Zeitschrift: Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries

Herausgeber: Vereinigung Schweizerischer Versicherungsmathematiker

Band: 76 (1976)

Artikel: Combinatorial summation

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DOI: https://doi.org/10.5169/seals-967188

Combinatorial Summation

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Abstract

An elementary but quite general combinatorial summation method is developed in this paper. It is based on the use of Kronecker's delta and a straightforward generalisation of that function.

In one illustration of the method, we calculate

$$E\left(\sum_{i,j,k,\ldots=1}^{n} a_{ijk\ldots}\, X_i X_j X_k \ldots\right), \tag{0}$$

where $X_1, X_2, X_3, \ldots$ are i.i.d. standardized normal variables.
With that result, we could give a simple demonstration of Craig's theorem (Craig, 1943) stating that two quadratic forms in i.i.d. normal standardized random variables are independent if and only if the product of the matrices of the forms is zero. In fact we proved the "only if" part only, but that is the most difficult one. The demonstrations known hitherto (Hotelling, 1944; Carpenter, 1950; Aitken, 1950; see also the demonstration of Lancaster published in Kendall and Stuart, 1958) used rather high-level tools (multidimensional transforms and advanced results in matrix theory). Hotelling (1944) found that Craig's original demonstration needed an improvement.
In another illustration, we calculate (0) in the case of i.i.d. Poisson variables. This illustration is based on a rather deep result proved previously in the paper.
In a last section, we show how previous results extend to all distributions with existing moment generating function. It appears, there, that a bridge between the method developed in this paper and Fisher's k-statistics is not excluded.

1. General case

1.1. Introduction to the method

Suppose we want to calculate

$$E(X_1 + X_2 + \ldots + X_n)^3, \tag{1}$$

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where $X_1, X_2, \ldots, X_n$ are i.i.d. random variables. We write this expression as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E(X_i X_j X_k).$$

From our general result follows that for any i, j, k:

$$E(X_i X_j X_k) = M_0 + M_1 (\delta_{ij} + \delta_{jk} + \delta_{ki}) + M_2 \delta_{ij} \delta_{jk},$$

where M_0, M_1, M_2 are polynomials in the first moments of X_1. Therefore (1) equals $M_0 n^3 + 3 M_1 n^2 + M_2 n$.
To be convinced of the advantage of the combinatorial method, the reader should calculate (1) by the usual algebraical method and observe how easy it is then to make mathematical slips.

1.2. Partitions of a positive integer

Let p be a positive integer. A *partition* of p is a sequence $\pi = (a, b, c, \ldots)$ of positive integers with properties:

$$p = a + b + c + \ldots; \quad a \geq b \geq c \geq \ldots$$

The partitions of 1, 2, 3, 4, 5, 6 are given in table 1.

Table 1. Partitions of 1, 2, 3, 4, 5, 6

1	2	3	4	5	6
1	11	111	1111	11111	111111
	2	21	211	2111	21111
		3	31	311	3111
			22	221	2211
			4	41	411
				32	321
				5	222
					51
					42
					33
					6

As is done in table 1, we shall often omit the comma's and brackets in the notation of a partition.

The set of partitions of p is partially ordered by the relation denoted "$\rightarrow$", where $(a, b, c, \dots) \rightarrow (a', b', c', \dots)$ means that a' is a sum of elements $a, b, c, \dots$, that b' is a sum of remaining elements, ... The partial order relation is illustrated in fig. 1, where arrows resulting from transitivity and reflexivity are omitted.

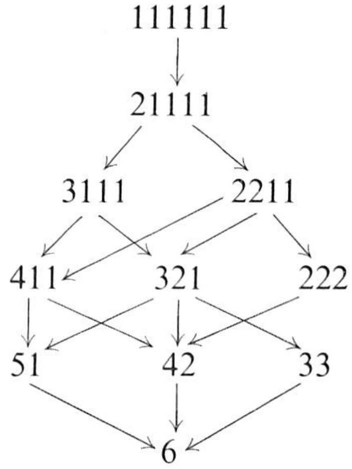

Fig. 1. Partial order relation in the set of partitions of 6

1.3. The fundamental symmetrical functions of indices

1.3.1. Generalized Kronecker's δ

The function $\delta_{ijk\dots}$ is defined to take the value 1 if and only if all the indices $i, j, k, \dots$ are equal and the value 0 otherwise. If there is only one index i, then $\delta_i = 1$ for each value of i.

