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Credibility for Esscher Premiums[1]

1 Introduction

In credibility theory, there are two distinct methods: the Bayesian approach and the distribution free approach. Both have been applied successfully to the estimation of net premiums.

Not much has been written about credibility methods in connection with principles of premium calculation other than the expected value principle. The Bayesian approach can be implemented for any principle of premium calculation. It simply amounts to applying the principle to the conditional distribution (given whatever information is available) of the risk in question. Implementing the distribution free approach is much more difficult; in [1] it has been done for the variance principle. The purpose of this note is to show that attractive results can be obtained with the Esscher principle.

In the following let H denote the Esscher principle with parameter $h>0$. If X is a risk, say with moment generating function $M(t)$, the premium is

$$H(X)=\frac{E[Xe^{hX}]}{E[e^{hX}]}=\frac{M'(h)}{M(h)}=\frac{d}{dh}\log M(h). \tag{1}$$

Note that $H(X)$ can be interpreted as the mean of the Esscher transform (parameter h) of the distribution of X. It is well known that for any X, $H(X)$ is an increasing function of h.

$H(X)$ can be characterized as the constant c that minimizes

$$E[(c-X)^2e^{hX}]. \tag{2}$$

More generally, let X be a risk and Y a random variable (or random vector). Then $H(X|Y)$ minimizes

$$E[(f-X)^2e^{hX}] \tag{3}$$

among all $f=f(Y)$. The distribution free approach will be based on this property.

[1] Paper presented at the meeting on risk theory in Oberwolfach October 12–18 1980.

2 The Bayesian Approach

We consider the two examples that have become classical in connection with expected values.

Example 1: Suppose that θ has a normal distribution with mean μ and standard deviation σ_0, and that $X_1, X_2, \ldots$ are (given θ) conditionally independent and normally distributed with mean θ and standard deviation σ_1. Then the marginal distribution of X is normal with mean μ and variance $\sigma_0^2+\sigma_1^2$,

$$\log M(t)=\mu t+\tfrac{1}{2}(\sigma_0^2+\sigma_1^2)t^2. \tag{4}$$

From (1) it follows that

$$H(X)=\mu+(\sigma_0^2+\sigma_1^2)h. \tag{5}$$

This is the premium that has to be charged for X_1, when no additional information about the value of θ is available. The other extreme is when we know the value of θ. Let $p(\theta)=H(X|\theta)$ denote the corresponding premium. In analogy to (5) it follows that

$$p(\vartheta)=\vartheta+\sigma_1^2 h. \tag{6}$$

Consider now an intermediate situation: Given $X_1, \ldots, X_n$, what is the premium for X_{n+1}? The conditional (or posterior) distribution of θ is normal with mean

$$\frac{\sigma_1^2}{n\sigma_0^2+\sigma_1^2}\,\mu+\frac{n\sigma_0^2}{n\sigma_0^2+\sigma_1^2}\,\overline{X}, \tag{7}$$

where $\overline{X}=(X_1+\ldots+X_n)/n$, and variance

$$\frac{\sigma_1^2}{n\sigma_0^2+\sigma_1^2}\,\sigma_0^2. \tag{8}$$

Hence the conditional distribution of X_{n+1} is normal with the same mean and variance

$$\frac{\sigma_1^2}{n\sigma_0^2+\sigma_1^2}\,\sigma_0^2+\sigma_1^2. \tag{9}$$

By analogy with (5) it follows that

$$H(X_{n+1}|X_1, \ldots, X_n)=\frac{\sigma_1^2}{n\sigma_0^2+\sigma_1^2}\,\mu+\frac{n\sigma_0^2}{n\sigma_0^2+\sigma_1^2}\,\overline{X} + \left(\frac{\sigma_1^2}{n\sigma_0^2+\sigma_1^2}\,\sigma_0^2+\sigma_1^2\right)h. \tag{10}$$

With the notation

$$Z = \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma_1^2}, \tag{11}$$

this formula can be written in the suggestive form

$$H(X_{n+1}|X_1, \ldots, X_n) = (1-Z)H(X) + Zp(\bar{X}) \tag{12}$$

which is reminiscent of the classical formula ($h=0$).

