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Ruin Probabilities for Mixed Poisson Claim Numbers without Laplace Transforms

There is a theorem attached to any risk process with claims occurring as a mixed (including simple) Poisson process and arbitrary independent claim sizes that allows the calculation of finite term ruin probabilities without the necessity of employing – and inverting – Laplace transforms. This is of some importance because, as far as the writer knows, the *only* claim occurrence input ever used in practice has been mixed (or simple) Poisson. We propose to prove the theorem and mention its consequences from a numerical viewpoint.

The Theorem

If $p_n(t)$ denotes the probability of n claims occurring in an interval of length t, the mean of the distribution being t, the mixed Poisson is defined by

$$p_n(t) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dF(\lambda) \tag{1}$$

where $F(\cdot)$ is any distribution function (*df*) with unit mean on the positive semi-axis. When $F(\cdot)$ is the gamma *df* $p_n(t)$ is the negative binominal; when $F(\cdot)$ is a single-step function increasing from 0 to 1 at $\lambda = 1$ $p_n(t)$ is the simple Poisson. Suppose now that n claims are known to have occurred in the interval of time $(0, t)$. The theorem provides the conditional joint probability density that the n claims occurred at epochs t_j $(j = 1, 2, \ldots n)$ where $0 < t_1 < t_2 < \ldots < t_n < t$. Define $\lambda_i(\tau)$ as the intensity (or force) of claim occurrence at epoch τ after i claims have occurred. O. Lundberg (1940, Ch. V) shows that

$$\lambda_i(\tau) = -\frac{p_0^{(i+1)}(\tau)}{p_0^{(i)}(\tau)}$$

the parenthetical indices denoting differentiation of a specified order. Hence

$$\begin{aligned}\exp\left[-\int_a^b \lambda_i(\tau) d\tau\right] &= \exp\left[\int_a^b \frac{p_0^{(i+1)}(\tau)}{p_0^{(i)}(\tau)} d\tau\right] \\ &= \exp\left[\ln p_0^{(i)}(\tau)\Big|_a^b\right] \\ &= p_0^{(i)}(b)/p_0^{(i)}(a).\end{aligned}$$

We note, too, that the probability density of survival from epoch t_i at which the ith claim occurred to epoch t_{i+1} and for a claim then to occur is

$$\exp\left[-\int_{t_i}^{t_{i+1}} \lambda_i(\tau)d\tau\right]\lambda_i(t_{i+1}).$$

The required conditional joint probability density is thus

$$\exp\left[-\int_0^{t_1} \lambda_0(\tau)d\tau\right]\lambda_0(t_1)\left\{\prod_{i=1}^{n-1}\exp\left[-\int_{t_i}^{t_{i+1}} \lambda_i(\tau)d\tau\right]\lambda_i(t_{i+1})\right\}\times$$

$$\times\exp\left[-\int_{t_n}^{t} \lambda_n(\tau)d\tau\right]/p_n(t) =$$

$$=\frac{p_0^{(0)}(t_1)}{p_0^{(0)}(0)}(-)\frac{p_0^{(1)}(t_1)}{p_0^{(0)}(t_1)}\times\left\{\prod_{i=1}^{n-1}\frac{p_0^{(i)}(t_{i+1})}{p_0^{(i)}(t_i)}(-)\frac{p_0^{(i+1)}(t_{i+1})}{p_0^{(i)}(t_{i+1})}\right\}\times$$

$$\times\frac{p_0^{(n)}(t)}{p_0^{(n)}(t_n)}\times\frac{n!}{(-t)^n p_0^{(n)}(t)}.$$

Observing that the terms in t_i $(i=1,2,\ldots n)$ are

$$\frac{p_0^{(i-1)}(t_i)}{p_0^{(i)}(t_i)}\times\frac{p_0^{(i)}(t_i)}{p_0^{(i-1)}(t_i)}$$

the whole expression reduces to $n!/t^n$, the uniform density for n *independent* random variables uniformly distributed over $(0,t)$. This theorem is proved on similar lines by Cane (1977) and quite differently by McFadden (1965).

