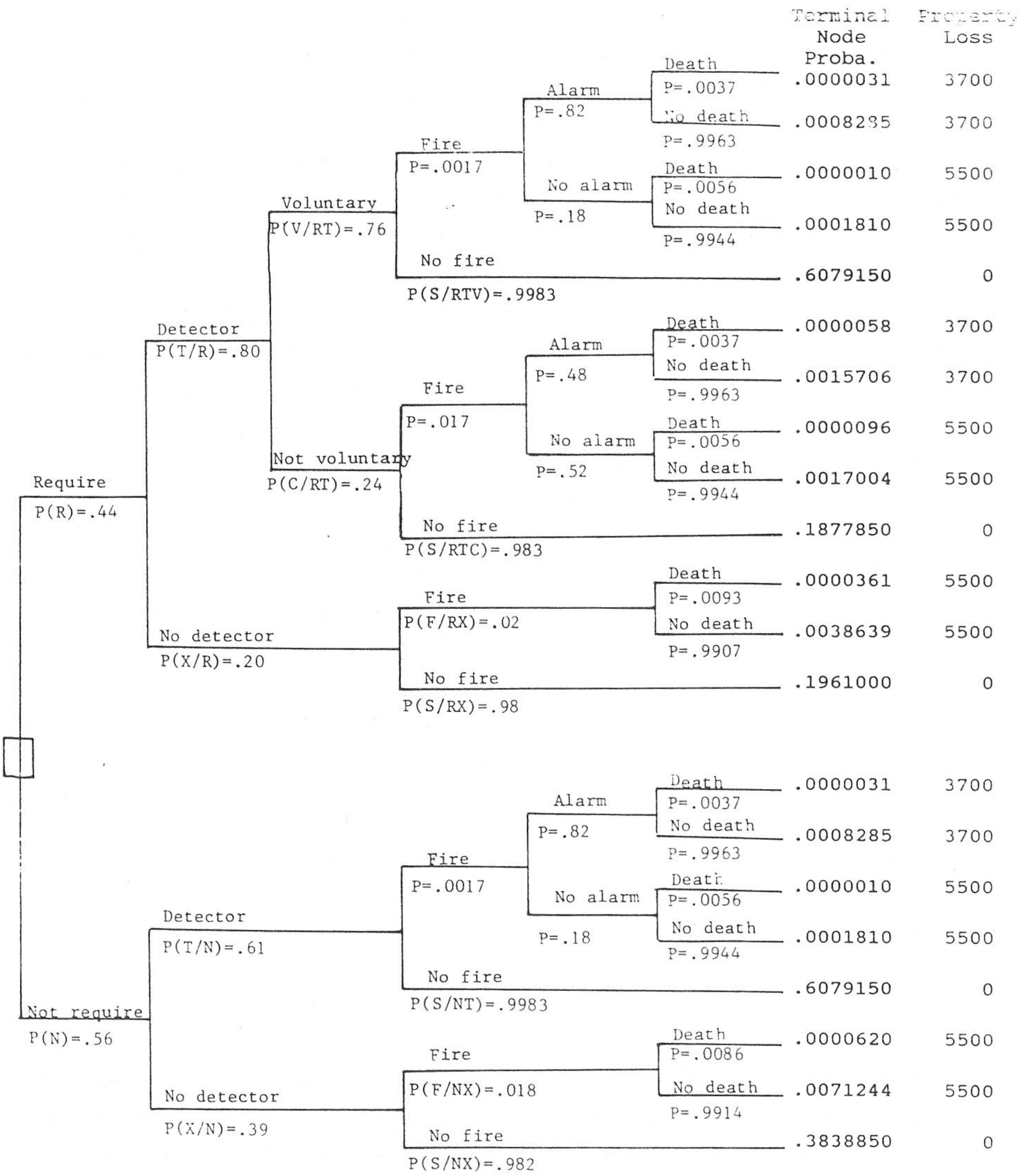
The decision tree is shown in figure 1. The first node ("Require" or "Not require" by law the installation of SD) can be considered as either a decision node (to decide whether a law should be enacted) or a probability node (to compute nationwide averages). All other nodes are probability nodes. The meaning of all notations is explained in the following table.

Letter	Branch	Meaning
R	Require	Detector required by law
N	Not require	Detector not required by law
T	Detector	Detector in residence
X	No detector	No detector in residence
V	Voluntary	Detector owned voluntarily
C	Not voluntary	Detector ownership coerced by law
F	Fire	Fire occurs in residence
S	No Fire	No fire occurs
A	Alarm	Detector sounds alarm
Q	No alarm	Detector fails
D	Death	Death occurs in fire
L	No death	No death occurs

The decision tree identifies three groups: people who (a) adopt SD voluntarily; (b) are coerced by a law into adopting SD; and (c) refuse to buy SD even when required by law. The probability of fire varies significantly among those groups.

"Voluntary" is used in the "Require" branch to identify residences situated in states that require SD, whose owners would have bought one even if no law had been enacted. So the homes in the "Require-Detector-Not voluntary" branch are the additional residences equipped with SD due to a law.