1.3.2. Definition of the fundamental symmetrical functions

Let p be the number of indices $i, j, \dots$ considered as integer variables (no values are assigned to them for the moment). Consider a partition of p:

$$\pi = (a, b, \dots) \tag{2}$$

Then we define the *fundamental symmetrical function*

$$S_\pi(i, j, \dots) = \frac{1}{K} \sum \delta_{\alpha_1 \dots \alpha_a} \delta_{\beta_1 \dots \beta_b} \cdots \tag{3}$$

where $(\alpha_1, \ldots, \alpha_a, \beta_1, \ldots \beta_b, \ldots)$ runs through the $p!$ permutations of $(i, j, k, \ldots)$. The constant K is unessential and could be dispensed with. However, we shall fix it in order to eliminate repetitions in the second member of (3).

The $p!$ functions $\delta_{\alpha_1 \ldots \alpha_a} \delta_{\beta_1 \ldots \beta_b} \ldots$ split in classes of equal, but differently represented functions. On grounds of symmetry each of these classes contains the same number of members. This number is K, by definition. It depends on the partition π.

The defining sequence $(a, b, c, \ldots)$ of π also splits in classes of equal numbers. If these classes have respectively $u, v, \ldots$ members, then it is clear from an elementary combinatorial argument (consider the permutations of the indices at each δ and also the permutations of δ's with the same number of indices) that $K = a!\, b!\, c! \ldots u!\, v! \ldots$ Thus, exactly

$$\frac{p!}{a!\, b!\, c! \ldots u!\, v! \ldots} \tag{4}$$

different functions, each with coefficient 1, appear in the second member of (3).

1.3.3. Examples (δ's with only one index are omitted):

$1 = S_1(i) = S_{11}(i,j) = S_{111}(i,j,k) = \ldots,$

$S_2(i,j) = \delta_{ij},$

$S_{21}(i,j,k) = \delta_{ij} + \delta_{jk} + \delta_{ik},$

$S_3(i,j,k) = \delta_{ijk},$

$S_{211}(i,j,k,l) = \delta_{ij} + \delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl} + \delta_{kl},$

$S_{22}(i,j,k,l) = \delta_{ij}\,\delta_{kl} + \delta_{jk}\,\delta_{il} + \delta_{ki}\,\delta_{jl},$ (cyclical permutations of three first indices)

$S_{31}(i,j,k,l) = \delta_{jkl} + \delta_{ikl} + \delta_{ijl} + \delta_{ijk},$

$S_4(i,j,k,l) = \delta_{ijkl}.$

1.3.4. Construction of $S_{222\ldots}(i,j,k,\ldots)$

This function will be of the utmost importance in the following sections. We show how to construct its development. Of course, similar devices can be found for other fundamental functions.

First, we show how to construct $S_{222}(i,j,k,l,m,n)$ from $S_{22}(i,j,k,l)$. Consider the array

i	j	k	l
2	2	1	1
2	1	2	1
2	1	1	2

To the lines 2211, 2121, 2112 correspond respectively the terms $\delta_{ij}\,\delta_{kl}$, $\delta_{ik}\,\delta_{jl}$, $\delta_{il}\,\delta_{jk}$ of $S_{22}(i,j,k,l)$. From any line $qrst$, we construct the array

$$\begin{array}{cccccc} 3 & 3 & q & r & s & t \\ 3 & q & 3 & r & s & t \\ 3 & q & r & 3 & s & t \\ 3 & q & r & s & 3 & t \\ 3 & q & r & s & t & 3 \end{array}$$

This construction being performed for each line we have the new array of table 2 and, for each line, the corresponding indicated term of $S_{222}(i,j,k,l,m,n)$.

Table 2

i	j	k	l	m	n	
3	3	2	2	1	1	$\delta_{ij}\,\delta_{kl}\,\delta_{mn}$
3	2	3	2	1	1	$\delta_{ik}\,\delta_{jl}\,\delta_{mn}$
3	2	2	3	1	1	$\delta_{il}\,\delta_{jk}\,\delta_{mn}$
3	2	2	1	3	1	$\delta_{im}\,\delta_{jk}\,\delta_{ln}$
3	2	2	1	1	3	$\delta_{in}\,\delta_{jk}\,\delta_{lm}$
3	3	2	1	2	1	$\delta_{ij}\,\delta_{km}\,\delta_{ln}$
3	2	3	1	2	1	$\delta_{ik}\,\delta_{jm}\,\delta_{ln}$
3	2	1	3	2	1	$\delta_{il}\,\delta_{jm}\,\delta_{kn}$
3	2	1	2	3	1	$\delta_{im}\,\delta_{jl}\,\delta_{kn}$
3	2	1	2	1	3	$\delta_{in}\,\delta_{jl}\,\delta_{km}$
3	3	2	1	1	2	$\delta_{ij}\,\delta_{kn}\,\delta_{lm}$
3	2	3	1	1	2	$\delta_{ik}\,\delta_{jn}\,\delta_{lm}$
3	2	1	3	1	2	$\delta_{il}\,\delta_{jn}\,\delta_{km}$
3	2	1	1	3	2	$\delta_{im}\,\delta_{jn}\,\delta_{kl}$
3	2	1	1	2	3	$\delta_{in}\,\delta_{jm}\,\delta_{kl}$

From this array, we can construct, in a similar manner, $S_{2222}(i,j\ldots)$, and so on.

