Kurzmitteilungen

D. Kurzmitteilungen

Hans U. Gerber, Lausanne

Verlustvortrag und Zufallswege

In der Notation von Strickler (1982) seien X_i die Schadenlast der Periode i, P_i die für die Periode i anrechenbare Prämie, und G_i der GA-pflichtige Gewinn (unter Verlustvortrag) aus Periode i ($i = 1, 2, \ldots$). Wir betrachten das «stabile» Modell, d.h. $X_1, X_2, \ldots$ seien unabhängige und gleichverteilte Zufallsvariablen, und $P_i = P$ sei unabhängig von i. Die folgende Formel wurde von Strickler (1982, Seiten 247–250) für $k = 2$ und 3 hergeleitet (und für beliebiges k vermutet):

$$E[G_k] = \frac{1}{k} E\left[\left(kP - \sum_{i=1}^{k} X_i\right)_+\right] \tag{1}$$

Jones und Gerber (1974, Formel (19) auf Seite 84) haben gezeigt, dass diese Formel tatsächlich allgemein gilt. Die Idee ist, den Zufallsweg $S_1, S_2 \ldots$ zu betrachten, wobei

$$S_k = \sum_{i=1}^{k} Y_i, \qquad \text{mit} \qquad Y_i = P - X_i. \tag{2}$$

Weiter sei

$$M_k = \max(0, S_1, S_2, \ldots, S_k). \tag{3}$$

Ein klassisches Resultat aus der Theorie der Zufallswege besagt, dass

$$E[M_k - M_{k-1}] = \frac{1}{k} E[(S_k)_+], \tag{4}$$

siehe beispielsweise Dinges (1963, Seite 287). Auf der anderen Seite prüft man leicht nach, dass $G_k = M_k - M_{k-1}$. Formel (1) ergibt sich also damit aus Formel (4).

Dinges hat meines Wissens als erster erkannt, dass Formel (4) mit einer kombinatorischen Überlegung hergeleitet werden kann. Sei $y = (y_1, y_2, \ldots, y_k)$, wobei die y_i beliebige Zahlen sind mit $y_1 + y_2 + \ldots + y_k > 0$. Wir setzen $s_i(y) = y_1 + \ldots + y_i$, und $m_i(y) = \max(0, s_1(y), \ldots, s_i(y))$. Dann gilt

$$\sum_z [m_k(z) - m_{k-1}(z)] = y_1 + \ldots + y_k, \tag{5}$$

wobei die Summation über alle zyklischen Permutationen der ursprünglichen Folge geht. Diese Formel sei anhand eines Beispiels ($k = 6$) illustriert:

Permutation						m_5	m_6	m_6-m_5
−1	2	4	−3	2	3	5	7	2
2	4	−3	2	3	−1	8	8	—
4	−3	2	3	−1	2	6	7	1
−3	2	3	−1	2	4	3	7	4
2	3	−1	2	4	−3	10	10	—
3	−1	2	4	−3	2	8	8	—
						Total		7

Das Total (7) ist tatsächlich $-1+2+4-3+2+3$! Im Falle, wo $y_1+y_2+\ldots+y_k \leq 0$, ist $m_k(z)=m_{k-1}(z)$ für alle z. Allgemein gilt daher

$$\sum_z [m_k(z)-m_{k-1}(z)]=(y_1+\ldots+y_k)_+ . \tag{6}$$

Formel (4) erhält man schliesslich, indem man zuerst den bedingten Erwartungswert, gegeben dass $(Y_1, Y_2, \ldots, Y_k)$ eine zyklische Permutation von $(y_1, y_2, \ldots, y_k)$ sei, unter Benützung von (6) ausrechnet.
Wegen des Verlustvortrages vermutet man, dass $E[G_k]$ eine fallende Folge in k sei. Anhand der ursprünglichen Definition

$$G_k=(kP-X_1-\ldots-X_k-G_1-\ldots-G_{k-1})_+ \tag{7}$$

ist es aber schwierig, diese plausible Vermutung direkt zu beweisen. Benützt man dagegen den Zusammenhang mit den Zufallswegen, so gilt es zu zeigen, dass die Folge

$$\frac{1}{k} E[(S_k)_+] \quad \text{fallend in } k \tag{8}$$

ist, was bedeutend einfacher ist: Seien $y_1, \ldots, y_k$ beliebige Zahlen, und A das Ereignis, dass $(Y_1, Y_2, \ldots, Y_k)$ eine Permutation von $(y_1, y_2, \ldots, y_k)$ sei. Aus Jensen's Ungleichung folgt, dass

$$E[(S_{k-1})_+|A]=\frac{1}{k}\sum_{i=1}^{k}\left(\sum_{j\neq i} y_j\right)_+ \tag{9}$$

$$\geq \frac{k-1}{k}\left(\sum_{j=1}^{k} y_j\right)_+$$

Indem man über A integriert, erhält man

$$E[(S_{k-1})_+] \geq \frac{k-1}{k} E[(S_k)_+], \tag{10}$$

was die Richtigkeit von (8) beweist.