Publications of the National Fire Protection Association [*Gankarski/Timoney* (1984), *Hall* (1985)] allow the estimation (sometimes rather crude) of most



Terminal Node Proba.
Property Loss
Require
P(R)=.44
Detector
P(T/R)=.80
Voluntary
P(V/RT)=.76
Fire
P=.0017
Alarm
P=.82
Death
P=.0037
.0000031
3700
No death
P=.9963
.0008285
3700
No alarm
P=.18
Death
P=.0056
.0000010
5500
No death
P=.9944
.0001810
5500
No fire
P(S/RTV)=.9983
.6079150
0
Not voluntary
P(C/RT)=.24
Fire
P=.017
Alarm
P=.48
Death
P=.0037
.0000058
3700
No death
P=.9963
.0015706
3700
No alarm
P=.52
Death
P=.0056
.0000096
5500
No death
P=.9944
.0017004
5500
No fire
P(S/RTC)=.983
.1877850
0
No detector
P(X/R)=.20
Fire
P(F/RX)=.02
Death
P=.0093
.0000361
5500
No death
P=.9907
.0038639
5500
No fire
P(S/RX)=.98
.1961000
0
Not require
P(N)=.56
Detector
P(T/N)=.61
Fire
P=.0017
Alarm
P=.82
Death
P=.0037
.0000031
3700
No death
P=.9963
.0008285
3700
No alarm
P=.18
Death
P=.0056
.0000010
5500
No death
P=.9944
.0001810
5500
No fire
P(S/NT)=.9983
.6079150
0
No detector
P(X/N)=.39
Fire
P(F/NX)=.018
Death
P=.0086
.0000620
5500
No death
P=.9914
.0071244
5500
No fire
P(S/NX)=.982
.3838850
0

Figure 1

conditional probabilities of the tree. All probabilities refer to the 1983 situation, and consequently all losses will be expressed in 1983 dollars. It is seen, for instance, that 61 % of homeowners install a SD if no law requires it. This percentage increases to 80 % if SD are mandated.

In case of a fire, SD may or may not sound an alarm. Malfunctions are primarily due to intentional disablement to prevent false alarms, and non-replacement of batteries by careless owners. Note that the probability of a malfunction is much higher if the installation of SD is not voluntary. Unfortunately, those who do not purchase SD, and those who only buy them when coerced by law, are precisely those who need them most. The probability that a fire erupts in their home is 10 times greater than the probability of a fire in the "Voluntary" groups. So the group least likely to install SD is also the group with the highest risk of fire. This is because detector possession varies with socioeconomic factors such as home ownership, education level, ethnicity, family stability, income, type of neighbourhood, age of dwelling,. . ., all variables correlated with fire rates. A recent multiple regression study (*Fahi/Norton,* 1989) showed that the three variables that are most correlated to fire rate are

(i) the percentage of people living under the poverty level,
(ii) the percentage of adults with less than 8 years of schooling, and,
(iii) the percentage of persons under 18 living with both parents.

Alone, the first of these variables explains over 50 % of the variance in fire rates. Table 1 clearly illustrates this point.

Table 1: Fire rates as a function of poverty level

% of persons living under poverty level	Median residential fire rate	Median residential fire death rate	Number of US cities*
>25.1	265	4.56	3
20.1 – 25	216	3.56	11
15.1 – 20	242	2.60	10
10.1 – 15	194	1.79	23
5.1 – 10	126	0.64	3

* Population > 250, 000 inhabitants

A. Estimation of the number of lives saved

Consider the initial mode of the tree as a decision mode. Terminal node probabilities are first computed by multiplying in chain all conditional probabilities of the path. Those probabilities add up to one in the "Require" and in the "Not require" branches of the tree. By simple addition, the probability of a death if SD are required is found to be 0.0000556. Without a law, this probability is 0.000066. Multiplying the difference by the estimated number of residences in the U.S. (80 million) leads to a figure of 832. This is an estimation of the annual number of lives saved if all states enact SD laws, compared to a situation with no SD laws. SD are extremely inexpensive. They cost as little as five dollars today, and have a life expectancy exceeding 10 years. Batteries, costing 69 cents, need to be replaced annually. Assume the annual cost of operating a SD is $1 (in 1983 dollars). From the decision tree, it is seen that 61 % of homeowners purchase SD without a law. SD laws induce 19 % of owners to purchase one. So, with 80 million residences, excess nationwide purchases of SD will cost annually $15, 200, 000. Assuming enforcement costs, such as residence inspections and education programs, of $0.50 per residence, the total annual cost of the program will be $55, 200, 000. This results in a cost of $66, 300 for each life saved. As shown in table 2, this figure compares extremely well with other public policies such as asbestos abatement measures or nuclear plants' regulations [*Chrostowski et al* (1991), *Wilson* (1975).

Table 2: Comparison of Program Cost Effectiveness

Regulation	Annual lives saved in the U.S.	Cost per life saved (1983 $)
Smoke detectors	832	66, 300
Steering column protection	1, 300	100, 000
Passive restraint belts	1, 850	300, 000
Children's sleepwear flammability	106	1, 300, 000
Asbestos abatement	396	7, 400, 000
Radioactivity level of nuclear plants	4	500, 000, 000
Vinyl chloride emissions from EDC-VCM plants	0.5	n.a.
Low arsenic copper smelters	< 0.1	n.a.

See *Graham* and *Vaupel* (1981) for a cost/benefit analysis of 57 life-saving programs].
Note that the figure of $66,300 per life saved does not take into account reductions in the number of injuries, and in property losses.

B. *Estimation of property damage reductions*

The average residential fire loss, when a SD sounds an alarm, is $3,700 (last column of figure 1). Without a SD, or with a malfunctioning SD, property losses average $5,500. Considering the initial node as a probability node, the average annual fire loss per residence (in 1983 dollars) is calculated to be $14.14 with SD, $103.84 without SD. Spending $1 a year to purchase and maintain SD results in a decrease of $89.4 in annual fire losses. The substantial homeowners' premium discount awarded to homes with SD is more than justified.
Obviously, a complete sensitivity analysis needs to be performed to evaluate the effects of parameter uncertainties. This sensitivity analysis shows a great fluctuation in the total number of lives saved. Still, even if many parameter values are uncertain, it seems a good idea for homeowners to install smoke detectors, and for states to require them.

3 Mortgage Selection

The following analysis extends research work by *Luna/Reid* (1986), and *Heian/Gale* (1988).
Consider a young couple moving in the Philadelphia area. They know they are going to be relocated four years from now. They buy an apartment. After the down payment, they still have to borrow $100,000. Three different lenders offer them the following mortgages.

