A lower bound for the probability of ruin, given a compound Poisson process

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For a compound Poisson process S_t, $t \geqslant 0$, specified by the Poisson parameter $\lambda > 0$ and the claim amount distribution Q on $(0, \infty)$ the probability of ruin, given the initial reserve $x \geqslant 0$ and the total of premiums $c > 0$ received in the time interval $(0, 1)$, is defined by

$$\psi(x) = P\{x + ct - S_t < 0 \quad \text{for some } t > 0\}.$$

One of the most famous results in ruin theory now is the inequality

$$\psi(x) \leqslant e^{-Rx}, \qquad x \geqslant 0,$$

where $R > 0$ is the so-called adjustment coefficient. This result, however, has certain shortages: From the mathematical point of view the underlying assumptions are very strong, i.e. one has to assume that the moment generating function of Q exists for some $t_0 > 0$, and, concerning applications, one has to realize that it is nearly impossible to calculate R explicitely.
It is the purpose of the present note to give a lower bound for $\psi(x)$ that can be computed easily. The only assumption which has to be made is a natural one, namely that the mean of Q exists. The result also appears to be of a certain interest for the user: If the lower bound is too large, then he knows that he has to increase the initial reserve x in order to obtain a sufficiently small probability of ruin.

Theorem: For $c > \lambda\mu$ and $x \geqslant 0$ we have

$$\psi(x) \geqslant 1 - (1-p)\, e^{px/\mu} \sum_{i=0}^{m} \frac{(-1)^i}{i!} \left[\left(\frac{x}{\mu} - i \right) p e^{-p} \right]^i,$$

where μ is the mean of Q,

$$m = \text{int}\left(\frac{x}{\mu}\right) \quad \text{and} \quad p = \frac{\lambda\mu}{c}.$$

Proof: Since

$$\psi(x) = P\{\sup_n \sum_{i=1}^{n} (X_i - c\,W_i) > x\}$$

$$= P\left\{\sup_n \sum_{i=1}^{n} \left(\frac{X_i}{\mu} - \frac{c}{\mu} W_i\right) > \frac{x}{\mu}\right\}$$

where $X_1, X_2, \ldots$ are i.i.d. with common distribution Q and $W_1, W_2, \ldots$ are i.i.d. according to an exponential distribution with mean $1/\lambda$, we may assume in the following that

$$\mu = 1$$

(This means that X_i, c and x are measured in μ-units.) Let Q_0 be the distribution concentrated in μ, i.e. $Q_0\{1\} = 1$. Then the means of Q and Q_0 are identical and Q is more dangerous than Q_0 in the sense of *Bühlmann* et al. [1]. If $\psi_0(x)$ is the probability of ruin, given the compound Poisson process with Poisson parameter λ, the claim amount distribution Q_0, and the premium rate $c > \lambda$, then by the results in *Bühlmann* et al. [1] and in *Michel* [4]

$$\psi_0(x) \leqslant \psi(x), \qquad x \geqslant 0.$$

The following proposition therefore gives the assertion.

Proposition: We have

$$\psi_0(x) = 1 - (1-p)\,e^{px} \sum_{i=0}^{[x]} \frac{(-1)^i}{i!} [(x-i)\,p\,e^{-p}]^i,$$

where $p = \lambda/c$ and $[x] = \text{int}(x)$.

Proof: Here we use the general representation (with $p = \lambda\mu/c < 1$)

$$\psi(x) = (1-p) \sum_{k=1}^{\infty} p^k D^{*k}(x, \infty), \qquad x \geqslant 0,$$

where D is the distribution on $(0, \infty)$ with the Lebesgue-density

$$y \to \frac{1}{\mu} Q(y, \infty)\, 1_{(0,\infty)}(y).$$

(This immediately follows from the renewal equation (3.7) in *Gerber* [3], p. 115, in connection with the known representation of the solution of such an equation.)
In our case, i.e. $Q_0\{1\} = 1$, D is the uniform distribution on (0, 1). Hence, we have for $k = 1, 2 \ldots$,

$$D^{*k}[0, x] = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} [(x-i)^+]^k,$$

where $a^+ = \max(0, a)$. This formula can be found in *Feller* [2], Theorem 1, p. 27. Therefore, with $p = \lambda/c$,

$$\begin{aligned} 1 - \psi(x) &= (1-p) \sum_{k=0}^{\infty} p^k D^{*k}[0, x] \\ &= (1-p) \sum_{k=0}^{\infty} p^k \sum_{i=0}^{k} \frac{(-1)^i}{i!\,(k-i)!} [(x-i)^+]^k \\ &= (1-p) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{k=i}^{\infty} \frac{p^k}{(k-i)!} [(x-i)^+]^k \\ &= (1-p) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} [p\,(x-i)^+]^i e^{p\,(x-i)^+} \end{aligned}$$

Hence, the result follows.

