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Actuarial Analysis of Dependent Lives

1 Introduction

Actuarial tables of multilife statuses are invariably based on the assumption of mutual independence of the component life lengths. Presumably, the popularity of the independence hypothesis is due to its computational feasibility rather than its realism; there are reasons to believe that the component life lengths may be interdependent. For instance, husband and wife are more or less exposed to the same risks, which may change for either of them when the spouse dies. Moreover, there may be certain selectional mechanisms in the matching of couples (birds of a feather flock together).
The present paper undertakes to investigate some alternatives to the independence hypothesis and to derive some consequences for actuarial computations. We shall focus on the bivariate case and find sufficient conditions for positive or negative dependence between life lengths and present values of insurances. In Section 2 we refer some definitions of different notions of positive and negative dependence between stochastic variables and some useful results on relationships between these definitions. Section 3 discusses how various forms of dependence affect present values of payments related to single- and multilife statuses. Section 4 and 5 present model assumptions that imply positive or negative dependence of life lengths; Section 4 treats a Markov model with forces of mortality depending on marital status, and Section 5 launches a heterogeneity model specifying that the component forces of mortality are stochastic processes that may be dependent.
Actuarial aspects of dependence between life lengths have been discussed previously by *Carrière and Chan* (1986). Their approach centers on quantifying the possible impact of dependence on actuarial values by establishing bounds for bivariate distributions, and is thus methodologically remote from the one taken here.

2 Notions of dependence between random variables

Let S and T be real random variables defined on some probability space. We say that S and T are *positively quadrant dependent* and write $PQD(S, T)$ if

$$P[S > s, T > t] \geq P[S > s]\, P[T > t] \quad \text{for all} \quad s, t. \tag{2.1}$$

Note that (2.1) is equivalent to $P[S > s \mid T > t] \geq P[S > s]$ for $P[T > t] > 0$, which is easy to interpret.
We say that S and T are *associated* and write $AS(S, T)$ if

$$\text{Cov}\,[g(S, T), h(S, T)] \geq 0 \tag{2.2}$$

for all pairs of functions g and h that are increasing in both arguments and for which the covariance exists. We could state this definition equivalently by saying that g and h are decreasing in both arguments or by reversing the inequality in (2.2) and saying that g is increasing and h is decreasing.
We say that S is *right tail increasing/decreasing* in T and write $RTI(S \mid T)$ / $RTD(S \mid T)$ if

$$P[S > s \mid T > t] \quad \text{is increasing/decreasing in } t \text{ for all } s. \tag{2.3}$$

Stochastic independence is included as a special case of each of the three notions of dependence: replace "$\geq$" by "$=$" in (2.1) and (2.2), and replace "increasing/decreasing" by "constant" in (2.3). We could distinguish between independence and genuine dependence by adding the qualification "strictly" to PQD, AS and RTI/RTD in the latter case.
A thorough analysis of these and other notions of dependence between random variables can be found in Sections 2.2 og 5.4 in *Barlow/Proschan* (1975). From this reference we pick the following useful result:

Lemma 2.1: $RTI(S \mid T) \;\Rightarrow\; AS(S, T) \;\Rightarrow\; PQD(S, T)$

3 Present values of endowments, annuities and insurances on dependent lives

We shall restrict our discussion to the bivariate case and, more specifically, consider a married couple which buys an insurance policy when the husband is at age x and the wife is at age y, say. Let S and T denote the remaining life lengths of husband and wife, respectively. We consider them as random variables.

In actuarial applications we are interested in the following single- and multilife statuses defined by S and T.

Status (z)	Life length U	Survival function $P[U > \tau]$	
Husband (x)	S	$P[S > \tau]$	(3.1)
Wife (y)	T	$P[T > \tau]$	(3.2)
Joint life (x, y)	$S \wedge T$	$P[S > \tau, T > \tau]$	(3.3)
Longest life $(\overline{x, y})$	$S \vee T$	$P[S > \tau] + P[T > \tau] - P[S > \tau, T > \tau]$	(3.4)

Random variables of particular interest are the present values of certain payments determined by the life lengths of these statuses. We list some that are commonly used, restricting to n-year payment schemes specifying that a unit amount is payable either immediately upon the survival of a status (z) to time n or death of the status, or annually, continuously at constant rate 1, during the existence of the status. As usual, $v = e^{-\delta}$ denotes the discounting factor corresponding to a fixed force of interest δ, and I_A denotes the indicator function of the event A.