Option 1. 30-year conventional fixed-rate loan (FIXED).
The nominal annual interest rate is $9\,{}^5\!/_8\,\%$. In addition, the borrowers have to prepay $2,000 as origination fee (2 "points"). Their monthly payment is calculated to be

$$100{,}000/a_{\overline{360}|\,9.625/12} = \$849.99$$

Option 2. 1-year adjustable-rate mortgage (ARM-1)

The initial nominal annual interest rate of this 30-year loan is $7.5\,\%$. It results in a monthly payment of \$699.21. In addition, the borrowers must pre-pay 3 points. The interest rate is then modified annually, depending on the fluctuations of the yield of 1-year U.S. Treasury Notes with constant maturities (1-year T-Bills). A margin of $2.75\,\%$ is added to the T-Bills index (TB). In addition, $(2\,\%, 6\,\%)$ caps limit annual fluctuations of the loan's interest rate (upward or downward) to $2\,\%$ annually and $6\,\%$ over the lifetime of the loan. Given those restrictions and the outstanding principal (OP) of the loan, the monthly payment is then recalculated. For instance, if the interest rate after one year is increased to $9.5\,\%$, the next monthly payment will be

$$(\mathrm{OP})_{t=12}/a_{\overline{348}|\,9.5/12}$$

if the time unit is one month. At time $t = 0$, the T-Bills index is $7.67\,\%$.

Option 3. 3-month adjustable-rate mortgage (PRIME)

The initial nominal annual interest rate of this 20-year loan is $6.95\,\%$. Consequently, the initial monthly payment is \$772.30. The borrowers must pre-pay 1 point. The interest rate is adjusted quarterly, following the Prime Rate (PR), with a margin of $1.5\,\%$. Each increase or decrease cannot however exceed $0.75\,\%$. Moreover, the interest rate will never be below $4.95\,\%$ or above $14.95\,\%$. At time $t = 0$, the Prime Rate is $8.5\,\%$.
The borrowers, confronted with several alternative mortgage contracts, use Decision Analysis to evaluate them. Their decision will depend on

(i) the features of the various loans and the evolution of interest rates;
(ii) the timing of payments and the time value of money, reflected in the borrowers' discount rate; and
(iii) the borrowers' attitude towards risk.

Modeling the subjective risk preferences of the borrowers is a step that cannot be avoided. Adjustable-rate mortgages (ARMs) transfer part of the risk inherent in the variation of market interest rates from the lender to the borrower. To compensate for this transfer, lenders need to offer attractive initial "teaser" rates, that reduce the first years' payments. Purchasers of ARMs are compensated for accepting a greater uncertainty by a reduction in the expected value of the mortgage obligation. They are trading expected costs for variability. Highly risk-averse consumers will be attracted by fixed-rate mortgages, for which all payments are known in

advance. Less risk-averse individuals might be willing to accept a certain degree of uncertainty of future payments, if the reduction of expected payments is large enough.
This decision problem can be structured in terms of a decision-flow diagram, to quantify the uncertain future behaviour of interest rates.
An analysis of the evolution, quarter by quarter, of the Prime Rate, between 1980 and 1991, resulted in a maximum of 21.5 %, a minimum of 7.5 %, a mean of 11.54 %, and a standard deviation of 3.51 %. What is more important is the analysis of the quarterly variations. 32 % of the time, the Prime Rate did not change, or changed only by 0.25 %, from one quarter to the next. 38 % of the time, it moved down, by an average amount of 1.71 %. 30 % of time, it moved up, by an average amount of 1.73 %. During the period under consideration, the trend of the evolution of the Prime Rate was slightly downward (regression line: Prime $= 15.98 - 0.18\,t$, $r = 0.714$). Since we do not want to introduce any systematic trend in the analysis, the quarterly evolution of the Prime Rate will be modelled in the following way. (The quarterly rate of change has been rounded to the nearest 0.25 %, to reflect traditional Prime Rate change increments.)

Any quarter	Following quarter	
Prime Rate	New Rate	Probability
	PR+1.75 %	1/3
PR	PR	1/3
	PR-1.75 %	1/3

A similar analysis of the evolution, year by year, of the 1-year T-Bills rates, resulted in a maximum of 14.32 %, a minimum of 5.78 %, an average of 9.27 %, and a standard deviation of 2.72 %. The analysis of the annual variations, excluding an obvious outlier, showed five increases (by an average amount of 1.36 %), and five decreases (by an average amount of 1.32 %). The evolution of the T-Bills rate will be modelled by the following closed binomial lattice.

Any quarter	Following quarter	
T-Bills rate	New Rate	Probability
	TB+1.34 %	1/2
TB		
	TB-1.34 %	1/2

The Prime Rate is more volatile than the T-Bills index. Indeed, the standard deviation of the *quarterly* variations of the Prime Rate is greater than the standard deviation of the *annual* variations of the T-Bills rate. Moreover, the caps on the two ARM options set more restrictions of the variability of the ARM-1 rate. Consequently, the PRIME loan is much more risky than the ARM-1 loan. Figure 2 presents the beginning of the decision tree. Small circles represent monthly payments. Calculations are illustrated for PRIME payment (B).