To prove that the method is valid, it is sufficient to prove that all terms obtained are different functions, since enough functions are constructed, as is seen from (4). That the functions are different is easily seen by an inductive argument.

1.3.5. Properties of the fundamental symmetrical functions

Hitherto, the indices $i, j, \ldots$ were considered as distinct variables. Of course, they may have the same value. Suppose now that $i, j, k, \ldots$ have each a fixed integer positive value. Then the sequence $(i, j, k, \ldots)$ splits in classes of equal numbers. If the number of members in these classes is $r, s, t, \ldots$, where we assume $r \geq s \geq t \ldots$, then we have the partition $(r, s, t, \ldots)$ of p. This partition is denoted by $\varrho(i, j, k, \ldots)$.

Examples: $\varrho(7, 8, 7, 9, 8) = (2, 2, 1)$, $\varrho(3, 3, 3, 1) = (3, 1)$.

The following properties are valid:

(i) $S_\pi(i, j, \ldots)$ depends only on $\varrho(i, j, \ldots)$.
(ii) If $\pi = \varrho(i, j, \ldots)$, then $S_\pi(i, j, \ldots) = 1$.
(iii) If $\pi \rightarrow \varrho(i, j, \ldots)$, then $S_\pi(i, j, \ldots) \neq 0$.
(iv) If not $(\pi \rightarrow \varrho(i, j, \ldots))$, then $S_\pi(i, j, \ldots) = 0$.

Property (i) is quite evident on grounds of symmetry. The reader can be convinced of the validity of the other properties by examining some particular cases based on developments in 1.3.3. Anyway, the validity results from the general theorem in 1.4.2.

We shall write S_π^ϱ for any $S_\pi(i, j, \ldots)$ where $\varrho = \varrho(i, j, \ldots)$.

Example: $S_{32111}^{431} = S_{32111}(1, 1, 1, 1, 2, 2, 2, 3)$.

If ϱ is the particular partition $\varrho = (p)$ of p, then S_π^p equals the number of terms in the general development of $S_\pi(i, j, \ldots)$, since, if $i = j = k = \ldots$, then each term in this development equals one. Thus S_π^p is given by (4). Note that, in that expression, $(u, v, \ldots) = \varrho(a, b, \ldots)$.

1.4. Central result

1.4.1. Theorem

Let f be a function of p integer positive variables $i, j, k, \ldots$ such that the value $f(i, j, k, \ldots)$ depends only on the partition $\varrho(i, j, k, \ldots)$. Then f can uniquely be expressed as

$$f(i,j,\ldots) = \sum c_\pi S_\pi(i,j,\ldots), \tag{5}$$

where π must run through the partitions of p. The coefficients c_π are obtained recursively from a system of linear equations obtained by giving particular values to $i, j, \ldots,$ in (5).

Demonstration.

The case $p = 6$ contains all the elements of a general demonstration. Then, since each member of (5) depends only on $\varrho(i,j,\ldots)$, it is necessary and sufficient to verify this relation for the particular sequences (i, j, k, l, m, n):

$$(1,2,3,4,5,6), (1,1,2,3,4,5), (1,1,1,2,3,4), (1,1,2,2,3,4)\ldots \tag{6}$$

The partitions $\varrho(i,j,k,l,m,n)$ associated with the sequences are respectively

$$(1,1,1,1,1,1), (2,1,1,1,1), (3,1,1,1), (2,2,1,1)\ldots,$$

i.e. those appearing, line after line, in the graph of figure 1. By the properties (ii) and (iv) of 1.3.5, (5) becomes for the sequences of (6):

$$f(1,2,3,4,5,6) = c_{111111},$$

$$f(1,1,2,3,4,5) = c_{111111}\, S^{21111}_{111111} + c_{21111},$$

$$f(1,1,1,2,3,4) = c_{111111}\, S^{3111}_{111111} + c_{21111}\, S^{3111}_{21111} + c_{3111},$$

$$f(1,1,2,2,3,4) = c_{111111}\, S^{2211}_{111111} + c_{21111}\, S^{2211}_{21111} + c_{3111}\, S^{2211}_{3111} + c_{2211},$$

$$\ldots \quad \ldots \quad \ldots$$

From these considerations, the theorem is clear.

