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Semi-continuous linear programming

By Fl. de Vylder

Abstract

The kind of problems that we consider in this paper is to maximize $\int b(x)dF(x)$ under the constraints $\int A_i(x)dF(x) \leq a_i (i = 1,2,...,n)$, where $b(x), A_i(x)$ are given functions and where $F(x)$ is an unknown distribution function. In order to emphasize the analogy with discrete linear programming problems, we write rather $A_i^x(b^x, F_x)$ instead of $A_i(x)(b(x), F(x))$ and we consider the set of doubly indexed elements A_i^x as a matrix with a finite number of rows i and an infinite number of columns x, but this interpretation has nothing essential. The paper is inspired by *Taylor, G.C.* (to be published). New and essential, in the present paper, is the simultaneous consideration of a problem and its dual problem. The dual can often be solved by methods of classical analysis and usual discrete linear programming theory. Then its solution gives, practically automatically, a solution to the original problem.

As a simple illustration, we consider the *Gagliardi/Straub* (1974) problem (in fact, the problem we treated is only a small part of that one solved by these authors) where a maximal stop-loss distribution is sought under certain constraints. For reasons of lenghtiness of the present note, a similar more difficult problem is solved in another paper *De Vylder* (probably to be presented to the ASTIN Colloquium in Sicily in 1978). In the latter paper it is also shown how equality constraints can be treated by the duality technique.

In practical numerical work, continuous problems are often replaced by close discrete problems. Then approximate solutions are obtained. Here the point of view is different: the purpose is to show how semi-continuous problems can be solved exactly if the number of constraints is small.

1. Discrete linear programming

1.1. Generalities. Notations

For a matrix with elements A_i^j, A_i^j is the element at the intersection of row i and column j, A_i is the row i, A^j is the column j, A is the matrix itself.

If A has m rows and n columns, then A is said to be a ${}^{n}_{m}$ matrix. Let M be a subset of the row-indices, N a subset of the column-indices. Then A^N_M is the submatrix of A obtained by retaining only the rows with index in M and the columns with index in N. If M is the set of all the row-indices, then M is omitted in the notation A^N_M. Similarly for N.
The relations $A > 0, A \geq B$, ..., mean that $A^j_i > 0$, $A^j_i \geq B^j_i$, ..., for each i, j. These agreements apply to ${}^{n}_{1}$ matrices (rows) and to ${}^{1}_{m}$ matrices (columns).

1.2. Problems in linear programming

1.2.1. Definition of the problems

The following problems are considered in the sequel. The dimensions of the matrices involved are those indicated here:

$$\underset{\substack{n\\m}}{A}, \quad \underset{\substack{1\\m}}{a}, \quad \underset{\substack{n\\1}}{b}, \quad \underset{\substack{1\\n}}{x}, \quad \underset{\substack{m\\1}}{y}$$

The unknown is x for the moment.

Problem $(A,a,b,x,\max)$: maximize bx under the constraints $x \geq 0$, $Ax \leq a$.
Problem $(A,a,b,x,\min)$: minimize bx under the constraints $x \geq 0, Ax \geq a$.
Problem $(A,a,b,x,\max, =)$: maximize bx under the constraints $x \geq 0, Ax = a$.

1.2.2. Dual problems. The problems $(A,a,b,x,\max)$, $(A', b', a', y', \min)$ are called *dual problems.* Of course, in the latter, the unknown is y.

1.2.3. Associated problems. We define the *augmented matrices*

$$\underset{\substack{m+n\\n}}{A.} = \underset{\substack{n\ m\\m\ m}}{(A,1)}, \quad \underset{\substack{m+n\\1}}{b.} = \underset{\substack{n\ m\\1\ 1}}{(b,0)}, \quad \underset{\substack{1\\m+n}}{\underset{\cdot}{x}} = \begin{pmatrix} x \\ \xi \end{pmatrix},$$

where ξ is a ${}^{1}_{m}$ column of unknown *slack variables.*

The problem $(A., a, b., \underset{\cdot}{x}, \max, =)$ is the *associated problem* to the problem $(A,a,b,x,\max)$. If x is a solution of the last problem and $\xi = a - Ax$, then $\underset{\cdot}{x}$ is a solution of the first problem. Conversely, if $\underset{\cdot}{x}$ is a solution of the first problem, then x is a solution of the last problem. Such solutions are *associated solutions.*

1.3. Fundamental results

1.3.1. The problem (A, a, b, x, max) has a solution iff its constraints can be satisfied and

$$\sup_{x \geq 0, Ax \leq a} bx < \infty.$$

1.3.2. The problem (A, a, b, x, min) has a solution iff its constraints can be satisfied and

$$\inf_{x \geq 0, Ax \geq a} bx > -\infty.$$

1.3.3. If the constraints of the problem (A, a, b, x, max) and of its dual (A', b', a', y', min) can be satisfied, then each has a solution.