PRIME interest rate at $t = 0$	: $6.95\,\%$
Monthly payment	: $\$772.30$
Outstanding principal after 3 months	: $(\mathrm{OP})_{t=3} = \$99,417.24$
Prime Rate at $t = 0$	: $8.5\,\%$
Prime Rate at $t = 3$	: $8.5\,\% + 1.75\,\% = 10.25\,\%$
PRIME rate, adding the margin	: $10.25\,\% + 1.5\,\% = 11.75\,\%$
PRIME rate, with the 0.75 % cap	: $6.95\,\% + 0.75\,\% = 7.70\,\%$
Monthly payment for second quarter	: $99,417.24/a_{\overline{237}\rceil 7.7/12} = \$817.45$
Outstanding principal after 6 months	: $\$98,875.21$
PRIME Rate at $t = 6$	: $10.25\,\% - 1.75\,\% = 8.5\,\%$
PRIME rate, adding the margin	: $8.5\,\% + 1.5\,\% = 10\,\%$
PRIME rate, with the 0.75 % cap	: $7.70\,\% + 0.75\,\% = 8.45\,\%$
MONTHLY payment for third quarter	: $98,875.21/a_{\overline{234}\rceil 8.45/12} = \$863.40$

Given the selected horizon of four years, the ARM-1 tree section has 8 branches for the fourth year, and calculations are straightforward. The PRIME tree section has over 4.78 million branches for the 16th quarter. This makes the use of simulation techniques to evaluate future interest rates and monthly payments a necessity.
Using simulation for PRIME, and exact calculations for FIXED and ARM-1, the present value (PV) at time $t = 0$ of each possible stream of payments was computed, under the following assumptions.

1. The annual interest rate for discounting is 15 %. This rate is assumed to reflect not only inflation, but also personal elements such as expected salary increases, other possible future income, and the borrowers' time preferences.
2. Points, as well as the interest portion of each monthly payment, are fully tax-deductible. The borrowers' marginal tax rate is 28 %.
3. Initial loan charges (title insurance, application, document preparation, closing, recording, and survey fees) amount to $1, 526, for all three options. They are not tax-deductible.
4. The mortgage selection decision is taken independently of other financial decisions such as investments and insurance purchases.
5.Not to borrow is not an option.

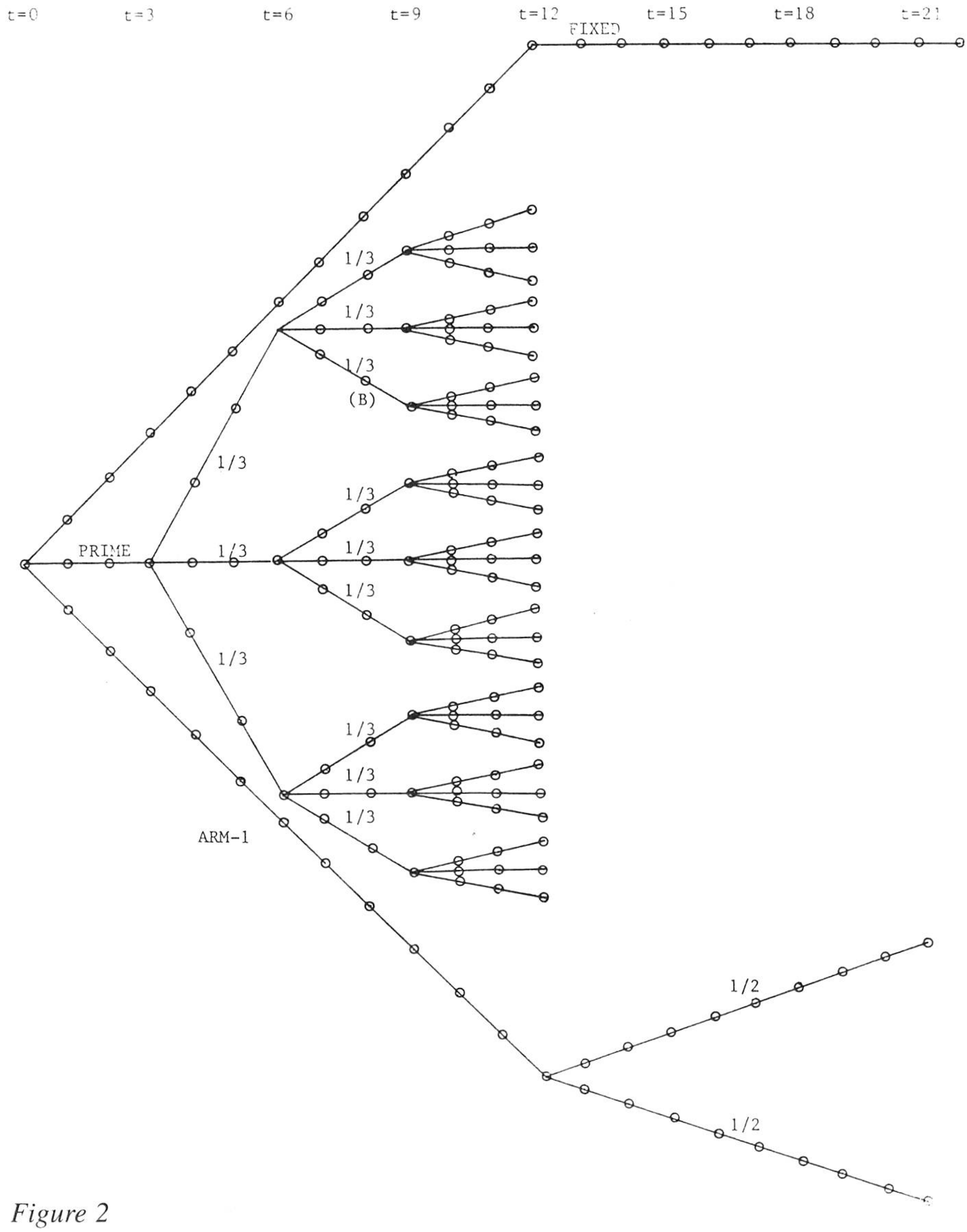


Figure 2

[In other words, we have attempted to model the decision process of a young couple, that has no other assets, and has to borrow $100,000. The model needs to be slightly revised to guide the selection of a wealthier decision-maker, who may consider the purchase of an apartment as a pure investment decision.]
6. Future variations of interest rates can be accurately estimated by analysing their past behaviour.
7. Interest rate changes form a random walk. The probability of interest rate moves does not change from period to period, and the outcome in each period is independent of the outcome in prior periods. Alternative formulations of the evolution of interest rates could include autoregressive or moving average models.

