Upper and lower bounds on stop-loss premiums in case of known expectation and variance of the risk variable

Autor(en): **De Vylder, F.**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **- (1982)**

Heft 1

PDF erstellt am: **27.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-966981

F. DE VYLDER, Denderleeuw, and M. GOOVAERTS, Leuven

Upper and Lower Bounds on Stop-loss Premiums in Case of Known Expectation and Variance of the Risk Variable

1 Primal problems

Let $[a,b]$ be a finite interval, e,m,m_2 real numbers. We consider the following problems

$$p_1(m,m_2)=\sup\left(\int_a^b (x-e)_+ dF(x) / \int_a^b x dF(x)=m, \int_a^b x^2 dF(x)=m_2, \int_a^b dF(x)=1\right),$$

$$q_1(m,m_2)=\inf\left(\int_a^b (x-e)_+ dF(x) / \int_a^b x dF(x)=m, \int_a^b x^2 dF(x)=m_2, \int_a^b dF(x)=1\right),$$

where the supremum (infimum) is over the distributions F on $[a,b]$ satisfying the constraints indicated after the slash. Thus, m is the first-order and m_2 the second order moment of the probability distribution F. The corresponding variance is $s^2=m_2-m^2$. We consider m,m_2 as independent parameters with domain C' to be specified later and s^2 as an abbreviation for m_2-m^2. Of course, we assume that the retention e is in the interval $[a,b]$. We assume a and b to be finite, but it is possible to let $b\uparrow\infty$ in most of the final results.
The *value* of problem $p_1(m,m_2)$ is the indicated supremum $p_1(m,m_2)$. No confusions arise from the fact that $p_1(m,m_2)$ denotes at the same time the whole problem and its value. A *solution* of problem $p_1(m,m_2)$ is a distribution F satisfying the constraints of that problem and such that

$$\int_a^b (x-e)_+ dF(x)=p_1(m,m_2).$$

Similar agreements and terminology are applied to problem $q_1(m,m_2)$ and to other problems to be considered later.
Using the methods developed in *De Vylder* (1982), we shall find the value and solution of problem $p_1(m,m_2)$. Of course, these methods also apply, after obvious adaptations, to problem $q_1(m,m_2)$. The just mentioned paper shall be abbreviated as DV in the rest of this note.

Some aspects of problem $p_1(m,m_2)$ are developed by *Bowers* (1969), *Taylor* (1977) and *Heilmann* (1981).
We say that a probability distribution F is *n-atomic* if all its probability mass is concentrated in n points at most. Then the latter are called the *atoms* of the distribution. From a general result by *Taylor* (1977) (or DV 2.6.2) results that the problems $p_1(m,m_2)$, $q_1(m,m_2)$ have 3-atomic solutions. We shall see that $p_1(m,m_2)$ has in fact a 2-atomic solution and $q_1(m,m_2)$ a 3-atomic one.
If α,β are two different atoms of the 2-atomic probability distribution F satisfying the first-order moment constraint $\int x dF = m$, then the corresponding probability masses p_α, p_β must necessarily be

$$p_\alpha = \frac{m-\beta}{\alpha-\beta}, \qquad p_\beta = \frac{m-\alpha}{\beta-\alpha}.$$

If α, β, γ are different atoms of the 3-atomic probability distribution F satisfying the moment constraints $\int x dF = m$, $\int x^2 dF = m_2$, then the corresponding probability masses can only be

$$p_\alpha = \frac{s^2+(m-\beta)(m-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \ p_\beta = \frac{s^2+(m-\alpha)(m-\gamma)}{(\beta-\alpha)(\beta-\gamma)}, \ p_\gamma = \frac{s^2+(m-\alpha)(m-\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Indeed (in case of the 3-atomic distribution), the moment constraints and the relation expressing that F is a probability distribution furnish three linear equations in p_α, p_β, p_γ with the unique indicated solution.
Thus the 2 and 3-atomic solutions of the problems $p_1(m,m_2)$, $q_1(m,m_2)$ are completely specified by the atoms of these solutions.
Before we state the unique theorem of this paper, it is a pleasure to mention that this note has been motivated by the intrest of *Dr. H. Schmitter*, as results from private correspondence with one of the authors.
The rest of this paper is devoted to the demonstration of the following theorem.

Theorem

For (m,m_2) belonging to the domain C' (defined and explicited in 3.1), the problems $p_1(m,m_2)$, $q_1(m,m_2)$ have the value and solution indicated in table 1 (at the end of the note).