1.3.4. Let x, y' satisfy the constraints of the problem (A, a, b, x, max) and of its dual (A', b', a', y', min) respectively. Then x, y' are solutions of these problems iff $bx = ay$.

1.3.5. The problem (A,a,b,x, max) has a solution iff its dual (A',b',a',y', min) has a solution.

1.3.6. If the problem (A, a, b, x, max, $=$) has a solution, then it has a solution with at most m ($=$ the number of rows of A) positive components x_i.

1.3.7. The problem (A, a, b, x, max, $=$) has a solution iff its constraints can be satisfied and

$$\sup_{x \geq 0, Ax = a} bx < \infty$$

1.3.8. Let the problem (A, a, b, x, max) and its dual (A', b', a', y', min) have solutions. Then they have solutions x, y' satisfying

$$x_M = (A^M)^{-1} a, \; x_{J-M} = 0, \; y = b^M (A^M)^{-1},$$

where J is the set of column-indices of A. and where $M \subseteq J$, M contains exactly m indices.

These results can be found e.g. in *Karlin, S.* (1959). By a "solution" a "finite solution" is always meant.

2. Semi-continuous linear programming

2.1. Semi-continuous matrices.

Suppose that the number A_i^x be defined for $i \in I = \{1,2,...,m\}$ and $x \in K$, where $K = [\alpha, \omega]$ is a closed finite interval in R. Then we may consider that A is a *semi-continuous matrix*, or a ${}^{c}_{m}$ matrix, where c stands for "continuous infinity". I is the set of row-indices of A and K the set of column-indices. For $M \subseteq I$, $N \subseteq K$, the matrices A_M^N, A_M, A^N are defined similarly as in the discrete case (1.1). We also consider ${}^{c}_{1}$ rows, i. e. *continuous rows* with elements $b^x (x \in K)$.
If A is a ${}^{c}_{m}$ matrix, the matrix $A. = (A, 1)$, where the 1 is the ${}^{m}_{m}$ unit matrix, is defined similarly as in 1.2.3. We imagine m supplementary indices $\sigma_1, \sigma_2, ...,$ σ_m, called *slack indices* for the indexing of the m last columns of $A.$, the *augmented matrix*. The *augmented row* $b.$ is the row b followed by m zero's, similarly as in 1.2.3.

2.2. Distribution

2.2.1. A *distribution function* on R is a function $F_x (x \in R)$, never decreasing and continuous on the right. Such a function defines a measure μ_F on the Borel sets of R. In fact we are interested only in the measure of sets in K. The measure of sets not in K is irrelevant. We call μ_F (or simply F) a distribution on K.
If $\mu_F(K) = 1$, $\mu_F(K) \leq 1$, $\mu_F(K) < 1$, the distribution is said, respectively, a *probability distribution*, a *defective distribution*, a *strictly defective distribution.* If the total mass in K is concentrated at a finite number of points, then the distribution is said to be *discrete* or *atomic.* An atom of a distribution F is a point x such that $\mu_F\{x\} > 0$.
All integrals in this paper are *Lebesgue/Stieltjes* integrals. Unless stated otherwise, they must be taken over K.
Only weak convergence of distribution functions is considered. The *Helly/Bray* lemma and the compactness theorem for distribution functions are supposed to be known.

2.2.2. It is easily proved that if each ${}_nF (n = 1,2,...)$ is atomic with at most m atoms and if ${}_nF \to F$, then F is atomic with at most m atoms.