Payment scheme	Present value	Expected present value	
Pure endowment	$C_n^e(U) = v^n I_{[U>n]}$	${}_nE_z = v^n P[U > n]$	(3.5)
Annuity	$C_n^a(U) = \int_0^n v^\tau I_{[U>\tau]}\, d\tau = \dfrac{1 - v^{U \wedge n}}{\delta}$	$\overline{a}_{z\,\overline{n}\rceil} = \int_0^n v^\tau P[U > \tau]\, d\tau$	(3.6)
(Term) insurance	$C_n^i(U) = v^U I_{[U \le n]}$	$\overline{A}^{\,1}_{z\,\overline{n}\rceil} = 1 - \delta \overline{a}_{z\,\overline{n}\rceil} - {}_nE_z$	(3.7)

Both $C_n^e(U)$ and $C_n^a(U)$ are increasing functions of U, whereas $C_n^i(U)$ is a decreasing function of U. Furthermore, each of the life lengths U defined in (3.1) – (3.4) are increasing functions of S and T. Combining these results with the definition (2.2), we obtain a number of results on interdependence of present values of payments related to associated lives.

Assume that S and T are associated, and let U and V be any two of the life lengths listed in (3.1) – (3.4). Then $C_m^\alpha(U)$ and $C_n^\beta(V)$ are positively correlated for $\alpha, \beta \in \{a, e\}$ and for $\alpha = \beta = i$, and they are negatively correlated for $\alpha \in \{a, e\}$ and $\beta = i$. For instance, $C_m^i(S)$ and $C_n^i(T)$ are positively correlated, and $C_m^i(S)$ and $C_n^a(T)$ are negatively correlated. Similar statements are valid also for expressions of the type

$$C_n^i(V) - \pi\, C_n^a(U), \tag{3.8}$$

which is the present value of benefits less premiums for a life insurance payable at time V if $V \le n$, with level premium payable at constant rate π until $U \wedge n$. Clearly, the expression in (3.8) is a decreasing function of S and T, and so we have, for instance, that $C_m^i(S) - \pi_x C_m^a(S)$ and $C_n^i(T) - \pi_y C_n^a(T)$ are positively correlated. General results of this kind cannot be obtained for present values that are not monotone in S and T, e.g. deferred payments of the type $C_\infty^\alpha(U) - C_m^\alpha(U)$, $\alpha \in \{a, i\}$, or for present values related to compound statuses, like $v^S I_{[S \le T \wedge n]}$ (contingent insurance on (x) payable if (y) outlives (x)) or $C_n^a(T) - C_n^a(S \wedge T)$ (reversionary annuity on (y) after the death of (x)). In any case, the covariance of any two given present values can be evaluated by integration.

Dependence between S and T affects not only variances and covariances of present values, but also expected present values of payments related to multilife and compound statuses. Let the topscript "ind" signify that a quantity is calculated under the independence hypothesis, that is, $P[S > s,\ T > t]$ is replaced by $P[S > s]\,P[T > t]$. Suppose that S and T are PQD as defined in (2.1). Inspection of (3.3) – (3.7) and use of (2.1) gives that