1.4.2. Practical calculation of S^p_π

For given ϱ and π, S^ϱ_π can be evaluated by combinatorial arguments.
First of all, we shall reinterpret the terms in the general development of $S_\pi(i,j,\ldots)$. For $\pi = (a,b,\ldots)$, we call a *π-partition* of the set $I = \{i,j,\ldots\}$ of indices, a decomposition

$$I = A + B + \ldots$$

of I in disjoint subsets A, B, ... of a, b, ... elements respectively. There is an obvious one-to-one mapping between the terms in the development of

$S_\pi(i,j,\ldots)$ and the π-partitions of I. For example, the term $\delta_{ikl}\,\delta_{jm}$ in $S_{32}(i,j,k,l,m)$ and the decomposition

$$\{i,j,k,l,m\} = \{i,k,l\} + \{j,m\}$$

correspond to each other.

Now, consider the case of S^{51}_{321}, for example. We consider the set of symbols

$$\{1_i, 1_j, 1_k, 1_l, 1_m, 2_n\} \tag{7}$$

for the following purpose. We shall have the value of S^{51}_{321} on putting $i=j=k=l=m=1$, $n=2$ in the development of $S_{321}(i,\ldots)$ and on counting the non zero terms. But, as we have seen, a one-to-one mapping between the terms in the development of $S_{321}(i,\ldots)$ and the 321-partitions of $\{i,j,k,l,m,n\}$ exists. On the other side, an obvious one-to-one mapping exists between these partitions and between the 321-partitions of (7). But, for the purpose of calculation of S^{51}_{321}, partitions of (7) such as

$$\{1_i, 1_j, 2_n\} + \{1_k, 1_l\} + \{1_m\}$$

may be neglected, since the corresponding value is $\delta_{112}\,\delta_{11}\,\delta_1 = 0$. So, it is easy to see, from (7) what partitions must be kept and what partitions must be dropped. One has $S^{51}_{321} = \binom{5}{3} = 10$.

To see how the method works in full generality, we consider S^{54}_{32211}. Then we consider the set

$$I = \{1_i, 1_j, 1_k, 1_l, 1_m, 2_n, 2_o, 2_p, 2_q\}$$

and we keep two kinds of 32211-partitions of I. A sample of each is

$$\{1_i, 1_j, 1_k\} + \{1_l, 1_m\} + \{2_n, 2_o\} + \{2_p\} + \{2_q\}$$
$$\{1_i, 1_j, 1_k\} + \{1_l\} + \{1_m\} + \{2_n, 2_o\} + \{2_p, 2_q\}.$$

The first corresponds to a 32-partition of the set of 5 elements $I_1 = \{1_i, 1_j, 1_k, 1_l, 1_m\}$ and to a 211-partition of the set of 4 elements $I_2 = \{2_n, 2_o, 2_p, 2_q\}$. The second sample corresponds to a 311-partition of I_1 and a 22-partition of I_2. We have to sum up all partitions of the first kind and all partitions of the second kind. Therefore $S^{54}_{32211} = S^5_{32}\cdot S^4_{211} + S^5_{311}\cdot S^4_{22}$, since, for each kind of partition, the possibilities combine multiplicatively.

From the preceding discussion, the following theorem should be clear:

Theorem.
If $\varrho = (r, s, t, \ldots)$, $\pi = (a, b, c, \ldots)$, then S^{ϱ}_{π} is the sum of the terms $S^{r}_{\pi_1} \cdot S^{s}_{\pi_2} \cdot S^{t}_{\pi_3} \ldots$

With $\pi_\nu = (a_\nu, b_\nu, \ldots)$, $(\nu = 1, 2, 3, \ldots)$, there corresponds exactly one term in this sum to each permutation $(a_1, b_1, \ldots, a_2, b_2, \ldots, a_3, b_3, \ldots)$ of $(a, b, c, \ldots)$ satisfying $a_1 + b_1 + \ldots = r$, $a_2 + b_2 + \ldots = s$, $a_3 + b_3 + \ldots = t$, $\ldots$ (and, of course, the usual condition for a partition of a number: $a_\nu \geq b_\nu \geq \ldots$; $\nu = 1, 2, \ldots$).
Finally, as an illustration of the correct application of the general rule given by the theorem, we calculate S^{422}_{221111}. Then all partitions to be considered are in the table:

4	2	2
22	11	11
211	2	11
211	11	2
1111	2	2

Therefore

$$S^{422}_{221111} = S^{4}_{22}\, S^{2}_{11}\, S^{2}_{11} + S^{4}_{211}\, S^{2}_{2}\, S^{2}_{11} + S^{4}_{211}\, S^{2}_{11}\, S^{2}_{2} + S^{4}_{1111}\, S^{2}_{2}\, S^{2}_{2} = 16$$

An alternative method for the calculation of S^{ϱ}_{π} shall be discussed in section 3.