2.2.3. Let F be a distribution on K. We show how to approximate it by a sequence of discrete distributions ${}_nF$. Let n be fixed for the moment. Partition $K = [\alpha, \omega]$ in n intervals ${}_nK_j (j = 1,2,...,n)$ of equal lenght $(\omega - \alpha)/n$. Take ${}_nK_1$ closed and each other ${}_nK_j$ open on the left and closed on the right. At the right end point ${}_nx_j = \alpha + j(\omega - \alpha)/n$ of ${}_nK_j$ place the mass $\mu_F({}_nK_j) = {}_nm_j$. Let ${}_nF$ be the corresponding distribution function. Then ${}_nF \to F$.

2.3. Problems in semi-continuous linear programming

2.3.1. Definitions

In the following problems, A is a semi-continuous matrix with elements $A_i^x (i \in I,\ x \in K)$, a is a column of elements $a_i (i \in I)$, b is a continuous row of elements $b^x (x \in K)$, F is an unknown distribution on K.

Problem $(A, a, b, F, \max)$: maximize $\int b^x dF_x$ under the constraints

$$\int A_i^x dF_x \leq a_i (i \in I).$$

Problem $(A, a, b, F, \max, =)$: maximize $\int b^x dF_x$ under the constraints

$$\int A_i^x dF_x = a_i (i \in I).$$

The *dual problem* of the problem $(A, a, b, F, \max)$ is the Problem $(A', b', a', y', \min)$: the unknown being the ${}_1^m$ row y, minimize ya under the constraints

$$y \geq 0,\ \sum_{i \in I} y^i A_i^x \geq b^x (x \in K).$$

2.3.2. The boundedness constraint

If for a fixed $i \in I$, the function A_i^x of $x \in K$ satisfies $A_i^x > \varepsilon (x \in K)$ for some $\varepsilon > 0$, then we call the constraint $\int A_i^x dF_x \leq a_i$ or $\int A_i^x dF_x = a_i$ a *boundedness constraint*. Indeed, then for any F satisfying that constraint, we must have $\varepsilon \mu_F(K) \leq \int A_i^x dF_x \leq a_i$, $\mu_F(K) \leq a_i/\varepsilon$ and the measures induced by these F are uniformly bounded. Among other things, such a constraint makes it possible to use the compactness theorem for distributions.

2.4. The existence of discrete solutions if any solutions exist

2.4.1. Th. Let the problem $(A, a, b, F, \max, =)$ have a solution F. Suppose that b^x and, for each $i \in I, A_i^x$ are continuous in $x \in K$. Then, if there is a boundedness constraint, the problem has a discrete solution G with at most $m (=$ number of rows of $A)$ atoms.

Demonstration. We consider the distribution ${}_nF$ with, for $j = 1,2,\ldots, n$, the mass ${}_nm_j$ attached to the point ${}_nx_j$, defined in 2.2.3. Since ${}_nF \to F$, we have by the *Helly/Bray* lemma

$$\sum_{j=1}^{n} {}_nB_i^j\, {}_nm_j = \int A_i^x d_nF_x \underset{n}{\to} \int A_i^x dF_x = a_i \quad (i\in I),$$

$$\sum_{j=1}^{n} {}_nc^j\, {}_nm_j = \int b^x d_nF_x \underset{n}{\to} \int b^x dF_x = \max,$$

where $\qquad {}_nB_i^j = A_i^{{}_nx_j},\ {}_nc^j = b^{{}_nx_j} (i\in I; j = 1,2,\ldots,n).$

For each n let us consider the discrete problem

$$(\underset{m\ n}{{}_nB},\underset{m\ 1}{{}_na},\underset{1\ n}{{}_nc},\underset{n\ 1}{z},\max, =) \quad \text{where} \quad {}_na_i = \int A_i^x d_nF_x. \tag{1}$$

The constraints of that problem are satisfied by $z = {}_nm$, the $\frac{1}{n}$ column of components ${}_nm_j (j = 1,2,\ldots,n)$. By 1.3.7 and the boundedness constraint, the problem (1) has a solution and even a solution with at most m positive components by 1.3.6. To such a solution corresponds a discrete distribution ${}_nG$ with at most m atoms (in an obvious way; see also 2.6.1). Then we have

$$\int A_i^x d_nG_x = {}_na_i = \int A_i^x d_nF_x, \quad \int b^x d_nG_x \geq \int b^x d_nF_x \left(= \sum_{j=1}^{n} {}_nc^j\, {}_nm_j\right),$$

because ${}_nm$ satisfies the constraints of (1) and because ${}_nG$ corresponds to a solution of (1). Going over to a subsequence, we may assume, by the compactness theorem that ${}_nG \to G$ for some G. Then in the limit, by the *Helly/Bray* lemma,

$$\int A_i^x dG_x = a_i (i\in I), \quad \int b^x dG_x \geq \max,$$

where only equality is possible in the last relation since F is a solution of the original problem. Then G is also a solution of that problem and the theorem results from 2.2.2.