For the three loans, the present value of each possible alternative was computed. Included in the calculation are

1. the initial charges
2. the origination fee (points)
3. 48 monthly payments, and
4. the outstanding principal after 4 years.

(Since we assumed that the couple will be relocated after four years, the apartment will be sold, and the loan will have to be paid off.) In a formula, taking tax-deductibility into account,

$$\begin{aligned}\text{PV} =&1526 + (\text{number of points}) \cdot (1\,000) \cdot (1 - 0.28)\\ &+ \sum_{t=1}^{48} [(\text{principal repaid})_t + (\text{interest})_t (1 - 0.28)] \cdot v^t\\ &+ (\text{OP})_{48} \cdot v^{48},\end{aligned}$$

where $v = 0.988421$ is the monthly discount factor corresponding to an annual interest rate of 15 %.
Table 3 presents the expectation and the standard deviation of the present value of all possible payment streams for the three loans. It illustrates the tradeoff decision between expected payments and their variability.

Table 3: Comparison of loans

Loan	Expectation	Standard deviation
FIXED	81,502.72	0
ARM-1	80,607.49	1,575.81
PRIME	80,181.96	3,536.95

This table will help a consumer that only reasons in terms of means and standard deviations to reach a decision. A more sophisticated borrower may rely on utility theory. Assume he evaluates his situation by means of an exponential utility function, with constant risk aversion coefficient c.

$$u(x) = \frac{1}{c}(1 - e^{-cx}) \text{ ,}$$

A high value of c indicates that the consumer is highly risk averse, and will select the FIXED loan. The PRIME option will be chosen by borrowers with a low c. Denoting w the borrowers' initial wealth, indifference between FIXED and PRIME occurs when

$$\frac{1}{c}[1 - e^{-c(w-81,502.72)}] = \sum p_i \frac{1}{c}[1 - e^{-c(w-x_i)}]$$

where x_i is the present value of a PRIME payment stream of probability p_i, and the summation is over all payment streams. This equation reduces to

$$e^{81,502.72c} = \sum p_i e^{cx_i}$$

Solving for c yields $c = 22.8 \times 10^{-5}$. A consequence of constant risk aversion is that the solution does not depend on the wealth w.
Similarly, indifference between PRIME and ARM-1 occurs when $c = 9.4 \times 10^{-5}$. The indifference point between FIXED and ARM-1 is $c = 93.7 \times 10^{-5}$. The knowledge of his own value of c will then enable a consumer to select the mortgage that best suits his preferences from table 4.

Table 4: Decision table

Rang of c ($\times 10^{-5}$)	Preferred option	Second best option	Least preferred option
[0 – 9.4]	PRIME	ARM-1	FIXED
[9.4 – 22.8]	ARM-1	PRIME	FIXED
[22.8 – 93.7]	ARM-1	FIXED	PRIME
[93.7 –]	FIXED	ARM-1	PRIME

Modelling the selection of a mortgage by a decision tree also enables to analyse other classical loan features and problems, not incorporated in the preceding.

Negative amortization

In many ARMs, borrowers may select not to increase their monthly payments, even if the loan interest rate goes up. One clause of the PRIME loan specifies that monthly payments can be kept at the initial level ($772.30) for three years. Afterwards, monthly payments only need to be modified once a year, and payment increases may be limited to 7.5 % each. A likely consequence of level payments and increased loan rates is negative amortization: payments do not suffice to pay accrued interest. Unpaid interest is then added to the outstanding principal, which may even increase above its initial value of $100,000.

The right not to increase payments adds a decision node to the tree at times 3, 6, 9, ... A simple modification of the "payments" column in the simulated amortization schedule allows the study of this option. Deferring payments at the expense of an outstanding principal increase after four years modifies all present values. Whether the borrower should elect to defer payments mostly depends on the discount rate. Since, in the preceding analysis, the interest rate used for discounting (15 %) is greater than the maximum possible loan rate (14.95 %), borrowers should always defer payments as much as they can. Savings achieved in early years always outweigh the outstanding principal increase, which is discounted by a factor v^{48}. For the selected model parameters, the negative amortization option reduces the expected PRIME mortgage obligations from $80,181.96 to $79,738.69. Fixed payments during the first three years also reduce the total variability of mortgage obligations; the standard deviation decreases from 3,536.95 to 3,330.94.

Right to prepay

Most mortgages nowadays give borrowers the right to make payments of principal at any time before they are due, without any prepayment penalty. The decision to make prepayments again depends on the comparison of the mortgage interest rate and the rate used for discounting. If the latter is greater than all possible loan rates, as in our example, it is not in the borrowers' interest to make any prepayment.

Fixed interest rate option

Most ARMs allow borrowers to convert their mortgage into a conventional fixed rate loan on given anniversary dates of the mortgage. There is usually a penalty for this conversion, such as 1 % of the outstanding principal. Exercising this option amounts to refinancing the loan.

Refinance

A large decrease in interest rates should prompt informed customers to repay entirely the original mortgage and refinance. Refinancing should take place whenever the present value of the benefits of lower monthly payments exceeds the costs of refinancing (initial charges and points). A decision to refinance can be made at any time. In essence, it requires to repeat the decision analysis formulated here each month, with updated data.

3. Life Insurance Underwriting

The following analysis extends the results of a discussion paper by *Jones* (1970). Consider a life insurance underwriter, who has to set rules for requesting medical reports. The usual practice is to request more detailed examinations as the amount of insurance M (expressed in thousands of dollars) increases. Assume the company uses three different mortality tables, according to the various possible states of health of applicants.