2 Related problems

To the problems $p_1(m,m_2)$, $q_1(m,m_2)$, we associate the dual problems (DV 1.4.2, 1.7.3, 2.3.1)

$$p_2(m,m_2)=\inf\left(y_1 m+y_2 m_2+y_3/y_1 x+y_2 x^2+y_3\geq(x-e)_+,\ (a\leq x\leq b)\right),$$

$$q_2(m,m_2)=\sup\left(y_1 m+y_2 m_2+y_3/y_1 x+y_2 x^2+y_3\leq(x-e)_+,\ (a\leq x\leq b)\right),$$

where the infimum (supremum) is over the triplets $(y_1, y_2, y_3)\in R^3$ satisfying the constraints indicated after the slash.
With the change of variables

$$y_1=z_1+\tfrac{1}{2},\ y_2=z_2,\ y_3=z_3-\tfrac{e}{2},$$

we have

$$p_2(m,m_2)=\tfrac{1}{2}(m-e)+p_3(m,m_2),\ q_2(m,m_2)=\tfrac{1}{2}(m-e)+q_3(m,m_2),$$

where

$$p_3(m,m_2)=\inf\left(z_1 m+z_2 m_2+z_3/z_1 x+z_2 x^2+z_3\geq\tfrac{1}{2}|x-e|,\ (a\leq x\leq b)\right),$$

$$q_3(m,m_2)=\sup\left(z_1 m+z_2 m_2+z_3/z_1 x+z_2 x^2+z_3\leq\tfrac{1}{2}|x-e|,\ (a\leq x\leq b)\right),$$

where the infimum (supremum) is over the triplets $(z_1, z_2, z_3)\in R^3$ satisfying the constraints indicated after the slash.
This change of variables has nothing essential, but it makes some discussions more symmetric.

3 Domain of parameters. Solution of the problems on the frontier of this domain

3.1

The *domain of the parameters* m,m_2 is defined to be the set

$$C'=\left\{\left(\int_a^b x\,dF,\ \int_a^b x^2 dF\right)/F \text{ probability distribution on } [a,b]\right\}$$

of all possible values of the couple (m,m_2) corresponding to some probability distribution F on $[a,b]$.

Let E' be the curve (fig. 1, see p. 154) with parametric equations

$$X = x, \; Y = x^2, \; (a \leq x \leq b).$$

Then the interior C'^0 of C' is the interior of the smallest convex set containing E' (DV 1.2.2, 1.10.3, *A*19). This smallest convex set is delimited by the curve E' and by the straight line segment joining the extremities a', b' of E'.
The equation of E' is $Y = X^2$ $(a \leq X \leq b)$. Writing m for X and m_2 for Y it is also $s^2 = 0$ $(a \leq m \leq b)$. The equation of the segment $a'b'$ is

$$Y(b-a) = (X-a)b^2 + (b-X)a^2, \qquad (a \leq X \leq b)$$

or

$$s^2 = (m-a)(b-m), \; (a \leq m \leq b).$$

Thus,

$$C'^0 = \{(m, m_2)/a < m < b, \; 0 < s^2 < (m-a)(b-m)\}.$$

We shall prove that

$$C' = \{(m, m_2)/a \leq m \leq b, \; 0 \leq s^2 \leq (m-a)(b-m)\},$$

i.e. that all frontier points of C'^0 belong to C'.

3.2

Let m, m_2 satisfy $a \leq m \leq b$, $s^2 = m_2 - m^2 = 0$. Then the 1-atomic distribution with probability mass 1 at m has m and m_2 for first and second-order moments resp. This means that all points of E' belong to C'.
We notice that the relation $s^2 = 0$ is characteristic of the 1-atomic probability distributions.

3.3

Let m, m_2 satisfy $a \leq m \leq b$, $s^2 = (m-a)(b-m)$. Then the 2-atomic distribution with probability masses

$$p_a = \frac{b-m}{b-a}, \; p_b = \frac{m-a}{b-a}$$

at the points a, b resp. has m and m_2 for first and second-order moments resp. This means that all points of the segment $a'b'$ belong to C'.
The relation $s^2 = (m-a)(b-m)$ is characteristic of the 2-atomic probability distributions with probability mass concentrated at the extremities of $[a,b]$.

Indeed, let F be any probability distribution on $[a,b]$ such that $s^2=(m-a)(b-m)$. Then

$$\int_a^b (x-a)(b-x)\,dF(x)=mb+am-ab-m_2=0.$$

Then the nonnegative function $(x-a)(b-x)$ on $[a,b]$ must be zero F-almost everywhere on $[a,b]$, i.e. that F has all its mass concentrated at a and b. Conversely, if the probability distribution F has all its mass concentrated at a and b, then it is easily verified that $s^2=(m-a)\,(b-m)$.

3.4

For (m,m_2) on the frontier of C', the direct verification of the validity of table 1 is easy now.
From now on (except, of course, in the final table 1) only points (m,m_2) in the interior C'^0 of C' shall be considered.