1.5. The case of $E(X_i\, X_j \ldots)$

1.5.1. General considerations

For simplicity, we shall assume that $X_1, X_2, \ldots$ are i. i. d. random variables, but, in fact, only exchangeability properties are essential. Then the value of $E(X_i\, X_j \ldots)$ depends only on that of $\varrho(i, j, \ldots)$. Indeed, if we denote the r-th moment of X_1 by m_r and if $\varrho(i, j, \ldots) = (r, s, \ldots)$, then $E(X_i\, X_j \ldots) = m_r m_s \ldots$ So (5) applies and it is seen that the c_π are polynomials in $m_1, m_2, \ldots$

1.5.2. Particular results

On particularisation of the general procedure, one has:

$E(X_i\, X_j) = m_1^2 + (m_2 - m_1^2)\, \delta_{ij}$,

$$E(X_i X_j X_k) = m_1^3 + (m_1 m_2 - m_1^3)(\delta_{ij} + \delta_{jk} + \delta_{ki}) + \\ + (m_3 + 2\,m_1^3 - 3\,m_1 m_2)\,\delta_{ijk}, \tag{8}$$

$$\begin{aligned} E(X_i X_j X_k X_l) = {} & m_1^4 + (m_2 m_1^2 - m_1^4)(\delta_{ij} + \delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl} + \delta_{kl}) \\ & + (m_2^2 + m_1^4 - 2\,m_2 m_1^2)(\delta_{ij}\,\delta_{kl} + \delta_{jk}\,\delta_{il} + \delta_{ki}\,\delta_{jl}) \\ & + (m_3 m_1 + 2\,m_1^4 - 3\,m_2 m_1^2)(\delta_{jkl} + \delta_{ikl} + \delta_{ijl} + \delta_{ijk}) \\ & + (m_4 - 6\,m_1^4 + 12\,m_2 m_1^2 - 3\,m_2^2 - 4\,m_1 m_3)\,\delta_{ijkl}. \end{aligned} \tag{9}$$

1.5.3. The symmetrical case

For a larger number of indices, the method becomes lengthy. But, if one has to calculate

$$\sum_{ijk\ldots} a_{ijk\ldots}\, E(X_i X_j X_k \ldots),$$

where $a_{ijk\ldots}$ is symmetrical in all its indices, then a great simplification occurs. Indeed, then, because of the dummy character of the indices in a summation, each $S_\pi(i, j, \ldots)$ may be replaced by just one term of its development, multiplied by the number of terms in this development.

For example, for symmetrical a_{ijklmn}:

$$\begin{aligned} \sum_{ijklmn} a_{ijklmn}\, S_{33}(i,j,k,l,m,n) &= 10 \sum_{ijklmn} a_{ijklmn}\,\delta_{ijk}\,\delta_{lmn} \\ &= 10 \sum_{ij} a_{iiijjj}. \end{aligned}$$

This observation can be used, with advantage, in the calculation of expressions like $E((\sum X_i^2)(\sum X_i)^2)$. If we write it as

$$\sum_{ijkl} \delta_{ij}\, E(X_i X_j X_k X_l),$$

then the remark does not yet apply. But it does if we write it

$$\frac{1}{6} \sum_{ijkl} (\delta_{ij} + \delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl} + \delta_{kl})\, E(X_i X_j X_k X_l).$$

2. The normal case and Craig's theorem

2.1. General combinatorial theorem

2.1.1. Assumptions

In this section 2., we assume that X_1, X_2, ... are i.i.d. normal standardized variates. Then

$$E(X_1^{2r}) = (2r-1)(2r-3)\dots 3.1$$
$$E(X_1^{2r+1}) = o \qquad (10)$$

Therefore, if the number of indices i, j, ..., is an odd number, we have $E(X_i X_j \dots) = o$. So we have only to consider the case of an even number $2p$ of indices.

2.1.2. Theorem

$$E(X_i X_j \dots) = S_{22\dots}(i,j,\dots). \qquad (11)$$

Demonstration.

First we prove that

$$p\,E(X_{i_1} X_{i_2} \dots X_{i_{2p}}) = \sum_{j_1 j_2} \delta_{j_1 j_2} E(X_{j_3} X_{j_4} \dots X_{j_{2p}}). \qquad (12)$$

In the second member of this formula, (j_1, j_2) must run through the $\binom{2p}{2}$ couples

$$(i_1, i_2), \dots, (i_1, i_{2p}), (i_2, i_3), \dots, (i_2, i_{2p}), \dots\dots (i_{2p-1}, i_{2p})$$

and, for each (j_1, j_2) fixed, $j_3, j_4, \dots, j_{2p}$ must be such that $(j_1, \dots, j_{2p})$ is a permutation of $(i_1, \dots, i_{2p})$.