2.5. The identification of solutions

2.5.1.Th. Suppose that F and y satisfy the constraints of the problem $(A,a,b,F,\max)$ and its dual $(A',b',a',y',\min)$ respectively and that

$$\int b^x dF_x = M = \sum_i y^i a_i. \tag{2}$$

Then F, y are solutions of these problems.

Demonstration. Since F satisfies the constraints of the original problem and y those of the dual, we have

$$\int A_i^x dF_x \leq a_i, \quad y^i \geq 0 (i \in I); \sum_i y^i A_i^x \geq b^x (x \in K).$$

Suppose that F is not a solution. Then there is a distribution G satisfying

$$\int A_i^x dG_x \leq a_i (i \in I), \quad \int b^x dG_x > M.$$

But this leads to the contradiction

$$M = \int b^x dF_x = \sum y^i a_i \geq \sum \int y^i A_i^x dG_x \geq \int b^x dG_x.$$

Similarly, if y is not a solution of the dual, then there is a z satisfying

$$z^i \geq 0 (i \in I); \sum z^i A_i^x \geq b^x (x \in K); \sum z^i a_i < M$$

and this leads to the contradiction

$$M = \sum y^i a_i = \int b^x dF_x \leq \int \sum z^i A_i^x dF_x \leq \sum z^i a_i.$$

2.5.2. Remark. Note the generality of the preceding theorem. There are no conditions on A, a or b. Even K might be quite arbitrary.

2.6. *The research for solutions*

2.6.1. Discrete distribution defined by a relation

Consider the ${}^{c}_{m}$ matrix A with elements $A_i^x (i \in I, x \in K)$ and the augmented matrix $A. = (A, 1)$. Suppose that for a subset M of the set of column-indices of $A.$, containing m indices, we have

$$\underset{\substack{m\\m}}{A^{\cdot M}}\ \underset{\substack{1\\m}}{z} = \underset{\substack{1\\m}}{a}, \tag{3}$$

where $z \geq 0$. For example, let $M = \{x_1, x_2, ..., x_{m-1}, x_m\}$, where $x_1, x_2, ..., x_{m-1} \in K$ and where x_m is a slack index. Then, if we place the masses $z_1, z_2, ..., z_{m-1}$

at the points $x_1, x_2, \ldots, x_{m-1}$ respectively, we have a discrete distribution F. From (3) and the relation $z_m \geq 0$ results that

$$\int A_i^x dF_x \leq a_i, \ (i \in I). \tag{4}$$

Moreover,

$$b_{.}^{M} z = \int b^x dF_x, \tag{5}$$

because the last m elements of $b.$ are 0. We call F the discrete *distribution defined by the relation* $A_{.}^{M} z = a$.

Suppose now that repetitions may occur in M, say $x_1 = x_2$. (Of course then M must rather be considered as being a sequence, rather than a set. However, there is no need for a change of terminology here, since the concept of a finite set with repetitions is logically clear, although unusual in modern mathematics.) Then we have just to place the mass $z_1 + z_2$ at the point $x_1 = x_2$. Similarly, in the case of more repetitions.

2.6.2.Th. Consider the problem $(A, a, b, F, \max)$ and its dual $(A', b', a', y', \min)$. Suppose that $a > 0$ and that there is a boundedness constraint. Suppose that b^x, $A_i^x (i \in I)$ are continuous functions of $x \in K$. Suppose that the constraints of the dual problem can be satisfied. Then the initial problem has a discrete solution F with at most m atoms, defined by a relation $A_{.}^{M} z = a$ and the dual problem has a corresponding solution y satisfying $y A_{.}^{M} = b_{.}^{M}$ The solutions satisfy $\int b^x dF_x = \sum y^i a_i$. (In M, *a* subset of the set of column indices of $A.$, repetitions may occur.)