4 General method of demonstration

Let E be the curve with parametric equations

$$X=x,\ Y=x^2,\ Z=\tfrac{1}{2}|x-e|,\ (a\leq x\leq b).$$

Let E_1 (E_2) be the part of E corresponding to the parameter values $x\leq e$ ($x\geq e$). See fig. 2. The projection of E on the XY-plane is the curve E' considered before. Let C be the smallest convex set containing E. Then the projection of C on the XY-plane is the domain of parameters C'.
Let (m,m_2) be a point in C'^0. The vertical through $(m,m_2,0)$ intersects the upper frontier of C in a point, say $P=(m,m_2,p)$. It intersects the lower frontier of C in a point, say $Q=(m,m_2,q)$. Then, by DV 1.10.1, $p=p_3\,(m,m_2)$ is the value of problem $p_3\,(m,m_2)$. Similarly, $q=q_3\,(m,m_2)$ is the value of problem $q_3\,(m,m_2)$.
Let $Z=z_1X+z_2Y+z_3$ be (the equation of) a plane through P tangent to C. Then, by DV 1.10.1, (z_1,z_2,z_3) is a solution of problem $p_3(m,m_2)$. Similarly, if $Z=z_1X+z_2Y+z_3$ is a plane through Q tangent to C, then (z_1,z_2,z_3) is a solution of problem $q_3\,(m,m_2)$. Only in exceptional cases, the considered planes and corresponding solutions are not unique.
From the value and solution of problem $p_3\,(m,m_2)$ ($q_3\,(m,m_2)$), we obtain the value and solution of problem $p_2\,(m,m_2)$ ($q_2\,(m,m_2)$).

Fig. 1
Domain of parameters C'

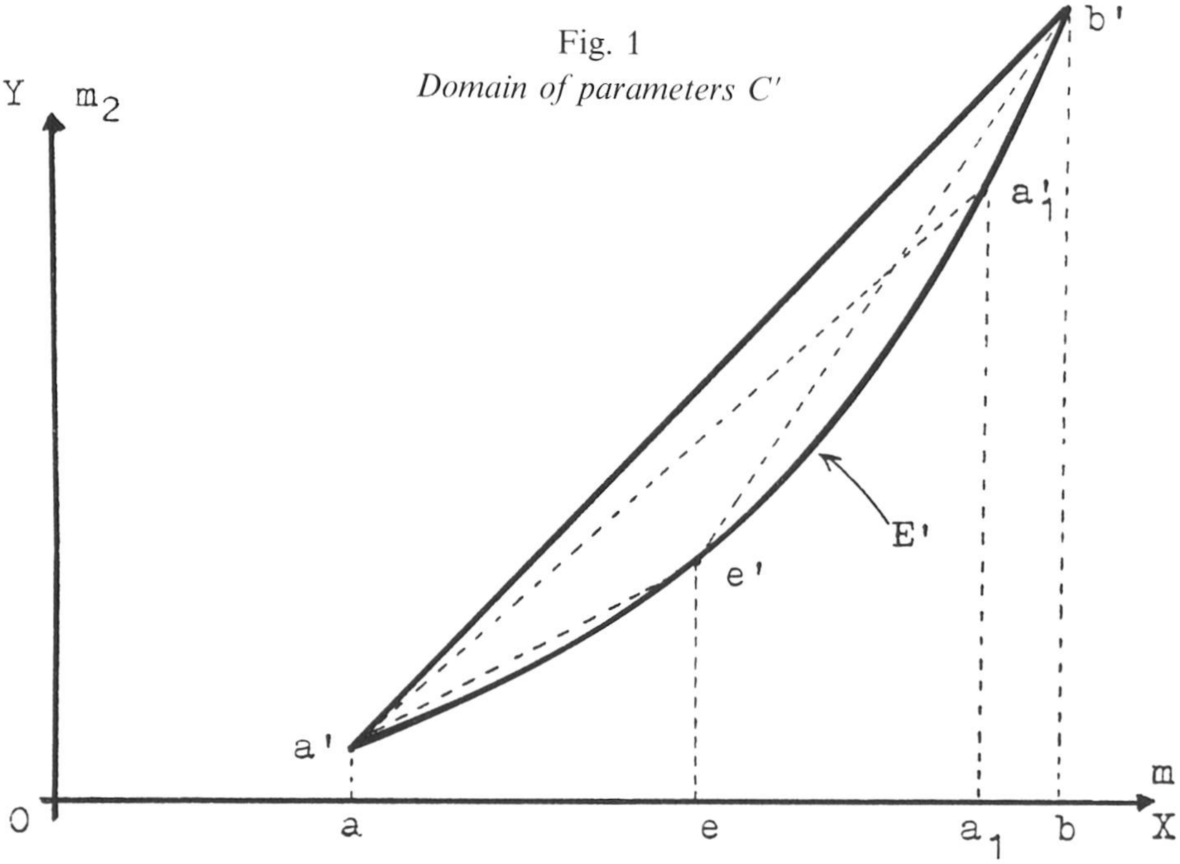
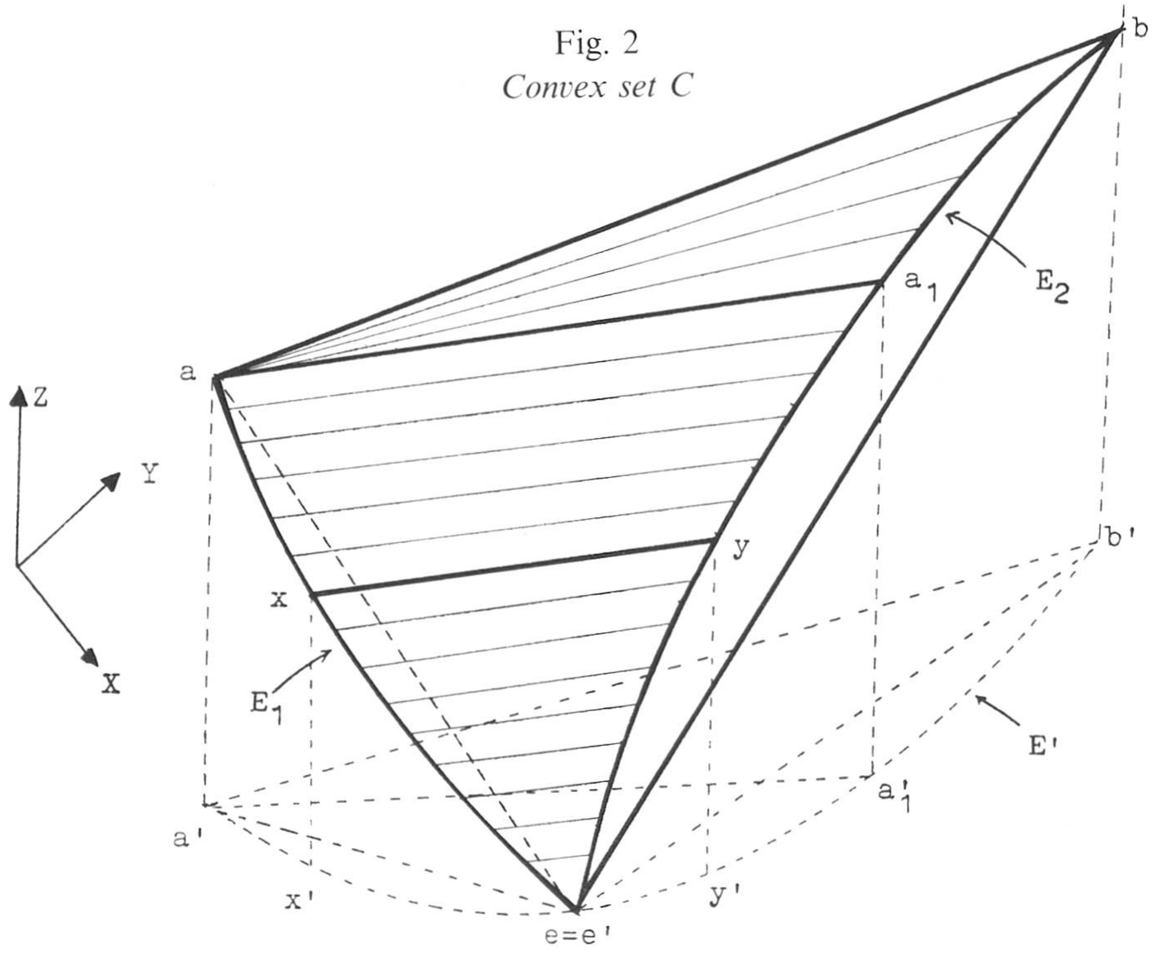