Hans U. Gerber
Institut de Sciences actuarielles
Bâtiment B.F.S.H.
1015 Lausanne-Dorigny

Literatur

Dinges, H.: «Eine kombinatorische Überlegung und ihre masstheoretische Erweiterung». *Zeitschrift für Wahrscheinlichkeitsrechnung, 1* (1963), 278–287.

Jones, D. A., und *Gerber, H. U.*: «Dividend Formulas in Group Insurance". *Transactions of the Society of Actuaries, 26* (1974), 77–93.

Strickler, P.: «In welchem Ausmass kann in der Lebensversicherung der Versicherungsnehmer am Risikogewinn einzelner Gruppen oder Verträge beteiligt werden?». *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, 1982* (1982), 239–260.

HILARY L. SEAL, Apples

Distributions of Claim Amounts – Continuous or Discrete?

Realizations of any random variable are necessarily discrete quantities. For example, the height of an individual is seldom measured more accurately than to the nearest centimetre. But some measurements are recognized as being convenient approximations to a variate that may run to many more decimal places than those shown. Thus an automobile claim in Switzerland is necessarily expressed in francs and centimes modulo 5; in Italy it is in lire modulo 20. This discreteness is an artifact of the observation of continuous variates.

The probability distribution of claim amounts is thus, in general, a continuous distribution. Several of the distributions fitted to claim sizes (e.g. Pareto, lognormal) have excellent credentials from successful fittings in other fields. There are, however, natural discrete distributions. In Group (one-year-term) Life insurance in the U.S. it is usual to fix sums assured modulo \$1000 (see e.g. Mereu, 1972). The individual claims of a 10-year period thus form a discrete distribution.

On the other hand, many operations on continuous distributions use selected ordinates therefrom. For example, in order to approximate $F(x, t)$, the (cumulative) distribution function of aggregate claim amounts during a period $(0, t)$, Pentikäinen (1947) successfully replaced the multi-humped distributions of actual individual claim amounts in three Finnish *Sterbekassen* by two spikes of probability at appropriate distances apart. Another example is the replacement of $\partial F(x, t)/\partial x \equiv f(x, t)$ by f_r and of $b(y)$, the density of individual claim amounts, by b_j. The well-known relation for the characteristic function of $f(x, t)$ with Poisson claim numbers (e.g. Bohman, 1971) can then be replaced by a relation connecting the probability generating functions of f_r and b_j, namely (Khatri and Patel, 1961)

$$\sum_{r=0}^{\infty} f_r z^r \equiv g(z) = \exp\left[-t + \sum_{j=0}^{\infty} b_j z^j\right] \equiv e^{-t+h(z)}.$$

Differentiating r times with respect to z

$$g^{(r)}(z) = \sum_{k=1}^{r} \binom{r-1}{k-1} h^{(k)}(z) g^{(r-k)}(z)$$

and putting $z=0$

$$f_r = \frac{1}{r} \sum_{k=1}^{r} k b_k f_{r-k} \qquad \text{with } f_0 = e^{-t+b_0}. \tag{1}$$

Relation (1) is a recursion formula for $f_1, f_2, f_3 \ldots$ in succession and has been known and used in biometry since 1939 (Beall and Rescia, 1953) but it may be derived in essentially the same form by midpoint quadrature of the integral involved in a similar manipulation of the characteristic function relation (Plackett, 1969; Seal, 1971).

There has recently been a tendency to replace the densities of a continuous $b(y)$ with discrete spikes b_j. The real difference between protagonists of continuous and discrete distributions is, I believe, the small number of j-values used compared with the large number of ordinates employed in quadratures. For example, Strauss (1976) used 158 terms for b_j, Seal (1973, Ch. 3) 79 terms, and Bertram (1981) about 375 terms. Compare these with Bohman's (1971) 1024 values of the function $b(y)$ in his fast-Fourier inversion and the same number in Seal's (1978) inversions. Perhaps we should conclude that there is no essential difference between continuous and discrete representations of the distribution of claim amounts provided about 1000 terms are used for the b_j's. However we mention that Bertram (1981) required 5 seconds to calculate a set of f_r-values using the negative binomial form of (1) but only one-twelfth of that time to produce the same number of values by means of a fast-Fourier inversion.

Prof. Hilary L. Seal
La Mottaz
1143 Apples

References

Beall, G. and Rescia, R. R. (1953): A generalization of Neyman's contagious distributions. *Biometrics 9*, 354–386.

Bertram, J. (1981): Numerische Berechnung von Gesamtschadenverteilungen. *Bl. Deuts. Gesell. Versich.-math. 15*, 175–194.