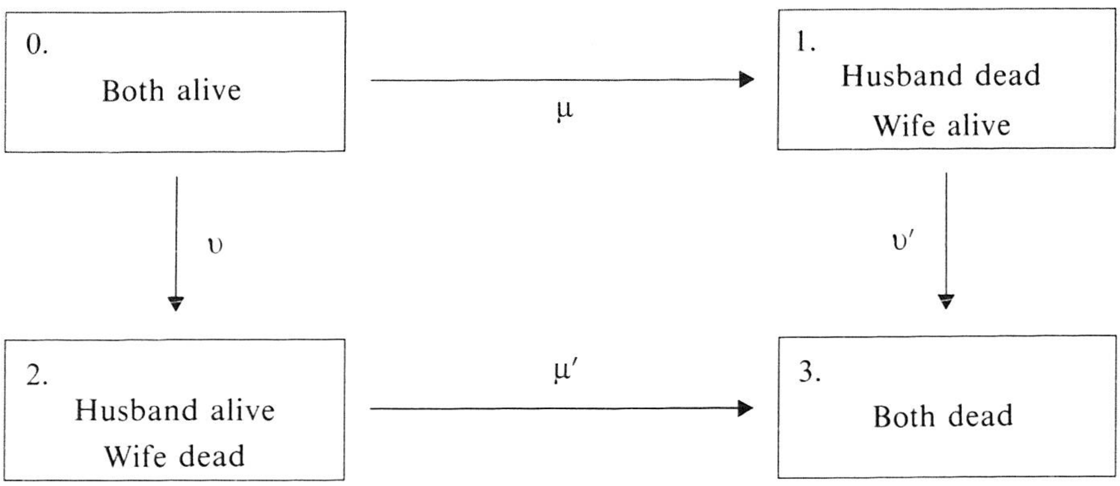
$$
\begin{aligned}
&{}_nE_{xy} \ge {}_nE_{xy}^{\text{ind}}, &&\bar{a}_{xy\,\overline{n}|} \ge \bar{a}_{xy\,\overline{n}|}^{\text{ind}}, &&\bar{A}_{\overset{1}{xy}\,\overline{n}|} \le \bar{A}_{\overset{1}{xy}\,\overline{n}|}^{\text{ind}}, \\
&{}_nE_{\overline{xy}} \le {}_nE_{\overline{xy}}^{\text{ind}}, &&\bar{a}_{\overline{xy}\,\overline{n}|} \le \bar{a}_{\overline{xy}\,\overline{n}|}^{\text{ind}}, &&\bar{A}_{\overset{1}{\overline{xy}}\,\overline{n}|} \ge \bar{A}_{\overset{1}{\overline{xy}}\,\overline{n}|}^{\text{ind}}.
\end{aligned}
$$

Moreover, let the level premium in (3.8) be determined by the equivalence principle, that is, by requiring that the expected value of the expression in (3.8) be zero. Then, from the inequalities above, we obtain that $\pi \ge \pi^{\text{ind}}$ e.g. for $V = S \vee T$ and $U \in \{S, T, S \vee T\}$, that is, the independence hypothesis yields an insufficient premium. On the other hand, $\pi \le \pi^{\text{ind}}$ e.g. for $V \in \{S, T, S \wedge T\}$ and $U = S \wedge T$.

4 A Markov model with forces of mortality depending on marital status

The interdependence of the life lengths of husband and wife could be explained by the change that incurs in the living conditions upon the loss of the spouse (grief, stress, etc.). Such effects can be accounted for simply by allowing the forces of mortality to depend on marital status (apart from age and sex). More specifically, assume that the husband's force of mortality at age $x+s$ is $\mu(s)$ if he is then still married and $\mu'(s)$ if he is a widower. Likewise, the wife's force of mortality at age $y+t$ is $\nu(t)$ if she is then still married and $\nu'(t)$ if she is a widow. (To simplify the notation, the dependence on x and y is not made visible. The prime should not be confused with differentiation here.)
Under these assumptions the future development of the marital status for an x years old husband and a y years old wife is a time continuous Markov chain $\{X_\tau(S,T);\, \tau \geq 0\}$ with state space and forces of transitions as shown in Fig. 1. The transition probabilities $P_{0j}(s,t) = P[X_t = j \mid X_s = 0]$, $j \in \{0,1,2\}$, $0 \leq s \leq t$, are given by

Figure 1.
Markov chain representation of the marital status of husband and wife



$$P_{00}(s,t) = e^{-\int_s^t \mu+\nu},$$

$$P_{01}(s,t) = \int_s^t e^{-\int_s^\tau \mu+\nu} \mu(\tau)\, e^{-\int_\tau^t \nu'}\, d\tau\,,$$

$$P_{02}(s,t) = \int_s^t e^{-\int_s^\tau \mu+\nu} \nu(\tau)\, e^{-\int_\tau^t \mu'}\, d\tau\,.$$