We may assume that $\varrho(i_1, i_2, \dots, i_{2p}) = (2s_1, 2s_2, \dots, 2s_k)$, for if there is an odd number in the sequence $\varrho(i_1, \dots, i_{2p})$, each member of (12) is zero. Then we may also assume that

$$(i_1, i_2, \dots, i_{2p}) = (1, 1, \dots, 1, 2, 2, \dots, 2, \dots\dots, k, k, \dots, k)$$

where the number of r's $(r = 1, 2, \dots, k)$ is $2s_r$. Then, in the second member of (12), j_1, j_2 must be equal and, owing to the possible choices of j_1, j_2 in the sequence $(r, r, \dots, r)$, it is seen that the second member of (12) equals

$$\sum_{r=1}^{k} \binom{2s_r}{2} E(X_1^{2s_1}) \dots E(X_1^{2s_{r-1}})\, E(X_1^{2s_r-2})\, E(X_1^{2s_{r+1}}) \dots E(X_1^{2s_k}). \tag{13}$$

But from (10) follows that $\binom{2s_r}{2} E(X_1^{2s_r-2}) = s_r\, E(X_1^{2s_r})$, and so, since

$\sum_r s_r = p$, (13) equals the first member of (12).

Applying (12) recursively, it follows, on grounds of symmetry, that

$$E\,(X_{i_1} \dots X_{i_{2p}}) = c\, S_{22\dots}(i_1, \dots, i_{2p}), \tag{14}$$

where c is some constant. On putting all indices equal to 1, it is seen from (4) and (10) that $c = 1$.

2.2. *Mean value of a multiform*

2.2.1. Examples

Let a_{ijkl} be defined for $i, j, k, l = 1, 2, \dots, n$ and consider the 4-form

$$\sum_{ijkl=1}^{n} a_{ijkl}\, X_i\, X_j\, X_k\, X_l, \tag{15}$$

where no assumption of symmetry is made. Then, by the preceding theorem, using the expression of $S_{22}(i, j, k, l)$ in 1.3.3:

$$\begin{aligned} E(\textstyle\sum a_{ijkl}\, X_i\, X_j\, X_k\, X_l) &= \textstyle\sum a_{ijkl}\, (\delta_{ij}\,\delta_{kl} + \delta_{jk}\,\delta_{il} + \delta_{ki}\,\delta_{jl}) \\ &= \sum_{jl} a_{jjll} + \sum_{kl} a_{lkkl} + \sum_{il} a_{ilil}. \end{aligned}$$

Thus

$$E(\textstyle\sum a_{ijkl}\, X_i\, X_j\, X_k\, X_l) = \displaystyle\sum_{i,j=1}^{n} (a_{iijj} + a_{ijji} + a_{ijij}) \tag{16}$$

For a 6-form we have, similarly, from table 2:

$$\begin{aligned} &E\left(\textstyle\sum a_{ijklmn}\, X_i\, X_j\, X_k\, X_l\, X_m\, X_n\right) = \\ &\sum_{ijk} (a_{iijjkk} + a_{ijijkk} + a_{ijjikk} + a_{ijjkik} + a_{ijjkki} \\ &\quad + a_{iijkjk} + a_{ijikjk} + a_{ijkijk} + a_{ijkjik} + a_{ijkjki} \\ &\quad + a_{iijkkj} + a_{ijikkj} + a_{ijkikj} + a_{ijkkij} + a_{ijkkji}). \end{aligned} \tag{17}$$

From these examples, the following theorem is clear.

2.2.2. Theorem

For an arbitrary $2p$-form, we have

$$E\Big(\sum_{i_1 i_2 \dots i_{2p}} a_{i_1 i_2 \dots i_{2p}} X_{i_1} X_{i_2} \dots X_{i_{2p}}\Big) = \sum_{i_1 \dots i_p} a_{j_1 j_2 \dots j_{2p}} \tag{18}$$

where $(j_1, j_2, \dots, j_{2p})$ is a permutation of $(i_1, i_2, \dots i_p, i_1, i_2, \dots, i_p)$. To each term in the development of $S_{22\dots}(i_1, i_2, \dots, i_{2p})$ corresponds one term in the second member of (18), in the following manner. Consider the term

$$\delta_{k_1 k_2} \delta_{k_3 k_4} \dots \delta_{k_{2p-1} k_{2p}}.$$

In

$$a_{\dots}$$

where there should be $2p$ dots, replace the k_1-th and k_2-th dot by i_1, the k_3-th and k_4-th by $i_2, \dots$

2.3. Mean value of a product of quadratic forms

2.3.1. Remark

The expressions that we shall find now could perhaps be obtained more simply by general methods based on multidimensional cumulant generating functions (See Kendall and Stuart, 1958, formula 15.51) or similar tools. However, this seems not to be the case for the general relations of 2.2.