Bohman, H. (1971): Ruin probabilities. *Skand. Aktuar.-tidskr. 54*, 159–163.

Khatri, C. G. and *Patel, I. R.* (1961): Three classes of univariate discrete distributions. *Biometrics 17*, 567–575.

Mereu, J. A. (1972): An algorithm for computing expected stop-loss claims under a group life contract (with discussion). *Trans. Soc. Actu. 24*, 311–347.

Pentikäinen, T. (1947): Einige numerische Untersuchungen über das risikotheoretische Verhalten von Sterbekassen. *Skand. Aktuar.-tidskr. 30*, 75–87.

Plackett, R. L. (1969): Stochastic models of capital investment (with discussion). *Jour. Roy. Statist. Soc. B 31*, 1–28.

Seal, H. L. (1971): Numerical calculation of the Bohman-Esscher family of convolution-mixed negative binomial distribution functions. *Mitt. Verein. schweiz. Versich.-mathr. 71*, 71–94.

Seal, H. L. (1978): *Survival Probabilities: The Goal of Risk Theory*. Wiley, Chichester.

Strauss, J. (1976): Computerverfahren zur Bestimmung der Gesamtschadenverteilung und ihre Anwendung zur Abschätzung der Grossschadenentwicklung. *Trans. 20th Intern. Cong. Actu. 3*, 763–772.

Baruch Berliner, Zürich

Reply to Bjørn Sundt's remarks on "Minimum Entropy in Risk Theory", BASA, vol. 2, 1982

In referring to the paper "On the Use of the Maximum Entropy Concept in Insurance" by Mr. B. Lev and myself, Mr. Sundt writes: "I have not the prerequisites for discussing the applicability of the maximum entropy principle in statistical mechanics and thermodynamics. However, these disciplines seem to have very little to do with risk theory. One should, therefore, not *uncritically* transfer a concept from these disciplines to risk theory". Just prior to making this statement Mr. Sundt maintains that Mr. Maeder in his article refers *uncritically* to our paper. I am afraid that Mr. Sundt is using the adverb "uncritically" uncritically often without any convincing argument.

The purpose of his remarks is to show that the principle of maximum entropy – which he calls unconvincingly "minimum" entropy, thus confusing an actuarial with a satirical remark – has nothing to do with risk theory. He maintains that axiom (c) (the composition law) in our paper "does not seem to be clearly motivated within the framework of risk theory" and without explaining this assertion he moves on to point out some inaccuracies in the paper that are completely irrelevant to the applicability of the maximum entropy principle in insurance.

This applicability is on the other hand described and interpreted in the paper's third section and should be easily explicable to every careful reader. Later, in section 7 for example, the derivation of the Poisson process as a result of the concept of maximum entropy subject to the constraints

$$\int_{t_0}^{\infty} f(t)\,dt = 1 \qquad \text{and} \qquad \int_{t_0}^{\infty} t f(t)\,dt = \bar{t} < \infty$$

gives the Poisson process a new interpretation and underlines the fundamental significance of the principle of maximum entropy in risk theory despite its being also important in statistical mechanics, information theory and many other fields.

Baruch Berliner
Swiss Reinsurance Company
Postfach
8022 Zürich

William S. Jewell, Berkeley

Corrigendum to "Exact Multi-dimensional Credibility", BASA, vol. 2, 1974

In the course of reviewing my above mentioned article the following typographical errors were found:

page 209, (64) n_{0i} should be n_{i0};

page 209, (66) $A\pi^{-1}A^{-1}$ should be $A\pi^{-1}A'$;

page 211, (75), (76) λ should be μ (twice);

page 211, (79) should be $N^{-1}(N+I)^{-1}U_0$.

In addition, a fundamental error in the reasoning on pages 210–211 for the enriched prior for the multinormal has been discovered, which invalidates (71), (72) and (81). Essentially, when the rank-p diagonal random precision π is "enriched" to the matrix precision ω, this induces constraints on ω, such as $\omega N = N'\omega$, and relationships between the hyperparameters, such as $NQ_0 = Q_0N'$. This in turn means that terms like $x_t x_t'$ are no longer sufficient statistics for ω^{-1}, and that "enlarged versions" of the data and prior mean, X_t, $\bar{X}$, and M must be substituted in (81), for x_t, $\bar{x}$, and m, where

$$X_t = A\,\mathrm{diag}(A^{-1}x_t);\ \text{etcetera.}$$

This new transformation is helped by the fact that the matrix A can be taken as the collection of p independent eigenvectors of N. The "richness" of Z, and the exactness of the credible mean (7) are unaffected.

This problem is treated in more detail in "Enriched Multinormal Priors Revisited", Report ORC 82-14, November, 1982 which is available from the author, and will also be published in the *Journal of Econometrics* in 1983.

William S. Jewell
IEOR, 4173 EH
University of California
Berkeley, CA 94720
USA