Zeitschrift: Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries

Herausgeber: Vereinigung Schweizerischer Versicherungsmathematiker

Band: - (1986)

Heft: 2

Artikel: Some elementary stop-loss inequalities

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DOI: https://doi.org/10.5169/seals-967056

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Some Elementary Stop-Loss Inequalities

1 Introduction

Suppose the aggregate claims, S, have a compound distribution with probability p_n of n claims and claims distribution function F. It is assumed that $F(0-)=0$ (no negative claims). So

$$Pr\{S \leq x\} = \sum_{n=0}^{\infty} p_n F^{n^*}(x).$$

We will construct bounds for the net stop-loss premium $E[(S-t)_+]$, where t is a certain deductible of interest. These bounds will depend only on

- $\mu = \int_0^\infty x\,dF(x)$, the mean claim amount
- $F(t)$, the probability that a claim is less than t
- $\mu(t) = \int_0^t x\,dF(x)/F(t)$, the average claim amount of a claim that is less than t

and on the distribution of the number of claims, which is supposed to be known. Thus the bounds will be particularly useful, where only partial information concerning the claim amount distribution is available.
In *Runnenburg/Goovaerts* (1985) similar ideas were developed. They used the conditional distribution of the claims not exceeding the retention to derive bounds on tail probabilities and stop-loss premiums of compound distributions.

* Acknowledgements: This research was supported by grants of the Boleslaw Monic Fund and the "Onderzoeksfonds K.U. Leuven" under No. OT/85/5. We thank referee Hans Gerber for suggesting an approach leading to proofs considerably easier to understand.

2 Stop-loss inequalities

To obtain inequalities on stop-loss premiums, we modify the original claim amount distribution by two methods. Following the method of mass concentration (see Example 3.1, page 98 of *Gerber,* 1979) we replace the mass between 0 and t by a point mass of $F(t)$ at an intermediate point, such that the mean is preserved. The result is the following distribution $G(x)$:

$$G(x)=\begin{cases} 0 & \text{for} \quad x<\mu(t) \\ F(t) & \text{for} \quad \mu(t)\leqslant x<t \\ F(x) & \text{for} \quad x\geqslant t \end{cases}$$

Following the method of mass dispersal (see Example 3.2 of *Gerber,* 1979), we replace the mass betwen 0 and t by two point masses at 0 and t, again such that the mean is preserved. The resulting distribution is as follows:

$$H(x)=\begin{cases} \dfrac{t-\mu(t)}{t}F(t) & \text{for} \quad 0\leqslant x<t \\ F(x) & \text{for} \quad x\geqslant t \end{cases}$$

Then G is a lower and H is an upper bound for F in the sense of the stop-loss ordering:

$$G \propto F \propto H$$

This means nothing other than that the stop-loss premiums with each retention are ordered correspondingly. In *Goovaerts et al.* (1984) it is proven that this property is preserved under convolution, so

$$G^{n*} \propto F^{n*} \propto H^{n*}$$

In particular this holds for the stop-loss premiums at retention t. So, using partial integration, we have

$$\int_t^\infty [1-G^{n*}(x)]\,dx \leqslant \int_t^\infty [1-F^{n*}(x)]\,dx \leqslant \int_t^\infty [1-H^{n*}(x)]\,dx \qquad (1)$$

The lower bound can be evaluated as follows:

$$\int_t^\infty [1-G^{n*}(x)]\,dx = \int_0^\infty [1-G^{n*}(x)]\,dx - \int_0^t [1-G^{n*}(x)]\,dx$$

$$= n\mu - t + \int_0^t G^{n*}(x)\,dx$$

$$= n\mu - t + (t - n\mu(t))_+ F(t)^n \tag{2}$$

The upper bound is

$$\int_t^\infty [1-H^{n*}(x)]\,dx = \int_0^\infty [1-H^{n*}(x)]\,dx - \int_0^t [1-H^{n*}(x)]\,dx$$

$$= n\mu - t + t\left[\frac{t-\mu(t)}{t}\; F(t)\right]^n \tag{3}$$

Now we multiply (1) by p_n and sum over n. Using (2) and (3) we get the desired bounds:

$$\mu E[N] - t\left\{1 - \sum_{n=0}^{\infty} p_n F^n(t)\left(1 - n\frac{\mu(t)}{t}\right)_+\right\} \tag{4}$$

$$\leqslant E[(S-t)_+]$$

$$\leqslant \mu E[N] - t\left\{1 - \sum_{n=0}^{\infty} p_n F^n(t)\left(1 - \frac{\mu(t)}{t}\right)^n\right\} \tag{5}$$

Remark 1. The upper bound (5) may be rewritten as follows using the probability generating function $\Phi_N(u) = E[u^N]$:

$$E[(S-t)_+] \leqslant \mu E[N] - t + t\,\Phi_N\left(F(t)\left[1 - \frac{\mu(t)}{t}\right]\right) \tag{6}$$

Remark 2. Using partial integration in the definition of $\mu(t)$, one may re-write upper bound (5) in another way as:

$$E[(S-t)_+] \leqslant \mu E[N] - t\left\{1 - \sum_{n=0}^{\infty} p_n \left[\frac{1}{t}\int_0^t F(y)\,dy\right]^n\right\} \tag{7}$$

Remark 3. In case X has a finite upper bound M, for $t > M$ (4) reduces to the following well-known lower bound (see *Goovaerts et al.*, 1984):

$$E[(S-t)_+] \geqslant \mu \sum_{n=[t/\mu+1]}^{\infty} p_n(n - t/\mu) \tag{8}$$

Example. Let the number of claims follow a Poisson (μ), the size of claims an exponential (1) distribution. Then we have the following upper bound for the stop-loss premiums:

$$E[(S-t)_+] \leqslant \mu - t\left\{1 - \sum_{n=0}^{\infty} \frac{e^{-\mu}}{n!}\left[\frac{\mu}{t}\int_0^t (1-e^{-y})\,dy\right]^n\right\}$$

$$= \mu - t + t\sum_{n=o}^{\infty} \frac{e^{-\mu}}{n!}\mu^n t^{-n}\{t - 1 + e^{-t}\}^n$$

$$= \mu - t\left[1 - \exp\left(-\frac{\mu}{t}\{1 - e^{-t}\}\right)\right]. \tag{9}$$

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Abstract

Surprisingly simple upper and lower bounds for stop-loss premiums of compound distributions are deduced. The information needed is the claim number distribution, the mean claim, the mean claim less than the retention t, and the probability of a claim less than t.

Zusammenfassung

Es werden überraschend einfache obere und untere Schranken für Stop-Loss-Prämien von zusammengesetzten Verteilungen hergeleitet. Die benötigten Informationen sind: Verteilung der Anzahl Schäden, der mittlere Schaden, der mittlere Schaden in der Höhe kleiner als der Selbstbehalt t, die Wahrscheinlichkeit eines Schadens in der Höhe kleiner als t.

Résumé

L'article présente des bornes inférieures et supérieures étonnement simple dans le cas des primes stop-loss liées à des lois de distribution composées. Sont nécessaires de connaître la distribution du nombre des sinistres, l'espérance mathématique du montant des sinistres, celle du montant des sinistres inférieurs au plein t et la probabilité d'obtenir un sinistre inférieur à t.