θ_1 : Standard
θ_2 : Substandard
θ_3 : Sub-substandard

For each application, the underwriter has four possible decisions.

a_0 : Return the application
a_1 : Accept at standard rates
a_2 : Accept at substandard rates
a_3 : Accept at sub-substandard rates

Table 5 is the payoff matrix, that provides the expected profit to the company for all pairs (θ_i, a_j). The profit is highest when the applicant has been correctly rated. Negative payoffs occur, due to operating expenses, when a policy is not purchased by the applicant, due to an incorrect substandard classification.

Table 5: Payoff matrix

States	Actions a_0	a_1	a_2	a_3
θ_1	0	$120\,M$	-300	-300
θ_2	0	$80\,M$	$120\,M$	-300
θ_3	0	$20\,M$	$60\,M$	$120\,M$

In addition, the underwriter can request medical examinations. He has a choice between three decisions.

	Decision	Cost
e_0 :	no request	\$ 0
e_1 :	request a routine examination R	\$ 30
e_2 :	request a special examination S (blood test, EKG, ...)	\$300

The special exam is more accurate than the routine exam, as shown by table 6, the table of conditional probabilities $P_R(T/\theta)$ and $P_S(T/\theta)$. $P_{\cdot}(T/\theta)$ is the probability that the medical exam indicates rating classification T_i, when the applicant in fact belongs to θ_j. The probability of an incorrect classification is always lower for the special exam.

Table 6: Conditional probabilities

Indicated class	True class $P_R(T/\theta)$ θ_1	θ_2	θ_3	$P_S(T/\theta)$ θ_1	θ_2	θ_3
T_1	.93	.035	.02	.97	.015	.01
T_2	.05	.93	.05	.02	.97	.02
T_3	.02	.035	.93	.01	.015	.97

Finally, the distribution of applicants is as follows.

Rating class θ	$P(\theta)$
θ_1	.90
θ_2	.07
θ_3	.03

Figure 3 is the decision-tree. In the e_0-branch of the tree, the underwriter has to select an action a_i without the benefit of a medical exam. The true state of the applicant then determines the payoff. In the e_1- and the e_2-branches, the underwriter first requests a medical report. Given the indication provided by the report (T_1, T_2, or T_3), he selects a rate (a_1, a_2, or a_3).
First, Bayes Theorem is used to compute the conditional probabilities $P_R(\theta/T)$ and $P_S(\theta/T)$ that the applicant is in rating class θ_i, while the medical exam indicated class T_1 (see table 7). For instance

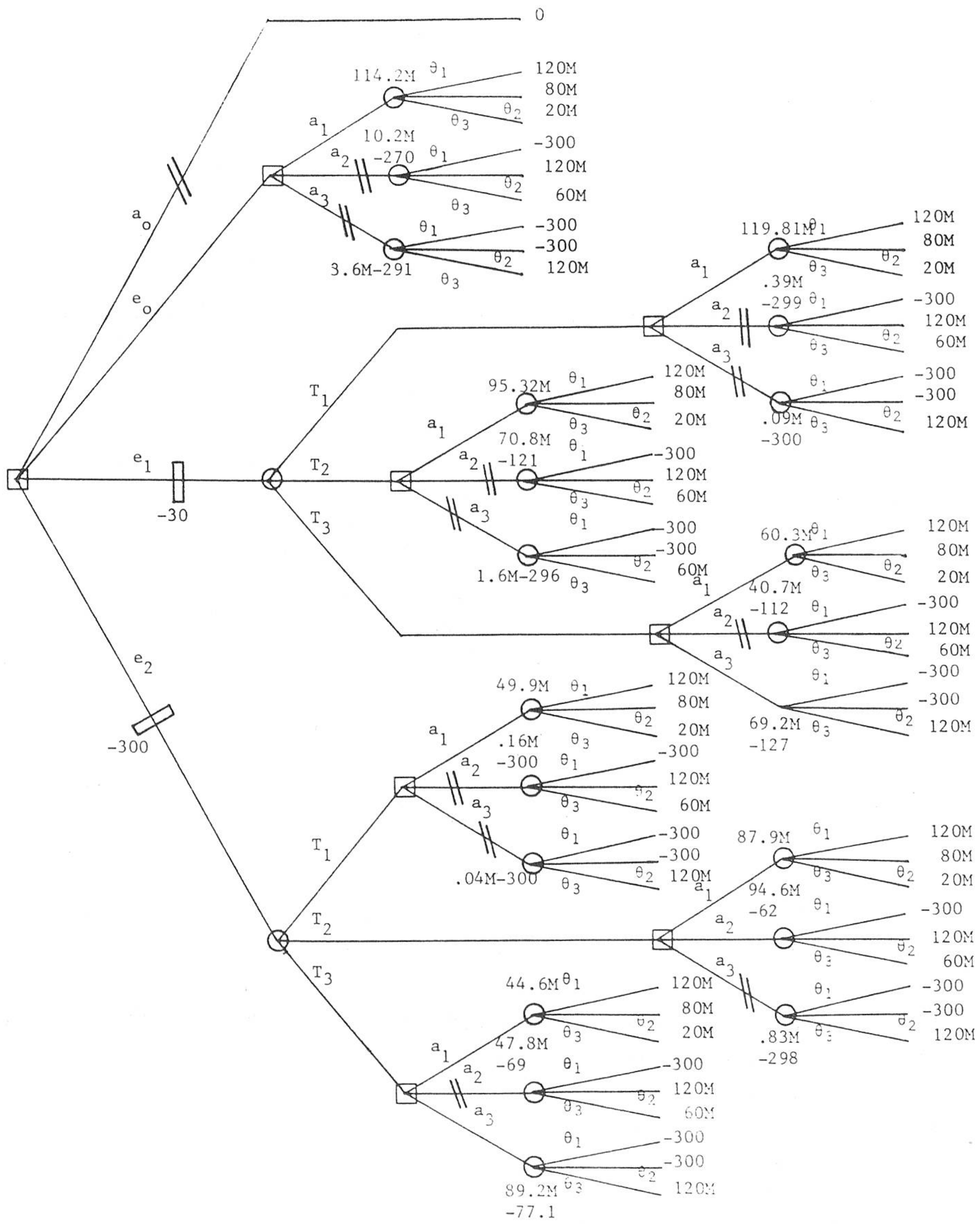
$$P_R(\theta_1/T_2) = \frac{P_R(T_2 \mid \theta_1)P(\theta_1)}{P_R(T_2 \mid \theta_1)P(\theta_1) + P_R(T_2 \mid \theta_2)P(\theta_2) + P_R(T_2 \mid \theta_3)P(\theta_3)}$$
$$= \frac{.05 \times .09}{(.05 \times .9) + (.93 \times .07) + (.05 \times .03)} = .0432$$