Fig. 2
Convex set C



By DV 2.4.1, the problems $p_2(m,m_2)$ and $p_1(m,m_2)$ have the same value: $p_2(m,m_2) = p_1(m,m_2)$. Similarly, $q_2(m,m_2) = q_1(m,m_2)$.
Finally, if (y_1,y_2,y_3) is a solution of problem $p_2(m,m_2)$ $(q_2(m,m_2))$, then by DV 2.5.1, the atoms of any atomic solution of problem $p_1(m,m_2)$ $(q_1(m,m_2))$ must be roots of the *atoms equation* in x,

$$y_1 x + y_2 x^2 + y_3 = \tfrac{1}{2}(x-e)_+ .$$

In most cases (exceptions follow), this equation completely determines the atoms of the solution and then also the solution itself (see discussion in section 1).
Once an atomic solution of problem $p_1(m,m_2)$ $(q_1(m,m_2))$ is found, the following verifications must be possible (they are left to the reader in most cases, in the sequel):

- the atoms are in $[a, b]$
- the corresponding probability masses are nonnegative
- the sum of all probability masses is 1
- the two moment constraints are satisfied
- for the solution, say F, the integral $\int (x-e)_+ dF$ equals the value of the problem (obtained more directly from the upper or lower frontier of C, as described before)
- the duality equality (see DV 1.6.1)

$$\int (x-e)_+ dF = y_1 m + y_2 m_2 + y_3 ,$$

where (y_1,y_2,y_3) is solution of $p_2(m,m_2)$ $(q_2(m,m_2))$, must be satisfied.
That duality equality can also be used in order to extend to C' results proved for C'^0. In section 3, we already justified in a more direct way the validity of table 1 on the frontier of C'.