Example 2: Suppose that θ has a Gamma distribution with parameters α and λ, and that $X_1, X_2, \ldots$ are conditionally independent with a common Poisson distribution (parameter θ). Then the marginal distribution of X is negative binomial,

$$M(t) = \left(\frac{\lambda}{\lambda + 1 - e^t}\right)^\alpha. \tag{13}$$

To assure the existence of $H(X)$ we assume that $\lambda + 1 - e^h > 0$. Then, according to (1) and (13),

$$H(X) = \frac{\alpha e^h}{\lambda + 1 - e^h}. \tag{14}$$

Given $X_1, \ldots, X_n$, the conditional distribution of θ is Gamma with parameters

$$\tilde{\alpha} = \alpha + (X_1 + \ldots + X_n), \quad \tilde{\lambda} = \lambda + n. \tag{15}$$

Hence,

$$H(X_{n+1}|X_1, \ldots, X_n) = \frac{\tilde{\alpha} e^h}{\tilde{\lambda} + 1 - e^h}. \tag{16}$$

Recalling the moment generating function of the Poisson distribution, we see that

$$p(\theta) = H(X|\theta) = \theta e^h. \tag{17}$$

Substituting (15) in (16), we see that (16) can be written as

$$H(X_{n+1}|X_1, \ldots, X_n) = (1-Z)H(X) + Zp(\bar{X}), \tag{18}$$

where

$$Z = \frac{n}{n + \lambda + 1 - e^h}. \tag{19}$$

Unlike in Example 1, this credibility coefficient is now a function of h; curiously, it is an increasing function of h (and with that of the loading).
In both examples the credibility premium is a linear function of $\bar{X}$. This is an encouragement to try the distribution free approach.

3 The distribution free approach

Let θ be a random variable, and $X_1, X_2, \ldots$ the claims in subsequent years. We assume that given θ, the X_i's are conditionally independent and identically distributed random variables. Let

$$\begin{aligned} \mu(\theta) &= E[X|\theta], \quad & \sigma^2(\theta) &= \text{Var}\,[X|\theta], \\ p(\theta) &= H(X|\theta), \quad & m(\theta) &= E[e^{hX}|\theta]. \end{aligned} \tag{20}$$

The idea is to replace the exact credibility premium $H(X_{n+1}|X_1, \ldots, X_n)$ by an expression of the form $a\overline{X}+b$. In view of (3), a and b shall be determined in order to minimize

$$E[(a\overline{X}+b-X_{n+1})^2 e^{hX_{n+1}}]. \tag{21}$$

Setting the derivative with respect to b equal to zero, we obtain

$$aE[\overline{X}e^{hX_{n+1}}]+bE[e^{hX_{n+1}}]=E[X_{n+1}e^{hX_{n+1}}], \tag{22}$$

or

$$aE[\mu(\theta)m(\theta)]+bE[m(\theta)]=E[p(\theta)m(\theta)]. \tag{23}$$

With the notation

$$E^*[Y]=\frac{E[Ym(\theta)]}{E[m(\theta)]}, \tag{24}$$

this equation can be written more elegantly as

$$aE^*[\mu(\theta)]+b=E^*[p(\theta)]. \tag{25}$$

Setting the derivative with respect to a equal to zero, we get the equation

$$aE[\overline{X}^2 e^{hX_{n+1}}]+bE[\overline{X}e^{hX_{n+1}}]=E[\overline{X}X_{n+1}e^{hX_{n+1}}], \tag{26}$$

which can be written as

$$a\left\{E^*[\mu(\theta)^2]+\frac{1}{n}E^*[\sigma^2(\theta)]\right\}+bE^*[\mu(\theta)]=E^*[\mu(\theta)p(\theta)]. \tag{27}$$

We solve (25) and (27) for a. With the notation

$$\text{Cov}^*(U,V)=E^*[UV]-E^*[U]E^*[V], \tag{28}$$

the result can be written in the form

$$a=\frac{n\,\text{Cov}^*\,(\mu(\theta),p(\theta))}{n\,\text{Var}^*\,[\mu(\theta)]+E^*[\sigma^2(\theta)]}. \tag{29}$$

Finally, b is obtained from (25).