The probability of survival

Let us now apply this theorem to a risk process in which claims are occurring in a mixed Poisson process and the concomitant claim amounts are independently distributed with *df* $B(\cdot)$ of unit mean. We assume that the initial risk reserve is w and we require $U(w,t)$, the probability of non-ruin, or survival, through the interval $(0,t)$. A unit unloaded premium is supposed payable continuously throughout the interval so that an aggregate premium of t, the expected claims, will have been paid by the end of the interval.

Suppose it is known that n independent claims have occurred in $(0,t)$. The first problem is to find the density of the times between these claims including the times between the origin and the first claim and between the nth claim and epoch t. Applying the theorem this is the problem of the random division of an interval of length t by means of n points inserted on it, discussed by Feller

(1971, Ch. I.7). He first shows that all $n+1$ subintervals formed by the n points have the same densities. To do this he adds a further random point and bends the line into a circle. Symmetry implies that all $n+1$ sub-intervals have the same distribution. Reverting to the straight line by cutting the circle at the added point and eliminating it, all n points are at, or to the right of, the first point supposed located at a distance y from the new origin. The probability of this is $(\overline{t-y}/y)^n$.

For every sub-interval the distribution function of the length of any sub-interval (including the first, measured from the origin) is thus

$$A(y)=1-\left(1-\frac{y}{t}\right)^n \qquad 0\leq y\leq t$$

and the density is

$$a(y)=\frac{n}{t}\left(1-\frac{y}{t}\right)^{n-1} \qquad n=1,2,3,\ldots$$

which it is convenient to write as $a_n(y)$ to indicate that n claims are involved. We now write $g_n(x)$ for the density of the *loss, x*, sustained by the risk business on the occurrence of one of n claims. This loss is equal to the amount of the claim *minus* the aggregate premium paid since the previous claim (or since the time origin in the case of the first claim).

Consider the probability of survival through epoch t based on an initial risk reserve of w when only the first of n independent claims has occurred in $(0,t)$. Write it as $W_{1n}(w)$.

The loss suffered on that claim must not exceed w. Remembering that the maximum loss possible is $-t$ (a gain of t) when the claim is (theoretically) zero

$$W_{1n}(w)=\int_{-t}^{w} g_n(x)\,dx \qquad n=1,2,3,\ldots.$$

Next suppose that the first two independent claims out of $n\geq 2$ have not led to ruin and that the first of these resulted in a loss of x. The risk reserve available for the second claim would then be $w-x$, and letting x vary up to its limit w we have

$$W_{2n}(w)=\int_{-t}^{w} W_{1n}(w-x)g_n(x)\,dx \qquad n=2,3,4,\ldots.$$

Continuing, we have generally

$$W_{jn}(w)=\int_{-t}^{w} W_{j-1,n}(w-x)g_n(x)\,dx \qquad j=1,2,3,\ldots n \tag{2}$$

defining

$$W_{0n}(w)=1 \qquad w\geq 0.$$

We observe that relation (2) is identical with that for Beard's survival probability through the occurrence of the nth claim (Seal, 1978, p. 52). However, in (2) the probability is conditional on the occurrence of n claims; hence the required unconditional probability of survival is

$$U(w,t)=\sum_{n=0}^{\infty} p_n(t)\, W_{nn}(w). \tag{3}$$

An important property of this relation is that $W_{nn}(w)$ may be calculated before it is known what member of the mixed Poisson class is to be used for $p_n(t)$. The survival probability $U(w,t)$ therefore does not depend on the actual properties of the stochastic process corresponding to the chosen $p_n(t)$ (e.g. the precise form of dependence of successive intervals between claims) but only on the probability distribution of the n claims occurring in the interval $(0,t)$. Claim occurrences of mixed Poisson type can accordingly join the class of (impractical) renewal processes to constitute what Beneš (1963, Ch. 4, § 2) would call "weakly stationary" claim processes justifying the use of his (4.4) and thus of (4.1) of Seal (1978) in the calculation of $U(w,t)$[1]. On the other hand it should be observed that the mixed Poisson process is not ergodic (McFadden, 1965) so it is not possible to select a λ at random from $F(\lambda)$ of (1) and continue with a simple Poisson process.
Reverting to $g_n(x)$ we note that it takes a different form depending on whether x is positive or negative. In an appropriate notation