5 Geometry of the curve E

5.1

The main problem left is the determination of the smallest convex set C containing E. This smallest convex set C is the intersection of all half-spaces containing E. (Any plane in R^3 divides R^3 in two half-spaces.) The determination of C shall be immediate from the considerations of this section.
For any *number* x in $[a,b]$, we also denote by x the *point* of E corresponding to the value x of the parameter, i.e. the point $(x, x^2, \frac{1}{2}|x-e|)$. Accents are systematically

used for projections on the XY-plane of points or sets in R^3. Thus, $a', x', \ldots, E', C', \ldots$ are the projections on the XY-plane of the points $a, x, \ldots$ and the sets $E, C, \ldots$ respectively.

5.2

The curve E_1 is in the plane $Z=\frac{1}{2}(e-X)$. The curve E_2 is in the plane $Z=\frac{1}{2}(X-e)$.

5.3

The plane through the points a, e, b of E is

$$Z=\frac{1}{2(b-a)}(-(a+b+2e)X+2Y+(a+b)e).$$

5.4

Let x be a point of E_1, y a point of E_2. Then the plane through x and y, tangent to E_2 at y, is

$$Z=\frac{1}{2(y-x)^2}(((x+y)^2-4ey)X+2(e-x)Y+e(y^2-x^2+2xy)-2xy^2).$$

Indeed, if $Z=z_1X+z_2Y+z_3$ is the equation of that plane, then z_1, z_2, z_3 must satisfy the relations

$$\begin{aligned} \tfrac{1}{2}(e-x) &= z_1x+z_2x^2+z_3 \quad \text{(the plane contains } x\text{)} \\ \tfrac{1}{2}(y-e) &= z_1y+z_2y^2+z_3 \quad \text{(the plane contains } y\text{)} \\ \tfrac{1}{2} &= z_1 \;\; +2z_2y \end{aligned}$$

The last equation, expressing the tangency at y, is the derivative in y of the preceding equation.

5.5

Let x be a point of E_1, y a point of E_2. Then the plane through x and y, tangent to E_1 at x, is

$$Z=\frac{1}{2(y-x)^2}((4ex-(x+y)^2)X+2(y-e)X+e(y^2-x^2-2xy)+2x^2y).$$

5.6

In 5.4 and 5.5, suppose that the points x and y are at the same height $\frac{1}{2}(e-x) = \frac{1}{2}(y-e)$. Then the plane of 5.4 is the same as the plane of 5.5. Its equation is

$$Z = \frac{1}{4(e-x)}\left(-2eX + Y + e^2 + (e-x)^2\right).$$

Thus, this plane contains x and y and is tangent to E_1 at x and tangent to E_2 at y.

6 Value and solution of the primal maximization problem

6.1. *Partition of the domain of parameters in the case $e \leq c$*

We use c as an abbreviation for $\frac{1}{2}(a+b)$.
Let us assume first that the point b is higher than the point a, i.e. $\frac{1}{2}(e-a) \leq \frac{1}{2}(b-e)$ or $e \leq c$. Let a_1 be the point of E_2 at the same height as a, i.e. $\frac{1}{2}(e-a) = \frac{1}{2}(a_1-e)$ or $a+a_1 = 2e$. In the variables $m = X$ and $m_2 = Y$, the equation of the straight line through the projections a', a_1' is $s^2 = (m-a)(a_1-m)$ (compare with the equation of the straight line through a', b' in 3.1), or

$$s_{me} = e - a,$$

where we use the abbreviation

$$s_{me} = +\left(s^2 + (m-e)^2\right)^{1/2}.$$

We denote by C_1', C_2' the parts of C' characterized by the relations

C_1': $s_{me} \leq e-a$ (delimited by E' and the segment $a'a_1'$),

C_2': $s_{me} \geq e-a$ (delimited by E_2' and the segments $a'a_1', a'b'$).

6.2. *Case C_1'*

On C_1' the upper frontier of C is composed of horizontal straight segments xy joining points x of E_1 and y of E_2. This follows from 5.6. The plane of 5.6 is tangent to C along the segment xy. The points of the projected segment $x'y'$ are characterized by the relation $s_{me} = e-x$ (compare with the equation of the straight line through a', a_1' in 6.1). Let (m,m_2) be a fixed point on the segment $x'y'$. Then

$$p_3(m,m_2)=\frac{1}{4s_{me}}\,(-2em+m_2+e^2+s_{me}^2)=\frac{1}{2}\,s_{me},$$

$$p_1(m,m_2)=p_2(m,m_2)=p_3(m,m_2)+\tfrac{1}{2}\,(m-e)=\tfrac{1}{2}\,(s_{me}+m-e).$$