The joint survival function of S and T is

$$P[S > s,\ T > t] = \begin{cases} P_{00}(0,t) + P_{00}(0,s)P_{01}(s,t)\,, & s \le t\,, \\ P_{00}(0,s) + P_{00}(0,t)P_{02}(t,s)\,, & s > t\,, \end{cases}$$

$$= \begin{cases} e^{-\int_0^t \mu+\nu} + \int_s^t e^{-\int_0^\tau \mu+\nu} \mu(\tau)\, e^{-\int_\tau^t \nu'}\, d\tau\,, & s \le t\,, \quad (4.1) \\ e^{-\int_0^s \mu+\nu} + \int_t^s e^{-\int_0^\tau \mu+\nu} \nu(\tau)\, e^{-\int_\tau^s \mu'}\, d\tau\,, & s > t\,. \quad (4.2) \end{cases}$$

The marginal survival function of T is

$$P[T > t] = P_{00}(0,t) + P_{01}(0,t)$$
$$= e^{-\int_0^t \mu+\nu} + \int_0^t e^{-\int_0^\tau \mu+\nu} \mu(\tau)\, e^{-\int_\tau^t \nu'}\, d\tau\,, \quad \tau \ge 0\,. \tag{4.3}$$

The expressions in (4.1) – (4.3) can be realized by direct reasoning.
We now prove that S and T are positively/negatively dependent if the mortality is higher/lower for widowed persons than for married persons. It is assumed throughout that all forces of mortality appearing in the model are continuous, so that the differentiations performed in the following are valid.

Theorem 4.1.: The following implications are valid:

$$\text{(I)} \qquad \mu'(\tau) \geq \mu(\tau) \quad \text{and} \quad \nu'(\tau) \geq \nu(\tau) \quad \text{for all} \quad \tau > 0 \tag{4.4}$$
$$\Longrightarrow \quad RTI(S \mid T) \quad \text{and} \quad RTI(T \mid S)$$
$$\Longrightarrow \quad AS(S, T) \quad \Longrightarrow \quad PQD(S, T),$$

$$\text{(II)} \qquad \mu'(\tau) \leq \mu(\tau) \quad \text{and} \quad \nu'(\tau) \leq \nu(\tau) \quad \text{for all} \quad \tau > 0 \tag{4.5}$$
$$\Longrightarrow \quad RTD(S \mid T) \quad \text{and} \quad RTD(T \mid S)$$
$$\Longrightarrow \quad AS(-S, T) \quad \Longrightarrow \quad PQD(-S, T),$$

$$\text{(III)} \qquad \mu'(\tau) = \mu(\tau) \quad \text{and} \quad \nu'(\tau) = \nu(\tau) \quad \text{for all} \quad \tau > 0 \tag{4.6}$$
$$\Longleftrightarrow \quad S \text{ and } T \text{ are stochastically independent.}$$

Proof: We first prove item (I). The last two implications are generally valid, confer Lemma 2.1. We need only to prove the first implication and, by symmetry, it suffices to establish that (4.4) implies $RTI(S \mid T)$.

First, for $s \leq t$ we find from (4.1) and (4.3) that

$$P[S > s \mid T > t] = \frac{e^{-\int_0^t \mu+\nu} + \int_s^t e^{-\int_0^\tau \mu+\nu} \mu(\tau) e^{-\int_\tau^t \nu'} \, d\tau}{e^{-\int_0^t \mu+\nu} + \int_0^t e^{-\int_0^\tau \mu+\nu} \mu(\tau) e^{-\int_\tau^t \nu'} \, d\tau}$$
$$= \frac{1 + \int_s^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau) \, d\tau}{1 + \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau) \, d\tau}.$$