$$g_{n+}(x)=\int_0^t b(x+y)a_n(y)dy=\frac{n}{t}\int_0^t b(x+y)\left(1-\frac{y}{t}\right)^{n-1} dy \qquad 0\leq x<\infty \tag{4}$$

$$g_{n-}(x)=\int_{-x}^t b(x+y)a_n(y)dy=\frac{n}{t}\int_{-x}^t b(x+y)\left(1-\frac{y}{t}\right)^{n-1} dy \qquad -t\leq x\leq 0 \tag{5}$$

where $b(\cdot)$ is the density corresponding to $B(\cdot)$, and the product in the first pair of integrands represents the density of a claim amounting to $x+y$ reduced by a unit premium paid for an inter-claim interval of length y. If the premium is risk loaded the changes in these integrals are seen from Seal (1978, p. 53).

[1] The three-parameter generalized Waring (G.W.D.) of Seal's book has not been proved to be "weakly stationary". The three or more parameters of a given mixed Poisson could be determined by equating the equivalent number of moments to the corresponding moments of the G.W.D. and the latter would then be "nearly weakly stationary".

Numerical calculations

The recursive relation (2) can, in general, only be evaluated by quadrature of the integral. Because w appears both in the integrand and as the superior limit of the integral successive values of w must proceed in equidistant steps from $w=0$ upwards. For given w the integration over the negative axis to obtain $W_{jn}(w)$, namely

$$\int_{-t}^{0} W_{j-1,n}(w-x)g_{n-}(x)dx=\int_{0}^{t} W_{j-1,n}(w+z)g_{n-}(-z)dz \qquad j=2,3,\ldots n$$

involves arguments of the prior set of $W(\cdot)$ through $w+t$ so that $W_{j-1,n}(w+z)$ must be "guessed" beyond the desired highest w. If this w is reasonably large for small t the guessed values may be set at unity for small n-values but this becomes invalid at larger n. The accuracy of $W_{nn}(\cdot)$ thus attenuates as n increases. To illustrate these ideas suppose that $b(y)=e^{-y}$, $0\leq y<\infty$, and choose $t=5$. If the eventual mixed Poisson claim number distribution is to be the negative binomal with $h=20$ (see Appendix) $p_n(5)=10^{-5}$ when $n=20$ and this may be chosen as the upper limit of n.
From (4) and (5)

$$g_{n+}(x)=\int_{0}^{t} b(x+y)a_n(y)dy=\frac{e^{-x}n}{t}\int_{0}^{t} e^{-y}\left(1-\frac{y}{n}\right)^{n-1}dy\equiv\frac{e^{-x}n}{t}I_n(0) \qquad 0\leq x<\infty$$

$$g_{n-}(x)=\frac{e^{-x}n}{t}\int_{-x}^{t} e^{-y}\left(1-\frac{y}{t}\right)^{n-1}dy=\frac{e^{-x}n}{t}I_n(-x) \qquad -t\leq x\leq 0$$

where

$$I_n(a)=\int_{a}^{t} e^{-y}\left(1-\frac{y}{t}\right)^{n-1}dy=e^{-a}\left(1-\frac{a}{t}\right)^{n-1}-\frac{n-1}{t}I_{n-1}(a) \qquad a\geq 0$$