Table 7: Posterior probabilities

Rating class	$P_R(\theta/T_1)$	$P_R(\theta/T_2)$	$P_R(\theta/T_3)$	$P_S(\theta/T_1)$	$P_S(\theta/T_2)$	$P_S(\theta/T_3)$
θ_1	.9964	.4032	.3723	.9985	.2081	.2299
θ_2	.0027	.5833	.0507	.0012	.7850	.0268
θ_3	.0007	.0134	.5770	.0003	.0069	.7433

Note the better performance of the special exam. For instance, the probability that the applicant is substandard if the exam indicates so is only .5833 for the routine exam, but .7850 for the special exam.
Those probabilities are then used to calculate expected values at each chance node. For instance, the expected payoff for the path (e_1, T_2, a_2) is

$$-300\, P_R(\theta/T_2) + 120\, M\ P_R(\theta/T_2) + 60\, M\ P_R(\theta/T_2)$$
$$= -300(.4032) + 120\, M(.5833) + 60\, M(.0134)$$
$$= 70.8\, M - 120.97$$



0
114.2M
120M
80M
20M
10.2M
-270
-300
120M
60M
3.6M-291
-300
-300
120M
119.81M
120M
80M
20M
.39M
-299
-300
120M
60M
.09M
-300
-300
-300
120M
95.32M
120M
80M
20M
70.8M
-121
-300
120M
60M
1.6M-296
-300
-300
60M
-30
60.3M
120M
80M
20M
40.7M
-112
-300
120M
60M
69.2M
-127
-300
-300
120M
-300
49.9M
120M
80M
20M
.16M
-300
-300
120M
60M
.04M-300
-300
-300
120M
87.9M
120M
80M
20M
94.6M
-62
-300
120M
60M
.83M
-298
-300
-300
120M
44.6M
120M
80M
20M
47.8M
-69
-300
120M
60M
89.2M
-77.1
-300
-300
120M

Figure 3

Expected payoffs are indicated on figure 3. They are used to "prune out the tree" by eliminating dominated branches. Two dashes indicate the actions that have been ruled out. Branch (e_2, T_3, a_2) is eliminated, since it is dominated by either (e_2, T_3, a_1) or (e_2, T_3, a_3), for all values of M. For some decision nodes, the optimal action depends on M. For instance, (e_1, T_3, a_1) results in a payoff of $60.27\,M$. The profit for (e_1, T_3, a_3) is $69.25\,M - 126.89$. The best action is a_1 if $M \leq 14.14$, and a_3 if $M > 14.14$. The standard rate should be offered when the face value of the policy is low; the sub-substandard rate should be offered when M exceeds 14.14.

The situation after elimination of branches is summarized in Figure 4. The underwriter should always offer the regular rates if either exam indicates it. For low values of M, the unreliability of the exams leads to offering standard rates, even if the exam indicates a non-standard applicant. Since e_1 is less precise, the break-even M is higher.

This "folding-back" process of "averaging out" at chance junctures and of selecting the best action at decision nodes continues until the base of the tree is reached. We obtain

$$\begin{aligned} P_R(T_1) &= P_R(T_1/\theta_1)P(\theta_1) + P_R(T_1/\theta_2)P(\theta_2) + P_R(T_1/\theta_3)P(\theta_3) \\ &= (.93)(.9) + (.035)(.07) + (.02)(.03) = .84005 \\ P_R(T_2) &= .11160 \\ P_R(T_3) &= .04835 \\ P_S(T_1) &= .87435 \\ P_S(T_2) &= .08650 \\ P_S(T_3) &= .03915 \end{aligned}$$

Denote a decision vector by (a_i, a_j, a_k), where a_i [resp. a_j, a_k] is the action taken when the medical exam indicates T_1 [resp. T_2, T_3]. The payoffs for all medical decisions are then as follows.

Medical exam	Payoff	Action	Condition
e_0	$114.2\,M$	a_1	
e_1	$114.2\,M$	(a_1, a_1, a_1)	$M \leq 14.14$
	$114.634\,M - 6.135$	(a_1, a_1, a_3)	$14.14 < M$
e_2	$114.2\,M$	(a_1, a_1, a_1)	$M \leq 1.73$
	$115.946\,M - 3.015$	(a_1, a_1, a_3)	$1.73 < M \leq 9.31$
	$116.526\,M - 8.415$	(a_1, a_2, a_3)	$9.31 < M$