The solution of problem $p_3(m,m_2)$ is

$$z_1=-\frac{e}{2s_{me}},\quad z_2=\frac{1}{4s_{me}},\quad z_3=\frac{e^2+s_{me}^2}{4s_{me}}.$$

The corresponding solution of problem $p_2(m,m_2)$ is

$$y_1=-\frac{1}{2s_{me}}\,(e-s_{me}),\quad y_2=\frac{1}{4s_{me}},\quad y_3=\frac{(e-s_{me})^2}{4s_{me}}.$$

The atoms equation is

$$-2(e-s_{me})x+x^2+(e-s_{me})^2=4s_{me}(x-e)_+$$

or

$$(x-(e-s_{me}))^2=4(x-e)_+s_{me}.$$

The roots of this equation are $e-s_{me}$ and $e+s_{me}$. These roots are in $[a,b]$ because $y-e=e-x=s_{me}$ and the points x, y are on E.
The points $e-s_{me}$, $e+s_{me}$ are the atoms of the 2-atomic solution of $p_1(m,m_2)$.

6.3. *Case C_2'.*

On C_2' the upper frontier of C is composed of the straight segments ay joining the point a of E_1 to a higher point y of E_2. Let $x=a$ in 5.4. Then the plane of 5.4 is tangent to C along the segment ay. The points of the projected segment $a'y'$ are characterized by the relation

$$s^2=(m-a)\,(y-m)\quad\text{or}\quad y=m+\frac{s^2}{m-a}.$$

Let (m,m_2) be a fixed point of $a'y'$. The height of a, y is respectively $\tfrac{1}{2}(e-a)$, $\tfrac{1}{2}(y-e)$. Then, by the linear interpolation formula (in symmetric form),

$$p_3(m,m_2)=\frac{1}{y-a}\,\left((m-a)\,\tfrac{1}{2}\,(y-e)+(y-m)\,\tfrac{1}{2}\,(e-a)\right)$$

$$=\frac{1}{2}\,\frac{1}{(m-a)^2+s^2}\,\left((m-a)^2(m-e)+s^2(m-a)+s^2(e-a)\right)$$

Then

$$p_1(m,m_2)=p_2(m,m_2)=p_3(m,m_2)+\tfrac{1}{2}(m-e)$$

and this gives the value indicated in table 1.
From the equation of the plane in 5.4 (with $x=a$), we find the solution (z_1,z_2,z_3) of problem $p_3(m,m_2)$. The corresponding solution of problem $p_2(m,m_2)$ is

$$y_1=\frac{a^2+y^2-2ey}{(y-a)^2},\quad y_2=\frac{e-a}{(y-a)^2},\quad y_3=\frac{2eay-ea^2-ay^2}{(y-a)^2}$$

The corresponding atoms equation is the equation in x

$$(a^2+y^2-2ey)x+(e-a)x^2+(2eay-ea^2-ay^2)=(x-e)_+(y-a)^2.$$

For $x\geq e$, this equation is $(e-a)\,(x-y)^2=0$. Its unique root is y. Clearly $e\leq y\leq b$ because the point y is on E_2. For $x\leq e$, the atoms equation becomes

$$(x-a)\,\big((e-a)x+y^2-2ey+ea\big)=0.$$

Only the root a is in $[a,b]$ (the other is always smaller than a).
Summarizing, the atoms equation has exactly the roots a, y in $[a,b]$. They are the atoms of the 2-atomic solution of problem $p_1(m,m_2)$.

6.4. *Partition of the domain of parameters in the case $e\geq c$*

In this case the point a is higher than the point b and some point b_1 of E_1 is at the same height as b. The domain C' is now partitioned in two parts by the straight line through b_1' and b'. The equation of that line is $s_{me}=b-e$. Thus we consider the subdomains C_3', C_4' of C' characterized by

$$C_3':\ s_{me}\leq b-e,$$

$$C_4':\ s_{me}\geq b-e.$$

6.5. *Case C_3'*

This case is treated in the same way as case C_1' and gives the same results.

6.6 *Case C_4'*

On C_4' the upper frontier of C is composed of the straight segments yb joining the point b of E_2 to a higher point y of E_1. In 5.5, substitute y for x and b for y. Then the plane of 5.5 is tangent to C along the segment yb. The points of the projected segment $y'b'$ are characterized by the relation

$$s^2 = (m-y)\,(b-m) \quad \text{or} \quad y = m - \frac{s^2}{b-m}.$$

Similarly as in case C_2', the value of problem $p_3(m,m_2)$ is found to be

$$p_3(m,m_2) = \frac{1}{2}\,\frac{1}{s^2+(b-m)^2}\,\big((b-m)^2(e-m)+s^2(b-m)+s^2(b-e)\big)$$