on integrating by "parts". Further

$$I_1(a)=\int_{a}^{t} e^{-y}dy=e^{-a}-e^{-t}$$

and successive values $I_2(a)$, $I_3(a)$, ... follow by recursion.
The computer program used to evaluate (2) was based on Seal's (1978) POLLAK recognizing that the recursions have to be calculated for each $n=1,2,3,\ldots 20$. The method of quadrature was retained, namely repeated Simpson at steps of 1/10 in w. The resulting values of $W_{nn}(w)$ for three selected values of w, namely $w=0$, 5 and 10, are given to five decimal places in Table 1. Values of $W_{nn}(w)$ for $n=1$ and 2 can be calculated exactly without too much algebraic effort and

they agreed with the three pairs of approximate values in Table 1. This looks promising, but the continued use of approximate values to produce further approximate values is bound to make itself felt for larger values of n.

The 20 approximate values of $W_{nn}(w)$, $n=1,2,3,\ldots 20$, were then combined with three selected sets of $p_n(5)$ values by means of relation (3).

These sets were, respectively, Poisson, negative binomial with $h=20$, and double Poisson

$$p_n(t)=ce^{-\lambda_1 t}\frac{(\lambda_1 t)^n}{n!}+(1-c)e^{-\lambda_2 t}\frac{(\lambda_2 t)^n}{n!} \qquad n=0,1,2,\ldots$$

with $\lambda_1 \neq \lambda_2$ and $c\lambda_1+(1-c)\lambda_2=1$ (choosing $c=0.6$ and $\lambda_1=0.9$), distributions. These are shown on the right of Table 1. The result of using (3) is printed at the foot of the Table, the designation S indicating repeated Simpson at 1/10 intervals.

In Seal (1978) Table 2.5 is believed to be a five decimal representation of a wide range of $U(w,t)$ values for the Poisson/exponential case with no risk loading in the premium. The three appropriate values from that Table to four decimal places were copied into Table 1 and indicate:

(i) the repeated Simpson quadrature of (2) produces barely two decimal results correctly for $w=0$ but this has improved to "almost" three decimal correctness by $w=10$;

(ii) because any selected form of $p_n(5)$ is applied to a fixed approximate set of $W_{nn}(w)$ values, a similar judgement may be applied to the negative binomial and double Poisson results even before their calculation by means of the computer programs GETUWT or GETBRM of Seal (1978) (using UINTEG for the double Poisson) – see Table 1.

The calculations of (2) and (3) for Poisson claim numbers were rather disappointing so in an attempt to improve them we halved the step in w which thus became 1/20 – with a more than tripled amount of computer time, namely 10 minutes! The relatively small reductions in $W_{nn}(5)$ and $W_{nn}(10)$ are shown on the left of Table 1 and these scarcely affected the survivorship probabilities $U(w,5)$. It was suspected that the quadrature over the negative axis might lend itself to amelioration so Dufton's Rule (Squire, 1970, p. 135) was applied there five times at intervals of 1/10. There was some improvement in the results (S & D in Table 1) so that the "almost" third decimal accuracy for $U(10,5)$ with Poisson claim numbers became "actual" third decimal accuracy. Nevertheless there is no sign that we are approaching accuracy to the fourth decimal. These calculations made for a small t-value, namely $t=5$, indicate that long

Table 1.