Remarks:

(i) If we write $\psi_0(x) = \psi_0(x, \lambda, c)$, then the lower bound in the theorem is $\psi_0(x/\mu, \lambda, c/\mu)$, and this in turn is the probability of ruin, given the claim amount distribution which is concentrated in μ, the rate of premiums c, and the initial reserve x.
Hence, the result of the theorem may be restated as follows: Given λ, $\mu > 0$ and $c > \lambda\mu$, the distribution concentrated in μ yields the smallest probability of ruin (for all $x \geqslant 0$) in the class of all claim amount distributions with mean μ.

(ii) According to formulas (5.27), p. 124, and (5.29), p. 125, in *Gerber* [3] we have the following asymptotic result for the lower bound in our theorem, say $b(x)$,

$$b(x) \sim C e^{-R_0 x} \quad \text{for} \quad x \to \infty,$$

where R_0 is the (uniquely determined) positive solution of the equation

$$\lambda e^{r\mu} = \lambda + cr$$

and

$$C = (c - \lambda\mu) / \left(\lambda + c R_0 - \frac{c}{\mu} \right).$$

(Since the denominator represents the derivative of $r \to (1/\mu)(\lambda e^{r\mu} - \lambda - cr)$ taken for $r = R_0$, it is positive.)

(iii) If the adjustment coefficient R for (Q, λ, c) exists, i.e. the positive solution of the equation

$$\lambda \int_0^\infty e^{rx} Q(dx) = \lambda + cr,$$

and if the mean of Q is μ, then we have

$$R \leqslant R_0,$$

where R_0 has been defined in (ii), i.e. R_0 is the adjustment coefficient for (Q_0, λ, c), where Q_0 is the distribution concentrated in μ. (This follows from the theorem, (ii), and the fact that $\psi(x) \leqslant e^{-Rx}$, $x \geqslant 0$.) Furthermore, the maximum value R_0 for R (in the class of all distributions with mean μ, for which R exists) is attained only for the distribution Q_0, i.e. if Q is different from Q_0, then we have

$$R < R_0.$$

To see this assume that $R = R_0$. Then R fulfills the equations

$$\lambda \int_0^\infty e^{Rx} Q(dx) = \lambda + cR \quad \text{and} \quad \lambda e^{R\mu} = \lambda + cR,$$

i.e. we have

$$\lambda e^{R\mu} = \lambda \int_0^\infty e^{Rx} Q(dx).$$

This, in turn, means equality in *Jensen*'s inequality

$$e^{R\mu} \leq \int_0^\infty e^{Rx} Q(dx).$$

As $x \rightarrow e^{Rx}$ is strictly convex, we therefore have

$$x = \int_0^\infty y Q(dy) = \mu \qquad Q\text{-}a.e.,$$

i.e.

$$Q = Q_0.$$

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Abstract

Under the assumption that the mean μ of the claim size distribution exists we derive a lower bound for the probability of ruin, given a compound Poisson process. This bound cannot be improved as it is attained for the claim size distribution that is concentrated in μ. Furthermore, it is shown that the claim size distribution which is concentrated in μ admits the largest adjustment coefficient in the class of all claim size distributions with mean μ for which the adjustment exists.

Zusammenfassung

Hergeleitet wird eine untere Schranke für die Ruinwahrscheinlichkeit, gegeben ein zusammengesetzter Poisson-Prozess, unter der Voraussetzung, dass der Mittelwert μ der Schadenhöhenverteilung existiert. Diese Schranke ist insofern nicht verbesserbar, als sie exakt ist für die in μ konzentrierte Schadenhöhenverteilung. Ferner wird gezeigt, dass die in μ konzentrierte Schadenhöhenverteilung in der Klasse aller Schadenhöhenverteilungen mit Mittelwert μ, für die der Anpassungskoeffizient existiert, den grössten Anpassungskoeffizienten besitzt.

Résumé

L'article présente une borne inférieure de la probabilité de ruine d'un processus de Poisson sous l'hypothèse de l'existence de la moyenne μ du montant d'un sinistre. Cette borne ne peut pas être améliorée en ce sens qu'elle est atteinte dans le cas de la distribution concentrée en μ. De plus, l'article montre que, dans la classe des distributions possibles des montants de sinistres de moyenne μ et pour lesquelles existe le coefficient d'ajustement, la distribution concentrée en μ possède le plus grand coefficient d'ajustement.