By the rule for differentiating a fraction, $\partial(u/v) = (v\partial u - u\partial v)/v^2$, the sign of $\frac{\partial}{\partial t} P[S > s \mid T > t]$ is the same as the sign of

$$\left\{1 + \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau\right\} \left[\mu(t) + \int_s^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau \{\mu(t) + \nu(t) - \nu'(t)\}\right]$$
$$- \left\{1 + \int_s^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau\right\} \left[\mu(t) + \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau \{\mu(t) + \nu(t) - \nu'(t)\}\right]. \tag{4.7}$$

Put

$$A(u) = \int_0^u e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau ,$$

and rewrite (4.7) as

$$\begin{aligned} &\{1 + A(t)\}\left[\mu(t) + \{A(t) - A(s)\}\{\mu(t) + \nu(t) - \nu'(t)\}\right] \\ &\quad - \{1 + A(t) - A(s)\}\left[\mu(t) + A(t)\{\mu(t) + \nu(t) - \nu'(t)\}\right] \\ &= A(s)\{\nu'(t) - \nu(t)\}. \end{aligned} \tag{4.8}$$

Second, for $s > t$ we find from (4.2) and (4.3) that

$$P[S > s \mid T > t] = \frac{e^{-\int_0^s \mu+\nu} + \int_t^s e^{-\int_0^\tau \mu+\nu} \nu(\tau) e^{-\int_\tau^s \mu'}\, d\tau}{e^{-\int_0^t \mu+\nu} + \int_0^t e^{-\int_0^\tau \mu+\nu} \mu(\tau) e^{-\int_\tau^t \nu'}\, d\tau}.$$

The sign of $\frac{\partial}{\partial t} P[S > s \mid T > t]$ is the same as the sign of

$$\left\{ e^{-\int_0^t \mu+\nu} + \int_0^t e^{-\int_0^\tau \mu+\nu} \mu(\tau) e^{-\int_\tau^t \nu'} d\tau \right\} \left\{ - e^{-\int_0^t \mu+\nu} \nu(t)\, e^{-\int_t^s \mu'} \right\}$$

$$- \left\{ e^{-\int_0^s \mu+\nu} + \int_t^s e^{-\int_0^\tau \mu+\nu} \nu(\tau)\, e^{-\int_\tau^s \mu'} d\tau \right\} \left[e^{-\int_0^t \mu+\nu} \left\{ - \mu(t) - \nu(t) \right\} \right. \tag{4.9}$$

$$\left. + e^{-\int_0^t \mu+\nu} \mu(t) + \int_0^t e^{-\int_0^\tau \mu+\nu} \mu(\tau)\, e^{-\int_\tau^t \nu'} d\tau \{-\nu'(t)\} \right].$$

Multiplication by $e^{\int_0^t \mu+\nu} e^{\int_0^s \mu+\nu}$ preserves the sign and transforms (4.9) into

$$\left\{ 1 + \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau \right\} \left\{ - e^{\int_t^s \mu+\nu-\mu'} \nu(t) \right\}$$

$$- \left\{ 1 + \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \nu(\tau)\, d\tau \right\} \left[- \nu(t) - \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau\, \nu'(t) \right]$$

$$= \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\, d\tau \left[- e^{\int_t^s \mu+\nu-\mu'} \nu(t) \right.$$

$$\left. + \left\{ 1 + \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \nu(\tau)\, d\tau \right\} \nu'(t) \right] \tag{4.10}$$

$$+ \nu(t) \left[- e^{\int_t^s \mu+\nu-\mu'} + 1 + \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \nu(\tau)\, d\tau \right].$$

Observe that

$$\int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \nu(t)\,d\tau = \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \{\mu(\tau)+\nu(\tau)-\mu'(\tau)\}\,d\tau$$

$$+ \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \{\mu'(\tau)-\mu(\tau)\}\,d\tau$$

$$= e^{\int_t^s \mu+\nu-\mu'} - 1 + \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \{\mu'(\tau)-\mu(\tau)\}\,d\tau\,,$$

and continue from (4.10):

$$= \int_0^t e^{\int_\tau^t \mu+\nu-\nu'} \mu(\tau)\,d\tau \left[e^{\int_t^s \mu+\nu-\mu'} \{\nu'(t)-\nu(t)\} \right.$$

$$\left. + \nu'(t) \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \{\mu'(\tau)-\mu(\tau)\}\,d\tau \right] \tag{4.11}$$

$$+ \nu(t) \int_t^s e^{\int_\tau^s \mu+\nu-\mu'} \{\mu'(\tau)-\mu(\tau)\}\,d\tau\,.$$