n	$W_{nn}(0)$	$W_{nn}(5)$	$W_{nn}(10)$	Poisson	Negative binomial $h=20$	Double Poisson		
0	1.00000	1.00000	1.00000	0.00674	0.01153	0.00794		
1	0.80135	0.99866	0.99999	0.03369	0.04612	0.03731		
2	0.60132	0.99385	0.99994	0.08422	0.09685	0.08853		
3	0.43177	0.98194	0.99971 (1)	0.14038	0.14204	0.14157		
4	0.29542	0.95794	0.99896	0.17547	0.16335	0.17187		
5	0.19199	0.91667	0.99696	0.17547	0.15681	0.16918		
6	0.11827	0.85448	0.99233	0.14622	0.13068	0.14078		
7	0.06900	0.77092	0.98292 (1)	0.10445	0.09707	0.10191		
8	0.03812 (1)	0.66960	0.96582 (1)	0.06528	0.06552	0.06553		
9	0.01994	0.55765	0.93780 (1)	0.03627	0.04077	0.03801		
10	0.00989	0.44408 (1)	0.89594 (2)	0.01813	0.02365	0.02012		
11	0.00466	0.33758 (1)	0.83848 (2)	0.00824	0.01290	0.00980		
12	0.00208	0.24479 (1)	0.76558 (3)	0.00343	0.00666	0.00443		
13	0.00089	0.16930	0.67959 (4)	0.00132	0.00328	0.00187		
14	0.00036	0.11173 (1)	0.58481 (4)	0.00047	0.00155	0.00074		
15	0.00014	0.07040	0.48684 (6)	0.00016	0.00070	0.00027		
16	0.00005	0.04240 (1)	0.39147 (6)	0.00005	0.00031	0.00010		
17	0.00002	0.02443	0.30377 (6)	0.00001	0.00013	0.00003		
18	0.00001	0.01348	0.22736 (6)	—	0.00005	0.00001		
19	—	0.00714	0.16412 (6)	—	0.00002	—		
20	—	0.00363	0.11426 (6)	—	0.00001	—		
				1.00000	1.00000	1.00000		
				0.2585	0.2722	0.2627 (1)	(3) S	
				0.2586	0.2723	0.2628	(3) S&D	$U(0,5)$
				0.2491	0.2637	0.2525	Seal (1978)	
				0.8722	0.8639	0.8696	(3) S	
				0.8738	0.8655	0.8712	(3) S&D	$U(5,5)$
				0.8822	0.8696	0.8780	Seal (1978)	
				0.9871	0.9834	0.9860	(3) S	
				0.9892	0.9856	0.9882	(3) S&D	$U(10,5)$
				0.9888	0.9848	0.9880	Seal (1978)	

The parenthetical results are the *reductions* in the fifth decimal place as a result of halving the interval of integration.

computer runs would be involved for t of the order of 100 or more because $p_n(t)$ would then extend to large n-values before evanescing. A possible alternative would be to calculate $U(w,t)$ by relation (3) for as large a t-value as computer time permits and to bridge the gap between this value and $U(w,\infty)=$ $=U(w)$ by interpolation using $1/t$ as the argument. Unfortunately more work needs to be done on eternal survival probabilities when claims are occurring as a mixed Poisson process. Ammeter's (1948) extension of the Lundberg limit for $1-U(w)$ (e.g. Seal, 1978, p. 60) to the mixed Poisson case should be read in conjunction with Thyrion's (1969) remark that the transformed mean of $B(\cdot)$ must remain finite. On the other hand Seal's (1974) expression for the limiting form of the Laplace transform of the survival probability $U(w,t)$ – equivalent to Prabhu's (1965, (7.162)) expression for the transform of the content of a dam – has not yet been inverted numerically.

Appendix

The negative binomial

Ever since O. Lundberg (1940) introduced it and applied it to accident and sickness policyholders, and Ammeter (1948, 1949) used it for an entire portfolio, the expanded negative binomial

$$\left(\frac{h+t}{h}-\frac{t}{h}\right)^{-h}$$

namely

$$\binom{h+n-1}{n}\left(\frac{h}{h+t}\right)^{h}\left(\frac{t}{h+t}\right)^{n} \qquad n=0,1,2,\ \ldots$$

has been the favourite claim occurrence distribution for actuaries. Its mean is t and its variance $t(1+t/h)$.

In the first – and only – published application of the negative binomial to a whole portfolio Ammeter (1949) showed that the annual deviations from the trend line of average claim numbers in 90 years' experience of a growing Swiss fire insurance company were well-fitted by a negative binomial with parameter $h=37.5$ (or 48.1 when 8 years of economic crisis were omitted). On the other hand when negative binomials are fitted to individual policyholders' experiences (Andreasson, 1966) h is as low as 0.2 and t is only 0.086.