2.3.2. General result

Here a, b, c, ... are symmetrical $n \times n$ matrices of elements a_{ij}, b_{ij}, c_{ij}, ... We consider the associated quadratic forms

$$X'aX = \sum_{i,j} a_{ij} X_i X_j, X'bX = \sum_{i,j} b_{ij} X_i X_j, \dots$$

and we want to evaluate

$$E(X'aXX'bXX'cX \dots) = \sum_{ijklmn\dots} a_{ij} b_{kl} c_{mn} \dots \; E(X_i X_j X_k X_l X_m X_n \dots). \tag{19}$$

To be precise, let there be exactly four quadratic forms. Then by (18) we have that (19) is a sum of terms of which samples are

$$\sum_{ijkl} a_{ik}\, b_{kj}\, c_{il}\, d_{jl} = \sum_{ijkl} a_{ik}\, b_{kj}\, d_{jl}\, c_{li} = tr\, abdc,$$

$$\sum_{ijkl} a_{jl}\, b_{ik}\, c_{ki}\, d_{jl} = \sum_{jl} a_{jl}\, d_{lj} \sum_{ik} b_{ik}\, c_{ki} = tr\, ad\, tr\, bc,$$

$$\sum_{ijkl} a_{ik}\, b_{ll}\, c_{kj}\, d_{ij} = \sum_{ijk} a_{ik}\, c_{kj}\, d_{ji} \sum_{l} b_{ll} = tr\, acd\, tr\, b.$$

Thus, the following theorem is clear.

Theorem.

$$E(X'aXX'bXX'cX\ldots) \tag{20}$$

is a sum of terms

$$tr\, AB \ldots tr\, CD \ldots tr\, E \ldots\ldots \tag{21}$$

multiplied by positive integer coefficients, where $(A, B, \ldots, C, D, \ldots, E, \ldots)$ is a permutation of $(a, b, c, \ldots)$.

2.3.3. A unicity result

The preceding theorem could be made more precise. However, the combinatorics become rather complicated. It is preferable to use methods similar to that employed in the demonstration of (25). That result uses the following fact, that could be stated for other powers than the fourth.

An expression for $E(X'dX)^4$ such as

$$c_1\, tr^4 d + c_2\, tr^2 d\, tr\, d^2 + c_3\, tr^2\, d^2 + c_4\, tr\, d\, tr\, d^3 + c_5\, tr\, d^4$$

supposed to be valid for any diagonal matrix d, is unique.
Indeed, if two such expressions exist and are valid for each diagonal d, then, by difference, we have a homogeneous linear relation in $tr^4\, d$, $tr^2\, d\, tr\, d^2, \ldots$ Then, if we take for d the particular matrices with diagonal

$$(1), (1, 1), (1, 1, 1), (1, 1, 1, 1), (1, -1),$$

we obtain a system of five equations with non-zero determinant. Therefore the homogeneous relation must be identically zero.

2.3.4. Properties of the trace of a product of matrices

The cyclical property is well known and easily proved. It states that

$$tr\,(ab\ldots cd) = tr\,(dab\ldots c).$$

If the matrices are symmetrical, the following property is also valid.

$$tr(abc\ldots de) = tr(aed\ldots cb).$$

Thus the matrices, the first excepted, may be written in reversed order. For example, in the case of five symmetrical matrices, the proof is:

$$\begin{aligned} tr(abcde) &= \sum_{ijklm} a_{ij}\, b_{jk}\, c_{kl}\, d_{lm}\, e_{mi} \\ &= \sum_{jimlk} a_{ji}\, e_{im}\, d_{ml}\, c_{lk}\, b_{kj} = tr(aedcb). \end{aligned}$$

Thus, in the case of three symmetrical matrices a, b, c, the only trace to be considered is $tr\,abc$. In the case of four symmetrical matrices a, b, c, d, only the following traces may differ from each other:

$$tr\,abcd,\; tr\,acbd,\; tr\,abdc.$$

2.3.5. Explicit formulas

$$E(X'aX) = tr\,a, \tag{22}$$

$$E(X'aXX'bX) = tr\,a\,tr\,b + 2\,tr\,ab, \tag{23}$$

$$\begin{aligned} &E(X'aXX'bXX'cX) = tr\,a\,tr\,b\,tr\,c \\ &+ 2(tr\,a\,tr\,bc + tr\,b\,tr\,ca + tr\,c\,tr\,ab) + 8\,tr\,abc, \end{aligned} \tag{24}$$

$$\begin{aligned} &E(X'aXX'bXX'cXX'dX) = tr\,a\,tr\,b\,tr\,c\,tr\,d \\ &+ 2(tr\,ab\,tr\,c\,tr\,d + tr\,ac\,tr\,b\,tr\,d + tr\,ad\,tr\,b\,tr\,c \\ &+ tr\,bc\,tr\,a\,tr\,d + tr\,bd\,tr\,a\,tr\,c + tr\,cd\,tr\,a\,tr\,b) \\ &+ 8(tr\,bcd\,tr\,a + tr\,acd\,tr\,b + tr\,abd\,tr\,c + tr\,abc\,tr\,d) \\ &+ 4(tr\,ab\,tr\,cd + tr\,ac\,tr\,bd + tr\,ad\,tr\,bc) \\ &+ 16(tr\,abcd + tr\,acbd + tr\,abdc) \end{aligned} \tag{25}$$