The corresponding value of problem $p_1(m,m_2)$ is indicated in table 1.
Using the plane of 5.5 (with the indicated substitutions), the solution (z_1,z_2,z_3) of problem $p_3(m,m_2)$ is easily written down and then also the solution (y_1,y_2,y_3) of problem $p_2(m,m_2)$. The corresponding atoms equation is the equation in x

$$-2y(b-e)x+(b-e)x^2+y^2(b-e) = (x-e)_+(b-y)^2.$$

For $x \leq e$, its unique root is y. Of course $a \leq y \leq e$, because the point y is on E_1. For $x \geq e$, the equation becomes

$$(b-x)\,\big((b-e)x-(y^2+eb-2ey)\big)=0.$$

Only the root b can be used (the other is always larger than b).
Summarizing, the atoms equation has exactly the roots y, b in $[a,b]$. They are the atoms of the 2-atomic solution of problem $p_1(m,m_2)$.

7 Value and solution of the primal minimization problem

7.1. *Partition of the domain of parameters*

We consider the points a',e',b' of E' corresponding to the value a,e,b of the parameter x, respectively. The equation of the straight line through a' and e', e' and b', a' and b' is, resp.

$$a'e':\quad s^2=(m-a)\,(e-m),$$

$$e'b':\quad s^2=(m-e)\,(b-m),$$

$$a'b':\quad s^2=(m-a)\,(b-m).$$

We consider the subdomains C'_5, C'_6, C'_7 of C' defined by:

C'_5: $s^2 \leq (m-a)\ (e-m)$, (delimited by E' and the segment $a'e'$)

C'_6: $s^2 \leq (m-e)\ (b-m)$, (delimited by E' and the segment $e'b'$)

C'_7: $s^2 \geq (m-a)\ (e-m)$, $s^2 \geq (m-e)\ (b-m)$,
(delimited by $a'e'$, $e'b'$, $b'a'$).

7.2. *Case* C'_5

On C'_5 the lower frontier of C is the corresponding part of the plane $Z = \frac{1}{2}(e-X)$ (see 5.2). Let (m,m_2) be fixed in C'_5. Then

$$q_3(m,m_2) = \tfrac{1}{2}(e-m),$$
$$q_1(m,m_2) = q_2(m,m_2) = q_3(m,m_2) + \tfrac{1}{2}(m-e) = 0.$$

The solution of $q_3(m,m_2)$ is $z_1 = -\frac{1}{2}$, $z_2 = 0$, $z_3 = \dfrac{e}{2}$. The corresponding solution of $q_2(m,m_2)$ is $y_1 = y_2 = y_3 = 0$.
The atoms equation is $(x-e)_+ = 0$. Its set of roots in $[a,b]$ is $[a,e]$. Because there are more than 3 roots in $[a,b]$, we cannot conclude to an atomic solution of problem $q_1(m,m_2)$. We shall try the 3-atomic solution with atoms a,y,e, where y is unspecified in (a,e) for the moment. The corresponding masses must be (see section 1):

$$p_a = \frac{s^2-(m-y)\ (e-m)}{(e-a)\ (y-a)},\quad p_y = \frac{(m-a)\ (e-m)-s^2}{(y-a)\ (e-y)},\quad p_b = \frac{s^2+(m-a)\ (m-y)}{(e-a)\ (e-y)}.$$

The points a,y,e shall be the atoms of a 3-atomic solution iff p_a,p_y,p_b are ≥ 0. This is the case for $y=m$ and then also for all y close enough to m (note that $s^2>0$ because we are always supposed to be in C'^0). We conclude that the problem $q_1(m,m_2)$ has the 3-atomic solution with atoms a,m,e but that this solution is not unique. It is easily verified that the atoms of a 3-atomic solution of problem $q_1(m,m_2)$ cannot be chosen arbitrarily in $[a,e]$.

7.3. *Case* C'_6

This case is similar to case C'_5. Now the plane $Z = \frac{1}{2}(X-e)$ is used. The value of problem $q_1(m,m_2)$, for (m,m_2) in C'_6, is found to be equal to $(m-e)$. The atoms equation is $(x-e) = (x-e)_+$. Again, it has more than 3 roots and we cannot conclude directly to a specific 3-atomic solution. Problem $q_1(m,m_2)$ has several 3-atomic solutions, in particular the one with atoms e,m,b. (For the verification of the latter fact, the already obtained value $m-e$ must be used.)

7.4. *Case* C_7'

On C_7', the lower frontier of C is the corresponding part of the plane considered in 5.3. Let (m,m_2) be a fixed point in C_7'. Then

$$q_3(m,m_2)=\frac{1}{2(b-a)}\left(-(a+b+2e)m+2m_2+(a+b)e\right).$$

The corresponding value of problem $q_1(m,m_2)$ is indicated in table 1.
From the plane in 5.3, we find the solution (z_1,z_2,z_3) of problem $q_3(m,m_2)$ and then the following solution of problem $q_2(m,m_2)$:

$$y_1=-\frac{a+e}{b-a},\quad y_2=\frac{1}{b-a},\quad y_3=\frac{ae}{b-a}.$$

The corresponding atoms equation is

$$-(a+e)x+x^2+ae=(x-e)_+(b-a).$$

Its roots are a,e,b. These are the atoms of the 3-atomic solution of problem $q_1(m,m_2)$.

