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On the Probability of Ruin in an Autoregressive Model

1. Introduction

In classical ruin theory it is assumed that the surplus process has independent increments. In a recent personal correspondence an actuary of a large insurance company pointed out that his assumption is too simplistic in many cases. In fact, a study of the annual earnings of his company showed that the correlation coefficient between the earnings of successive years was approximately one half. He then proposed an autoregressive model for the annual earnings. The purpose of this note is to examine the probability of ruin in such a model.

2. The model

We assume that U_n, the company's surplus at the end of year n, is of the form

$$U_n = u + G_1 + \dots + G_n, \tag{1}$$

where $U_0 = u$ is the initial surplus and G_i is the gain from year i. We assume that

$$G_i = X_i + a_1 G_{i-1} + \dots + a_m G_{i-m}, \tag{2}$$

where $a_1, \dots, a_m$ are certain constants, and the X_i's are independent and identically distributed random variables. It is assumed that $E[X_i] > 0$, and that the distribution of the X_i's is sufficiently regular at the tails. In particular, the equation

$$E[e^{-rX}] = 1 \tag{3}$$

should have a unique positive solution $r = R$, which is called the adjustment coefficient.

In addition to $U_0 = u$, one has to specify the initial values $g_0, \dots, g_{-m+1}$, where $G_i = g_i$ $(i = -m+1, \dots, -1, 0)$, or alternatively, the values of $u_{-1}, \dots, u_{-m}$, where $U_i = u_i$ $(i = -m, \dots, -1)$.

Let

$$T = \inf\{n \colon U_n \leq 0\} \tag{4}$$

denote the time of ruin (with the understanding that $T=\infty$ if ruin does not occur), and let

$$\Psi(u, u_{-1}, \ldots, u_{-m}) = \Pr(T<\infty) \tag{5}$$

denote the probability of ruin. Note that this is a function of $m+1$ variables.

3. Results

For notational convenience we introduce the function v, a function of $m+1$ variables, defined as

$$v(x_0, x_1, \ldots, x_m) = \exp(-Rx_0 + Ra_1x_1 + \ldots + Ra_mx_m). \tag{6}$$

We shall examine conditions under which

$$\Psi(u, u_{-1}, \ldots, u_{-m}) \leq v(u, u_{-1}, \ldots, u_{-m}) \tag{7}$$

and

$$\Psi(u, u_{-1}, \ldots, u_{-m}) = \frac{v(u, u_{-1}, \ldots, u_{-m})}{E[v(U_T, \ldots, U_{T-m}) \mid T<\infty]} \tag{8}$$

In the special case $a_1 = \ldots = a_m = 0$, i.e. in the classical case, both formulas are well known. In view of (8) an asymptotic formula like

$$\Psi(u, u_{-1}, \ldots, u_{-m}) \sim C \cdot v(u, u_{-1}, \ldots, u_{-m}) \tag{9}$$

for $u \to \infty$ is plausible. However, we shall not derive such a formula rigorously, and the formula remains a conjecture.

4. A martingale

The following result will be the key to derive (7) and (8).

Lemma: $\{v(U_n, U_{n-1}, \ldots, U_{n-m})\}$ is a martingale.

Proof: Since

$$\begin{aligned} U_{n+1} &= U_n + G_{n+1} \\ &= U_n + X_{n+1} + a_1G_n + \ldots + a_mG_{n-m+1} \\ &= U_n + X_{n+1} + a_1(U_n - U_{n-1}) + \ldots + a_m(U_{n-m+1} - U_{n-m}), \end{aligned} \tag{10}$$

it follows from the specific form of the function v that

$$v(U_{n+1}, \ldots, U_{n-m+1}) = e^{-RX_{n+1}}\, v(U_n, \ldots, U_{n-m}). \tag{11}$$

From this and the definition of R it follows that

$$E[v(U_{n+1}, \ldots, U_{n-m+1}) \mid U_n, U_{n-1}, \ldots] = v(U_n, \ldots, U_{n-m}) \tag{12}$$

for $n = 0, 1, 2, \ldots$. Q.E.D.