Demonstration. For $n = 1, 2, \ldots$ let ${}_nN \subseteq K$ be the set ${}_nN = \{{}_nx_1, {}_nx_2, \ldots, {}_nx_n\}$ where ${}_nx_j = \alpha + j(\omega - \alpha)/n, (j = 1, 2, \ldots, n)$.

Consider the discrete problem

$$(\underset{m}{\overset{n}{A}}{}^{{}_nN}, \ \underset{m}{\overset{1}{a}}, \ \underset{1}{\overset{n}{b}}{}^{{}_nN}, \ \underset{n}{\overset{1}{Z}}, \ \max) \tag{6}$$

and its dual. Since the constraints of the semi-continuous dual can be satisfied, those of the discrete dual can be, à fortiori. Since $a > 0$, the discrete dual has a solution by 1.3.2. Then (6) has a solution by 1.3.5. By 1.3.8, there exist solutions ${}_nZ, {}_ny$' of (6) and its dual respectively, satisfying

$$A_{.}^{{}_nM} {}_nz = a, {}_ny A_{.}^{{}_nM} = b_{.}^{{}_nM}, b_{.}^{{}_nM} {}_nz = {}_n y a, \tag{7}$$

where

$${}_nz = {}_nZ_{.{}_nM}, \ {}_nM \subseteq {}_nN + S, \ S = \text{set of slack indices.}$$

Let ${}_nM = \{{}_nt_1, {}_nt_2, ..., {}_nt_m\}$, where ${}_nt_1 < {}_nt_2 < ... < {}_nt_m$. For a fixed $j = 1,2, ..., m$ consider the sequence ${}_1t_j, {}_2t_j, {}_3t_j, ...$ Some elements of that sequence may be in K, others may be slack indices. Anyway, a subsequence exists with limit x_j, where $x_j \in K$ or where x_j is a slack index. Indeed, note that K is a compact subset of R. On the other side, if e.g. for some n all ${}_nt_j, {}_{n+1}t_j, {}_{n+2}t_j, ...$ are slack indices, then at least one of them must occur infinitely often. Thus, the indicated subsequence with limit x_j always exists. So, taking successively subsequences, we may assume that ${}_nt_1 \to x_1, {}_nt_2 \to x_2, ..., {}_nt_m \to x_m$. Note that if $x_j \in K$, then $A_{\cdot}^{{}_nt_j} \to A_{\cdot}^{x_j}$, $b_{\cdot}^{{}_nt_j} \to b_{\cdot}^{x_j}$ by the continuity assumptions. The same limit relations are evidently true if x_j is a slack index. From the boundedness constraint it results that the components of ${}_nz$ remain bounded when $n = 1, 2, \ldots$. Therefore, making use of a further subsequence, we may assume that ${}_nz \to z$ where z is a finite vector. Going over to the limit in the last relation (7) we have $b_{\cdot}^{M}z = \lim {}_nya$, where $M = \{x_1, x_2, ..., x_m\}$. Then, from $a > 0$ it results that the components of ${}_ny$ also remain bounded when $n = 1,2,....$ Therefore, by using a further subsequence, we may assume that ${}_ny \to y$, where y is a finite vector. Finally, in the limit, the relations (7) become

$$A_{\cdot}^{M}z = a,\ yA_{\cdot}^{M} = b_{\cdot}^{M},\ b_{\cdot}^{M}z = ya. \tag{8}$$

The first relation (8) defines a discrete distribution F and from the discussion in 2.6.1 results that the constraints (4) of the initial problem are satisfied by F. From (5) and the last relation (8) follows that $\int b^x dF_x = \sum y^i a_i$. Then, by 2.5.1, the theorem is demonstrated if we verifie that the constraints of the semi-continuous dual problem are satisfied by y. Since ${}_ny$ is a solution of the dual problem of (6), we have

$$\sum_i {}_ny^i A_i^{{}_nx_j} \geq b^{{}_nx_j} (n = 1,2,...; j = 1,2,...,n). \tag{9}$$

By the continuity assumptions and since any point $x \in K$ can be approached as closely as wanted by points ${}_nx_j$, we have $\sum y^i A_i^x \geq b^x (x \in K)$ from (9). Finally, since ${}_ny \geq 0 (n = 1,2,...)$, we have $y \geq 0$. This terminates the proof.