By inspection of (4.8) and (4.11), it is seen that $\mu' \geq \mu$ and $\nu' \geq \nu$ implies $\frac{\partial}{\partial t} P[S > s \mid T > t] \geq 0$ for all s and t, hence $RTI(S \mid T)$. This proves item (I). Item (II) follows immediately.

Item (III) follows by noting that $RTI(S \mid T)$ and $RTD(S \mid T)$ together is equivalent to stochastic independence. Q.E.D.

Reliability theorists would speak of (4.4) as the WBF-condition (the system (S, T) is "Weakened By Failures"). *Arjas/Norros* (1984) have proved association for multicomponent WBF systems by use of refined counting process theory. The present proof presents some interest of its own since it is elementary and, moreover, shows $RTI(S \mid T)$, which is stronger than $AS(S, T)$. The case $(\mu, \nu) \neq (\mu', \nu')$ could be taken as a definition of a causal relationship between the two events underlying the process, here death of husband and death of wife, see *Schweder* (1970).

5 A heterogeneity model with dependence between component forces of mortality

So-called frailty or heterogeneity models are commonly used in biomedical statistics to describe variation in mortality between individuals. A heterogeneity model specifies that the survival function or, equivalently, the force of mortality of a randomly selected individual is a random process.

In the bivariate case – and we stick to the example with the married couple – we assume that the forces of mortality of S and T are stochastic processes, $\{\mu(\tau);\ \tau > 0\}$ and $\{\nu(\tau);\ \tau > 0\}$, respectively. By assuming that these processes are dependent, we can model such effects as selective matching and exposure to common risk factors.

The joint survival function of S and T is

$$P[S > s,\ T > t] = E\left[e^{-\int_0^s \mu - \int_0^t \nu}\right],$$

and the marginal survival functions are

$$P[S > s] = E\left[e^{-\int_0^s \mu}\right], \qquad P[T > t] = E\left[e^{-\int_0^t \nu}\right].$$

It is readily seen that if the cumulative intensities $\int_0^s \mu$ and $\int_0^t \nu$ are associated for each s and t, then we have $PQD(S, T)$ as defined in (2.1). This implies that Cov $[g(S), h(T)] \geq 0$ for each g and h that are increasing.

As a special case, assume that the intensities are of the form

$$\mu(\tau) = \Theta m(\tau), \qquad \nu(\tau) = \Lambda n(\tau),$$

with Θ and Λ associated positive random variables and m and n non-random intensity functions. Then $\int_0^s \mu = \Theta \int_0^s m$ and $\int_0^t \nu = \Lambda \int_0^t n$ are associated for each s and t.

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Summary

The present paper discusses how actuarial analyses of multilife statuses are affected if we drop the traditional hypothesis of independent component life lengths. Some notions of dependence, well known from reliability theory, are presented, and conditions for positive dependence are found in a Markov model as well as in a heterogeneity model for the bivariate case.

Zusammenfassung

In der vorliegenden Arbeit wird die traditionelle Voraussetzung über die Unabhängigkeit der Sterblichkeiten bei Versicherungen auf mehrere Leben ersetzt durch verschiedene Formen der Abhängigkeit zwischen zukünftigen Lebensdauern. Es werden Konsequenzen diskutiert und im Fall von zwei Leben ein Markov- sowie ein Heterogenitäts-Modell präsentiert.

Résumé

Le présent article discute de l'effet, dans le cas des risques-vie sur plusieurs têtes, d'un renoncement à l'hypothèse traditionnelle de l'indépendance des durées de vie. On y présente quelques notions de dépendance bien connues en théorie de la fiabilité, ainsi que – dans le cas de risques sur deux têtes – des conditions de dépendance positive d'un modèle markovien et d'un modèle d'hétérogénéité.