The martingale is not really new. From the recursive formula (11) we see that

$$v(U_n, \ldots, U_{n-m}) = e^{-R(X_1 + \ldots + X_n)}\, v(u, u_{-1}, \ldots, u_{-m}). \tag{13}$$

Thus it is a multiple of the well-known exponential martingale.

5. An inequality

If we stop the martingale at T, we obtain another martingale. It follows that

$$\begin{aligned} &v(u, u_{-1}, \ldots, u_{-m}) = \\ &E[v(U_T, \ldots, U_{T-m})\, I_{T \leq n}] + E[v(U_n, \ldots, U_{n-m})\, I_{T > n}] \end{aligned} \tag{14}$$

for all n. Hence

$$v(u, u_{-1}, \ldots, u_{-m}) \geq E[v(U_T, \ldots, U_{T-m})\, I_{T \leq n}]. \tag{15}$$

For $n \to \infty$, we obtain

$$\begin{aligned} v(u, u_{-1}, \ldots, u_{-m}) &\geq E[v(U_T, \ldots, U_{T-m})\, I_{T < \infty}] \\ &= E[v(U_T, \ldots, U_{T-m}) \mid T < \infty]\, Pr(T < \infty) \end{aligned} \tag{16}$$

Thus we have the following upper bound for $Pr(T < \infty)$:

Theorem 1: $\Psi(u, u_{-1}, \ldots, u_{-m}) \leq \dfrac{v(u, u_{-1}, \ldots, u_{-m})}{E[v(U_T, \ldots, U_{T-m}) \mid T < \infty]}$

Since $U_T \leq 0, U_{T-1} > 0, \ldots, U_{T-m} > 0$, the denominator will be greater than one if

$$a_i \geq 0 \text{ for } i = 1, \ldots, m. \tag{17}$$

Then we obtain a simplified upper bound for the probability of ruin:

Corollary: Conditions (17) imply that inequality (7) holds.

It is instructive to formulate inequality (7) in terms of $u, g_0, g_{-1}, \ldots. g_{-m+1}$. Since

$$u_{-k} = u - g_0 - g_{-1} - \ldots - g_{-k+1} \tag{18}$$

($k = 1, 2, \ldots, m$), inequality (7) can be restated as follows:

$$\Psi(u, g_0, \ldots, g_{-m+1}) \leq \exp(-R(1-\alpha_m)u - R\alpha_m g_0 - \ldots - R\alpha_1 g_{-m+1}), \quad (19)$$

where

$$\alpha_i = a_{m-i+1} + \ldots + a_m. \quad (20)$$

Since the left hand side of (19) is a decreasing function of u, the inequality is not satisfactory unless the right hand side is a decreasing function of u, i.e. $\alpha_m < 1$.

6. Equality

We shall now turn to the question under which conditions the inequality sign in Theorem 1 can be replaced by an equality sign.
First we need an assumption about the a_i's. We assume that all the roots of the equation

$$a_m x^m + \ldots + a_1 x - 1 = 0 \quad (21)$$

lie outside the unit circle in the complex plane. For simplicity we assume that these roots are distinct and we denote their reciprocals by γ_k. Thus our assumption can be stated as the condition that

$$|\gamma_k| < 1 \text{ for } k = 1, \ldots, m. \quad (22)$$

In time series analysis, this condition is known as the stationarity condition, see Box and Jenkins (1970). Note that (22) implies that $\alpha_m < 1$ (if $\alpha_m \geq 1$, (21) has a real root ζ with $0 < \zeta \leq 1$).
We shall also assume that the range of the common distribution of the X_i's is bounded, i.e. that

$$|X_i| \leq d \quad (23)$$

for some constant d. This assumption is not believed to be essential, but it will facilitate the proof of the following result.

Theorem 2: If (22) and (23) are satisfied, (8) holds.