2.6.3. The duality technique.

Some conditions are very probably superfluous in the statement of the preceding theorem, e.g. the condition $a > 0$. Evidently, the reason for such a condition is the simplification in the demonstration. Here we describe a technique for resolving the problem $(A, a, b, F, \max)$ based on that theorem, but we suggest to make use of it even if all conditions are not satisfied. Indeed, if step 4 is conclusive, we are sure, by the very general theorem in 2.5.1 that we have a solution, that the conditions in the preceding theorem be satisfied or not. The following technique has been tested in various cases, in cases where some $a_i < 0$, in cases where $K = [0, \infty[$ ..... It was always successful, provided step 1 could be executed.

Step 1. Find a solution y of the dual problem (A', b', a', y', min). This can often be done by classical analysis or by discrete linear programming, or by combination of both.

Step 2. Find a set M of m column indices of A. satisfying $yA^M_{\cdot} = b^M_{\cdot}$. Note that M can be constructed element after element. For each x, slack or not, satisfying $yA^x_{\cdot} = b^x_{\cdot}$, we may have $x \in M$. Often, multi-choices are possible for M and lead to different or identical final solutions. In M repetitions may occur, but, in view of step 3, they should be avoided if possible.

Step 3. Find $z \geq 0$ satisfying $A^M_{\cdot} z = a$. Note that if $A^M_{\cdot}$ is inversible, a frequent case, then z can only be $z = (A^M_{\cdot})^{-1} a$.

Step 4. Let F be the discrete distribution defined by the relation $A^M_{\cdot} z = a$ (2.6.1) and verify that $\int b^x dF_x = \sum y^i a_i$. Then F is a solution of the problem $(A, a, b, F, \max)$.

2.6.4. Equality constraints.

The constraint $\int A_i^x dF_x = a_i$ is equivalent to the couple of constraints $\int A_i^x dF_x \leq a_i, \int (-A_i^x) dF_x \leq -a_i$. Thus, the duality technique can be used if equality constraints are present. An illustration of such a situation is given in *De Vylder* (probably to be presented to the *ASTIN* Colloquium in Sicily in 1978). Surprisingly and fortunately, it is only apparently that the equality constraints augment the number of constraints when they are replaced by inequality constraints. This is due to a simplification in the duality technique in such cases. For more details, see the paper just mentioned.

3. Illustrations of the duality technique

3.1. The Gagliardi/Straub problem

3.1.1. Initial problem

We search for a probability distribution F (a claim size distribution) on $K = [0, \omega]$, constrained by $\int x dF_x \leq \mu$ (μ fixed, $\mu \leq \omega$) such that the stop-loss premium

$$\int_e^{\omega} (x - e) dF_x ,$$

corresponding to a fixed excess e $(0 \leq e \leq \omega)$, be maximal. This is nearly the *Gagliardi/Straub* (1974) problem, where the constraint is $\int x dF_x = \mu$. However, it will turn out that our solution is at the same time a solution of the latter problem.

3.1.2. Transformed problem

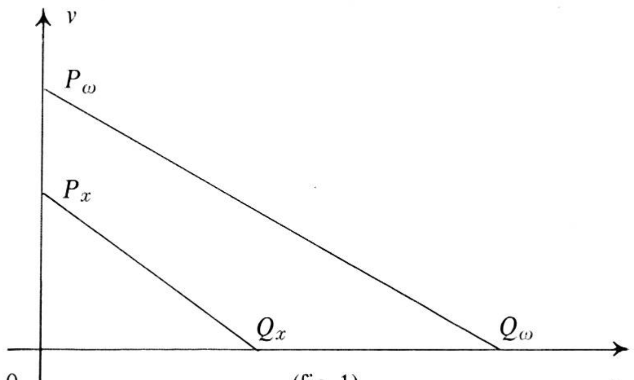
The quantities $\int x dF_x$, $\int_e^\omega (x-e)dF_x$ are not influenced by the probability mass at the origin of K. Therefore it is equivalent to look for a defective distribution, because, if a strictly defective distribution were obtained for solution, it would be sufficient to place the missing probability mass at the origin. Thus, our problem is the problem

$$(A,a,b,F,\max) \quad \text{where} \quad A_1^x = 1,\ A_2^x = x,\ a = \begin{pmatrix} 1 \\ \mu \end{pmatrix},\ b^x = \begin{cases} 0 & \text{for} \quad x < e \\ x-e & \text{for} \quad x \geq e \end{cases}$$

3.1.3. Solution

We apply the duality technique explained in 2.6.3.