8 Remark about verifications

The atoms of any atomic solution of problem $p_1(m,m_2)$ $(q_1(m,m_2))$ must necessarily satisfy the atoms equation. From this fact and the discussion in section 1 it easily follows that, if the atoms equation has 3 roots at most in $[a,b]$, they lead to an atomic solution of the problem. Then the verifications indicated in section 4 may be interesting, but are in fact superfluous. In that case, the value of the problem can also be calculated from its solution. It is not necessary then to calculate it independently.
The situation is different if the atoms equation has more than 3 roots in $[a,b]$ (see cases C_5', C_6').

Table 1

Value and solution of the primal problems

Abbreviations: $c=\frac{1}{2}(a+b)$, $s^2=m_2-m^2$, $s_{me}^2=s^2+(m-e)^2$
Domain of parameters: $a\leq m\leq b$, $0\leq s^2\leq(m-a)\ (b-m)$.

Conditions	Value of problem	Atoms of solution	
$e\leq c,\ s_{me}\leq e-a$	$\frac{1}{2}(s_{me}+m-e)$	$e-s_{me},\ e+s_{me}$	Maximization
$e\leq c,\ s_{me}\geq e-a$	$(m-a)\dfrac{s^2+(m-e)\ (m-a)}{s^2+(m-a)^2}$	$a,\ m+\dfrac{s^2}{m-a}\ (\geq e)$	
$e\geq c,\ s_{me}\leq b-e$	$\frac{1}{2}(s_{me}+m-e)$	$e-s_{me},\ e+s_{me}$	
$e\geq c,\ s_{me}\geq b-e$	$\dfrac{(b-e)s^2}{s^2+(b-m)^2}$	$m-\dfrac{s^2}{b-m}\ (\leq e),\ b$	
$s^2\leq(m-a)\ (e-m)$	0	$a,\ m,\ e$	Maximization
$s^2\leq(m-e)\ (b-m)$	$m-e$	$e,\ m,\ b$	
$\left.\begin{matrix} s^2\geq(m-a)\ (e-m) \\ s^2\geq(m-e)\ (b-m)\end{matrix}\right\}$	$\dfrac{s^2+(m-a)\ (m-e)}{b-a}$	$a,\ e,\ b$	

F. De Vylder
315, Steenweg op Ninove
B 9470 Denderleeuw
Belgium

M. Goovaerts
2, Dekenstraat
B 3000 Leuven
Belgium

References

Bowers, Jr., N. L. (1969): An upper bound on the stop-loss net premium. Transactions of the Society of Actuaries.

De Vylder, F. (1982): Best upper bounds for integrals with respect to measures allowed to vary under conical and integral constraints. To appear in the 2nd issue of Insurance: Mathematics and Economics.

Heilman, W.-R. (1981): Improved methods for calculating and estimating maximal stop-loss premiums. Blätter Deutschen Ges. Vers. Math.

Taylor, G. C. (1977): Upper bounds on stop-loss premiums under constraints on claim size distribution.

Summary

We consider a risk R with values in a given interval and with known expectation and variance. Under that incomplete information on the distribution function F of R, we solve the following problems:
- Find the maximum value of the stop-loss premium $E(R-e)_+$ corresponding to the retention limit e.
- Find the minimum value of $E(R-e)_+$.
- Find the distribution F leading to the maximum of $E(R-e)_+$.
- Find the distribution F leading to the minimum of $E(R-e)_+$.

Zusammenfassung

Die Autoren betrachten ein Risiko R mit Werten in einem gegebenen Intervall und mit bekannten Erwartungswert und Varianz. Bei diesen unvollständigen Informationen werden die folgenden Probleme gelöst:
- Man finde den maximalen Wert der Stop-Loss-Prämie $E(R-e)_+$ mit Selbstbehalt e.
- Man finde den minimalen Wert von $E(R-e)_+$.
- Man finde die Verteilung F, die zum Maximum von $E(R-e)_+$ führt.
- Man finde die Verteilung F, die zum Minimum von $E(R-e)_+$ führt.

Résumé

Les auteurs considèrent un risque R prenant des valeurs dans un intervalle donné et au sujet duquel on connaît l'espérance mathématique et la variance. Ils résolvent, sur la base de cette information incomplète, les problèmes suivants:
- Trouver la valeur maximum de la prime stop-loss $E(R-e)_+$ correspondant à un plein de conservation e.
- Trouver la valeur minimum de $E(R-e)_+$.
- Trouver la distribution F conduisant au maximum de $E(R-e)_+$.
- Trouver la distribution F conduisant au minimum de $E(R-e)_+$.