Proof: Equation (2) can be written as

$$(1 - a_1 B - \ldots - a_m B^m)\, G_i = X_i, \quad (24)$$

where B is the backward shift operator. While this equation is originally for $i=1,2,\ldots$, it can be extended to all values of i by setting $G_j = X_j = 0$ for $j \leq -m$ and by defining X_i appropriately for $i = -m+1, \ldots, 0$, i.e.

$$\begin{aligned} X_{-m+1} &= g_{-m+1} \\ X_{-m+2} &= g_{-m+2} - a_1 g_{-m+1}, \text{ etc.} \end{aligned} \tag{25}$$

In this sense, there is a 1-1-correspondence between the G_i's and the X_i's. Using the method of partial fractions, we can write

$$(1 - a_1 B - \ldots - a_m B^m)^{-1} = \sum_{k=1}^{m} \frac{C_k}{1-\gamma_k B} \tag{26}$$

By expanding in the corresponding geometric series we obtain

$$(1 - a_1 B \ldots - a_m B^m)^{-1} = \sum_{l=0}^{\infty} \beta_l B^l, \tag{27}$$

where

$$\beta_l = \sum_{k=1}^{m} C_k \gamma_k^l. \tag{28}$$

Now we apply this operator on both sides of (24) to get

$$G_i = \sum_{l=0}^{i+m-1} \beta_l X_{i-l} \tag{29}$$

From this formula, and conditions (22) and (23), it follows that there is a constant C such that

$$|G_i| \leq C \text{ for all } i. \tag{30}$$

To prove (8) we have to show that the last term in (14) vanishes for $n \to \infty$. We write

$$\begin{aligned} &v(U_n, \ldots, U_{n-m})\, I_{T>n} = \\ &\exp(-R(1-\alpha_m)U_n - R\alpha_m G_n - \ldots - R\alpha_1 G_{n-m+1})\, I_{T>n} \end{aligned} \tag{31}$$

In view of (30), these random variables converge to zero pointwise for $n \to \infty$. Furthermore, they are bounded by the constant

$$\exp(RC \sum_{i=l}^{m} |\alpha_i|). \tag{32}$$

Hence their expectation vanishes for $n \to \infty$. Q.E.D.

7. An interpretation

We shall see that inequality (19) can be thought of to be of the classical form

$$\Psi \leq e^{-\tilde{R}\tilde{u}} \tag{33}$$

if only $\tilde{R}$ and $\tilde{u}$ are defined appropriately.
A given X_j will contribute $\beta_l X_j$ to G_{j+l}, see (29). The sum of these terms generated by X_j is

$$(\beta_0+\beta_1+\beta_2+\ldots)\, X_j = (1-a_1-\ldots-a_m)^{-1}\, X_j = \frac{X_j}{1-\alpha_m}, \tag{34}$$

see (27). Hence the adjustment coefficient is reduced by the corresponding factor and is

$$\tilde{R} = (1-\alpha_n)\, R. \tag{35}$$

Then (19) is of the form (33) if we define a modified "initial" surplus as follows:

$$\tilde{u} = u + \frac{\alpha_m}{1-\alpha_m} g_0 + \ldots + \frac{\alpha_1}{1-\alpha_m} g_{-m+1}. \tag{36}$$

The additional terms are the sum of all the deterministic components of G_i ($i = 1, 2, \ldots$), i.e. the components that are due to $g_0, \ldots, g_{-m+1}$. This can be verified algebraically from (25) and (29).

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Summary

The random walk model for the surplus of an insurance company is replaced by an autoregressive model. Results for the probability of ruin are derived in this more general model.

Résumé

L'auteur propose, pour décrire l'évolution des excédents d'une compagnie d'assurance, de remplacer le modèle du cheminement aléatoire par un modèle autorégressif. Il établit dans ce cas plus général quelques résultats concernant la probabilité de ruine.

Zusammenfassung

Das Irrfahrten-Modell für die Überschüsse einer Versicherungsgesellschaft wird durch ein autoregressives Modell ersetzt. In diesem allgemeineren Modell werden einige Resultate über die Ruinwahrscheinlichkeit hergeleitet.