Since the sum of N negative binomial random variables with the same h and same t is still a negative binomial random variable with unchanged t/h but an index $-Nh$ (Andreasson, 1966), a company insuring a portfolio of N automobile policies, for example, would have, on Andreasson's figures, to base its survival calculations on a negative binomial with index $-0.2\,N$, mean $0.086\,N$ and variance $0.086\,N\,(1+0.086/0.2)$, the latter being 1.43 times the mean, the same as for the individual policies and well in excess of the Poisson value, unity. It is, perhaps, as well to emphasise this because the well-known theorem that the negative binomial tends to the Poisson for large h does not apply when t/h remains constant. The actuarial propensity to use the negative binomial is thus amply justified even for large negative binomial indices, say one-fifth of the size of the portfolio.

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References

Ammeter, H. (1948) A generalization of the collective theory of risk in regard to fluctuating basic-probabilities. Skand. Aktu. Tidskr. 31, 171–198.
(1949) Die Elemente der kollektiven Risikotheorie von festen und zufallsartig schwankenden Grundwahrscheinlichkeiten. Mitt. Verein. schweiz. Versich. Mathr. 49, 35–95.
Andreasson, G. (1966) Distribution free approximations in applied risk theory. Astin Bull. 4, 11–18.
Beneš, V. E. (1963) General Stochastic Processes in the Theory of Queues. Addison-Wesley, Reading, Mass.
Cane, V. R. (1977) A class of non-identifiable stochastic models. J. Appl. Prob. 14, 475–482.
Feller, W. (1971) An Introduction to Probability Theory and Its Applications, II. Wiley, New York.
Lundberg, O. (1940) On Random Processes and their Application to Sickness and Accident Statistics. Almqvist & Wiksell, Uppsala.
McFadden, J. A. (1965) The mixed Poisson process. Sankhyā A 27, 83–92.
Prabhu, N. U. (1965) Queues and Inventories. Wiley, New York.
Seal, H. L. (1974) The numerical calculation of $U(w, t)$, the probability of non-ruin in an interval $(0, t)$. Scand. Actu. J. 1974, 121–139.
(1978) Survival Probabilities: The Goal of Risk Theory. Wiley, London.
Squire, W. (1970) Integration for Engineers and Scientists. American Elsevier, New York.
Thyrion, P. (1969) Extension of the collective risk theory. Skand. Aktu. Tidskr. 52 Suppl., 84–98.

Summary

The introductory paragraph provides a summary of the paper.

Zusammenfassung

Risikoprozesse, deren Schadenanzahlen nach einem zusammengesetzten (oder einfachen) Poisson-Gesetz verteilt sind und deren Schadenhöhen einer beliebigen Verteilung folgen, verfügen über die Eigenheit, dass die Berechnung von Ruinwahrscheinlichkeiten über eine begrenzte Dauer ohne Zuhilfenahme von Laplace-Transformationen erfolgen kann. Diese Tatsache ist von grosser Bedeutung, weil – soweit dem Autor bekannt – bis heute in der Praxis für die Schadenereignisse nur solche Poisson-Prozesse verwendet wurden. Der Autor schlägt einen Beweis für die erwähnte Eigenheit vor und zeigt die sich hieraus für die numerische Berechnung ergebenden Konsequenzen.

Résumé

Les processus aléatoires comportant un nombre de sinistres distribué selon une loi de Poisson pondérée (ou pure) et un montant des sinistres de distribution quelconque jouissent d'une propriété qui permet de déterminer la probabilité de ruine durant une période limitée sans avoir recours à la théorie des transformations le Laplace. Ce fait est d'une grande importance car – au su de l'auteur – seuls des processus de type Poisson pondéré ont été considérés jusqu'à ce jour par les actuaires. L'auteur propose une démonstration de la dite propriété et signale les conséquences qu'on peut en retirer au point de vue du calcul numérique.