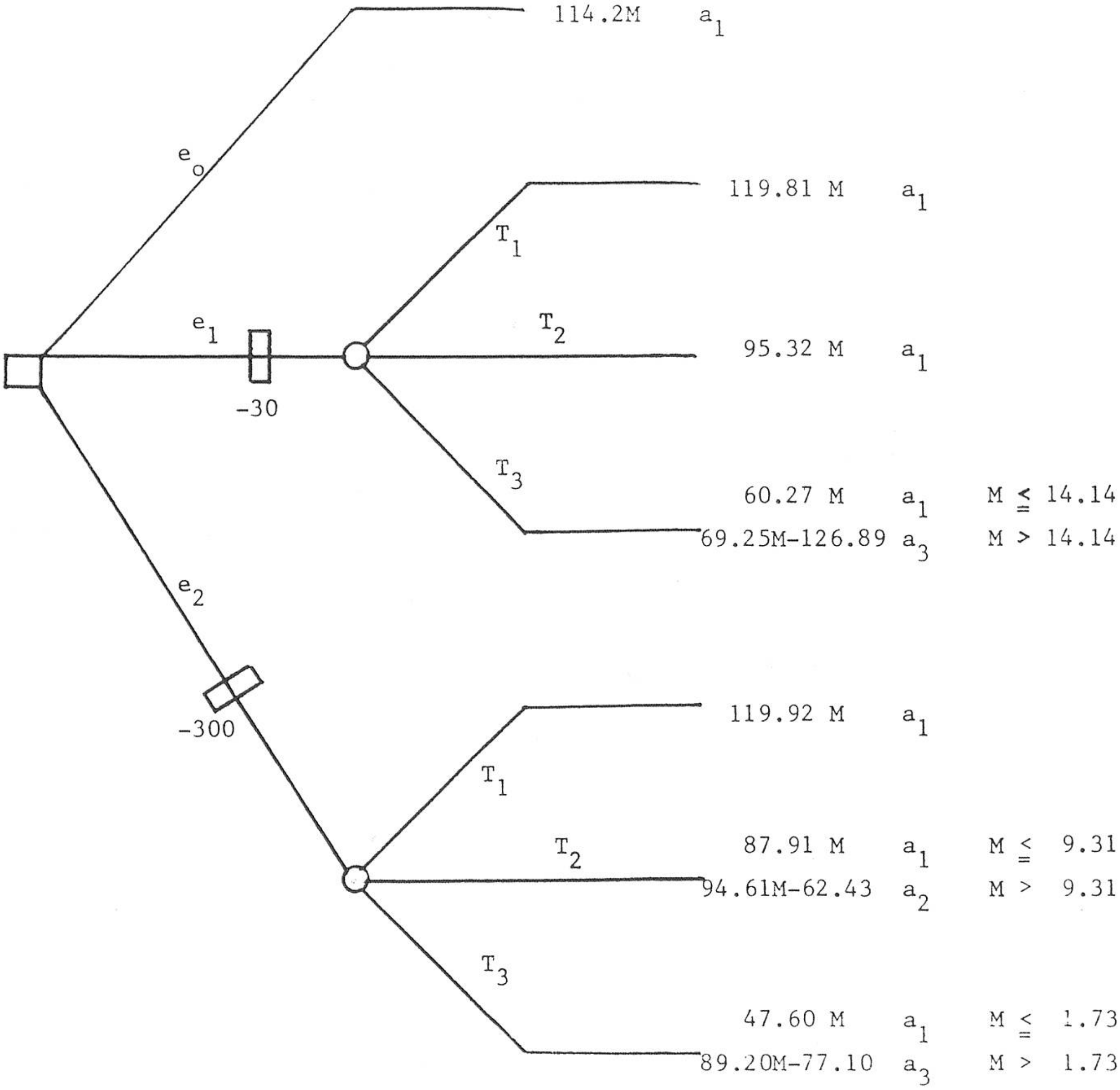

114.2M a1
e0
119.81 M a1
T1
e1
T2
95.32 M a1
-30
T3
60.27 M a1 M ≦ 14.14
69.25M-126.89 a3 M > 14.14
e2
-300
119.92 M a1
T1
T2
87.91 M a1 M ≦ 9.31
94.61M-62.43 a2 M > 9.31
T3
47.60 M a1 M ≦ 1.73
89.20M-77.10 a3 M > 1.73

Figure 4

Including the cost of the respective medical exams leads to the optimal strategy.

Condition	Optimal Strategy	Payoff
$0 < M \leq 83.26$	e_0	$114.2\,M$
$83.26 < M \leq 143.76$	$e_1 : (a_1, a_1, a_3)$	$114.63\,M - 36.14$
$143.76 < M$	$e_2 : (a_1, a_2, a_3)$	$116.53\,M - 308.42$

* For small policies ($M \leq 83.26$), the underwriter should always offer the standard rate. Any medical exam would be too expensive. Its cost would more than offset the benefits achieved through information provided by the exam.
* For middle-size policies ($83.26 < M \leq 143.76$), the underwriter should request a routine medical exam. He should offer the sub-substandard rate if the medical exam indicates it, and the standard rate otherwise. The use of the routine report is a trade-off between cost and accuracy. The relatively low cost of the exam compensates for its imprecision.
* For large policies ($143.36 < M$), the underwriter should request the special exam, and follow its recommendations. The high cost of the exam is fully justified by its accuracy, and the higher resulting profits.

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Summary

Three different actuarial applications of decision analysis are presented, to evaluate the benefits of smoke detectors laws, compare fixed-rate and adjustable-rate mortgage loans, and help a life insurance underwriter select a medical examinations strategy.

Zusammenfasung

Drei Anwendungen der Entscheidungstheorie werden präsentiert: die Beurteilung des Nutzens von obligatorischen Rauchmeldern, der Vergleich von Hypotheken mit festen und variablen Zinssätzen sowie eine Vorgehensweise für Antragsprüfer in der Lebensversicherung zur Anordnung von ärztlichen Untersuchungen.

Résumé

Trois applications actuarielles de la théorie de la décision sont présentées: l'évaluation d'une loi imposant un détecteur de fumée dans chaque résidence, la comparaison de prêts hypothécaires à taux fixes ou variables, et le choix d'un examen médical lors de la souscription d'une assurance vie.