4 Discussion

We assume that $\mu(\vartheta)$ is a monotone function of ϑ and denote by $\mu^{-1}(\cdot)$ the inverse function. Since

$$\bar{X}\to\mu(\theta) \quad \text{for} \quad n\to\infty, \tag{30}$$

it follows that

$$H(X_{n+1}|X_1, \ldots, X_n)\to p(\theta) \quad \text{for} \quad n\to\infty, \tag{31}$$

and that our credibility premium $a\bar{X}+b$ cannot be asymptotically correct unless $p(\vartheta)$ is a linear function of $\mu(\vartheta)$,

$$p(\vartheta)=A\mu(\vartheta)+B. \tag{32}$$

Consider now the special case where this condition is satisfied. We find that $a=AZ$, where

$$Z=\frac{n\,\mathrm{Var}^*\,[\mu(\theta)]}{n\,\mathrm{Var}^*\,[\mu(\theta)]+E^*\,[\sigma^2(\theta)]}, \tag{33}$$

and

$$\begin{aligned} b&=E^*\,[p(\theta)]-aE^*\,[\mu(\theta)]\\ &=(1-Z)E^*\,[p(\theta)]+BZ. \end{aligned} \tag{34}$$

Hence

$$a\bar{X}+b=(1-Z)E^*\,[p(\theta)]+Zp(\mu^{-1}(\bar{X})). \tag{35}$$

Note that $E^*\,[p(\theta)]$ is $H(X)$. Thus this formula has the same appealing interpretation as the classical formula. Furthermore, since $Z\to1$ for $n\to\infty$, the formula is asymptotically correct.

When is condition (32) satisfied? Of course, in the classical case ($h=0$) it is always satisfied (with $A=1$, $B=0$). It is satisfied for an arbitrary $h>0$ if

$$\log E[e^{tX}|\theta]=\theta\varphi_1(t)+\varphi_2(t) \tag{36}$$

for certain functions φ_1, φ_2, in which case

$$p(\vartheta)=\vartheta\varphi_1'(h)+\varphi_2'(h). \tag{37}$$

This was in fact the case in the two examples: In Example 1,

$$\varphi_1(t)=t, \qquad \varphi_2(t)=\tfrac{1}{2}\sigma_1^2t^2, \tag{38}$$

in Example 2,

$$\varphi_1(t)=e^t-1, \qquad \varphi_2(t)=0. \tag{39}$$

More generally, it is satisfied, if the conditional distribution of X (given θ) is compound Poisson with Poisson parameter θ and known claim amount distribution.

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Reference

[1] Bühlmann, H., Mathematical Methods in Risk Theory. New York, Springer, 1969.

Summary

Credibility formulas are derived for premiums that are calculated according to the Esscher principle. Some of the resulting formulas, see the expressions (12), (18), and (35), are of the same appealing type as in the classical case of net premiums.

Zusammenfassung

Credibility-Formeln werden für Prämien hergeleitet, die nach dem Esscher Prinzip bestimmt sind. Einige der resultierenden Formeln, siehe die Ausdrücke (12), (18) und (35), sind vom gleichen attraktiven Typus wie im klassischen Fall von Nettoprämien.

Résumé

L'auteur dérive des formules de crédibilité pour des primes determinées selon le principe d'Esscher. Certaines de ces formules, voir les expressions (12), (18) et (35), ont la même forme attractive que dans le cas classique des primes nettes.