Step 1. Writing $y = (u, v)$, the dual problem is to find u, v minimizing $u + v\mu$ under the constraints $u, v \geq 0, u + v\,x \geq b^x (x \in K)$. Of course, the last constraint can be replaced by $u + v\,x \geq x - e (e \leq x \leq \omega)$, since for $0 \leq x < e$ it is satisfied anyway. We consider x as a parameter and represent the portion $P_x\,Q_x$ of the straight line $u + vx = x - e$ situated in the positive quadrant of the (u, v) plane (fig. 1). When x varies from e to ω, P_x moves upside and Q_x moves



(fig. 1)

to the right. Therefore it is clear that the point (u,v) minimizing $u+v\mu$ under the given constraints must be on $P_\omega Q_\omega$. For reasons of linearity, it must be at P_ω or at Q_ω. It easily seen that it is at P_ω. Thus the solution of the dual problem is $u = 0$, $v = (\omega-e)/\omega$ and the corresponding minimum of $u+v\mu$ is $\min = (\omega-e)\frac{\mu}{\omega}$.

Step 2. We look for $x \in K$ (it will not be necessary to try slack indices here) satisfying $yA^x = b^x$, i.e.

$$\left(0, \frac{\omega-e}{\omega}\right)\begin{pmatrix}1\\x\end{pmatrix} = \begin{cases}0 & \text{if } 0 \le x < e\\ x-e & \text{if } e \le x \le \omega\end{cases}.$$

The solutions are $x = 0$ and $x = \omega$. Thus, we can try $M = \{0,\omega\}$ and then the corresponding matrix

$$A^M = \begin{pmatrix}1 & 1\\0 & \omega\end{pmatrix}$$

Step 3. We have to find $z = (z_1, z_2)'$ satisfying $A^M z = a$, i.e.

$$\begin{pmatrix}1 & 1\\0 & \omega\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}1\\\mu\end{pmatrix}$$

The solution is $z_1 = 1-\frac{\mu}{\omega}$, $z_2 = \frac{\mu}{\omega}$.

Thus we have obtained the discrete distribution F with 2 atoms 0 and ω and respective masses

$$m_0 = 1-\frac{\mu}{\omega},\ m_\omega = \frac{\mu}{\omega}.$$

Step 4. We have $\int_e^\omega (x-e)dF_x = (\omega-e)\frac{\mu}{\omega}$.

Since this is the value min found in step 1, we have the result: The discrete probabiity distribution F with atoms at 0 and ω and probabilities m_0, m_ω given in step 3 maximizes the stop-loss premium corresponding to the excess e, under the constraint $\int x dF_x \le \mu$. The maximal stop-loss premium is $(\omega-e)\frac{\mu}{\omega}$.

3.1.4. Observations

For the distribution F, we verifie that $\int x dF_x = \mu$. So we have a solution of the *Gagiardi/Straub* problem.

In step 2, we did not try slack indices. In fact we can use the first slack index σ_1. Then, with $M = \{\omega, \sigma_1\}$, step 3 gives the strictly defective distribution with single atom ω and corresponding mass $m_\omega = \mu/\omega$. By 3.1.2, this gives the same final answer.

3.2. A special illustration

It is the purpose of the following illustration to show simultaneously two facts: that the duality technique may succeed even if K is an infinite interval and that it may happen that no repetitions can be avoided in M. These facts are unrelated: examples can be given with K an infinite interval and no repetitions in M, others can be constructed with K a finite interval and repetitions in M unavoidable.
The problem is to maximize $\int x dF_x$ under the constraints $\int dF_x \leq 1, \int x^2 dF_x \leq v$, in the case $K = [0, \infty[$.

Step 1. With $y = (u, v)$, the dual problem is to minimize $u + vv$ under the constraints $u, v \geq 0,\ u + vx^2 \geq x\,(x \geq 0)$. If we imagine the straight line $u + vx^2 = x$ represented in a (u, v) plane, for each value of the parameter x, we find a superior envelope with equation $v = 1/4u$. It is clear that the solution of the dual problem is on this envelope.
It is easily found to be $u = \sqrt{v}/2$, $v = 1/2\sqrt{v}$ and the corresponding minimum is min $= \sqrt{v}$.

Step 2. We look for $x \geq 0$ (clearly no slack index can suit here) satisfying

$$\left(\frac{\sqrt{v}}{2}, \frac{1}{2\sqrt{v}}\right)\begin{pmatrix}1\\ x^2\end{pmatrix} = x.$$

The unique solution is $x = \sqrt{v}$ and so M can only be $M = \{\sqrt{v}, \sqrt{v}\}$

Step 3. We look for z_1, z_2 satisfying $\begin{pmatrix}1 & 1\\ v & v\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \begin{pmatrix}1\\ v\end{pmatrix}$

The system is indeterminate, but not impossible. Any $z_1, z_2 \geq 0$ satisfying $z_1 + z_2 = 1$ can be used, but the resulting distribution F is anyway that one with unique atom $\sqrt{v}$ and mass 1 at that atom.

Step 4. For this F we have $\int x dF_x = \sqrt{v}$ and since this is the value min found above, F is a solution of the problem. Indeed, as already noted in 2.5.2, there is no restriction on K in the theorem of 2.5.1.

Remark. Note that the preceding considerations include a proof of the well-known inequality

$$\left[\int_0^\infty x dF_x\right]^2 \leq \int_0^\infty x^2 dF_x,$$

valid for any defective distribution F on $[0, \infty]$.

4. Extensions

The theory extends in the following directions:

- K must not necessarily be a compact interval in R;
- the functions A_i^x, b^x of x may have discontinuities;
- x may interpreted as being a point $x \in R^n$, rather than a point $x \in R$ (multi-dimensional semi-continuous linear programming).

Of course, in these more general situations, the preceding results must be adapted.

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Zusammenfassung

Das Problem ist die Maximierung von $\int b(x)dF(x)$ unter den Nebenbedingungen $\int A_i(x)dF(x) \leq a_i$ $(i = 1, 2, \ldots n)$ worin F die unbekannte Verteilungsfunktion ist. Das zugehörige duale Problem lautet: Minimiere $\sum y^i a_i$ unter den Bedingungen $y^i \geq 0, \sum y^i A_i(x) \geq b(x)$ worin nun die y^i die Unbekannten sind. Die Lösung des dualen Problems führt fast automatisch auf eine Lösung des ursprünglichen Problems.

Résumé

Le problème initial consiste à rendre maximum $\int b(x)dF(x)$ sous les contraintes $\int A_i(x)dF(x) \leq a_i$ $(i = 1, 2, \ldots, n)$ où F est la fonction de distribution inconnue. A ce problème est associé un problème dual consistant à rendre minimum $\sum y^i a_i$ sous les contraintes $y^i \geq 0, \sum y^i A_i(x) \geq b(x)$ où les y^i sont maintenant les quantités inconnues. La solution du problème dual donne, assez automatiquement, une solution au problème initial.

Riassunto

Il problema è di trovare il massimo di $\int b(x)dF(x)$ alle condizioni $\int A_i(x)dF(x) \leq a_i$ $(i = 1, 2, \ldots, \mathrm{n})$ se F indica la legge di distribuzione incognita. Il problema duale consiste nel trovare il minimo di $\sum y^i a_i$ per $y^i \geq 0$ e $\sum y^i A_i(x) \geq b(x)$ dove ora gli y^i sono le variabili incognite. La soluzione del problema duale conduce quasi automaticamente a una soluzione del problema originale.

Summary

The initial problem is to maximize $\int b(x)dF(x)$ under the constraints $\int A_i(x)dF(x) \leq a_i$ $(i = 1, 2, \ldots, n)$ where F is the unknown distribution function. To this problem is associated a dual problem consisting in minimizing $\sum y^i a_i$ under the constraints $y^i \geq 0, \sum y^i A_i(x) \geq b(x)$ where now the y^i are the unknown quantities. The solution to the dual problem gives, quite automatically, a solution to the initial problem.