Relation (22) is immediate. Relations (23), (24) are particular cases of (16), (17) but can of course be obtained by the method that we shall use for (25). This formula can be found from the general formula (18). This takes a couple of hours since the 105 terms in the development of $S_{2222}(i,j,\ldots)$ (or better: the lines in the associated array) must be considered. The following method is more rapid.

Formula (9), used for the squares of the variables, gives

$$E(X_i^2 X_j^2 X_k^2 X_l^2) = 1 + 2(\delta_{ij} + \delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl} + \delta_{kl})$$

$$+ 4(\delta_{ij}\,\delta_{kl} + \delta_{jk}\,\delta_{il} + \delta_{ki}\,\delta_{jl}) + 8(\delta_{jkl} + \delta_{ikl} + \delta_{ijl} + \delta_{ijk}) + 48\,\delta_{ijkl}$$

Therefore, if d is diagonal:

$$E(X'dX)^4 = \sum_{ijkl} d_{ii}\,d_{jj}\,d_{kk}\,d_{ll}\,E(X_i^2 X_j^2 X_k^2 X_l^2) =$$

$$\sum_{ijkl} d_{ii}\,d_{jj}\,d_{kk}\,d_{ll}(1 + 12\,\delta_{ij} + 12\,\delta_{ij}\,\delta_{kl} + 32\,\delta_{ijk} + 48\,\delta_{ijkl}) =$$

$$\sum_{ijkl} d_{ii}\,d_{jj}\,d_{kk}\,d_{ll} + 12\sum_{jk\,l} d_{jj}^2\,d_{kk}\,d_{ll} + 12\sum_{jl} d_{jj}^2 d_{ll}^2 + 32\sum_{kl} d_{kk}^3\,d_{ll} + 48\sum_{l} d_{ll}^4\,.$$

$$E(X'dX)^4 = tr^4 d + 12 tr^2 d\, tr\, d^2 + 12 tr^2 d^2 + 32 tr\, d\, tr\, d^3 + 48 tr\, d^4. \quad (26)$$

Now, on grounds of symmetry and because of the general theorem of 2.3.2, there is only the problem of the coefficients $1, 2, 8, 4, 16$ in the second member of (25). But since we have (26), the coefficients must be those indicated, for otherwise, by putting $a = b = c = d$ (diagonal) in (25), we should contradict a result in 2.3.3.

2.4. Craig's theorem

Craig's theorem (Craig, 1943) states that $X'aX$, $X'bX$ are independently distributed if and only if $ab = o$ (under the assumptions of 2.1.1). In the case of general symmetrical matrices a, b (not necessarily semidefinite positive), it is the «only if» part that was most difficult to prove[1]. Here follows an elementary demonstration based on the explicit expressions of 2.3.5. They give

$$Cov\,(X'aX, X'bX) = 2\,tr\,ab, \quad (27)$$

[1] *Hotelling* (1944) notes about *Craig's* original proof: «The proof given that the condition is sufficient is adequate, but Craig's treatment of its necessity consists essentially in its assertion».

$$Cov\,(X'aX, X'bXX'bX) = 4\;tr\;b\;tr\;ab + 8\;tr\;ab^2, \tag{28}$$

$$Cov\,(X'aXX'aX, X'bX) = 4\;tr\;a\;tr\;ab + 8\;tr\;a^2b, \tag{29}$$

$$Cov\,(X'aXX'aX, X'bXX'bX) = 8\;tr\;ab\;tr\;a\;tr\;b\; + \;8\;tr^2\;ab$$

$$+ 16\;tr\;a^2b\;tr\;b + 16\;tr\;b^2atr\;a + 32\;tr\;a^2b^2 + 16\;tr\;abab. \tag{30}$$

Now suppose that $X'aX$, $X'bX$ are independent. Then the covariances (27) to (30) are zero and we have successively,

$$tr\;ab = o,\; tr\;ab^2 = o,\; tr\;a^2b = o,$$

$$4\;tr\;a^2b^2 + 2\;tr\;abab = o. \tag{31}$$

Considering the matrix $ab = c$, we have, since a, b are symmetrical

$$tr\;a^2b^2 = tr\;baab = tr\;(ab)'ab = tr\;c'c = \sum_{i,j} c_{ij}^2 = \sum_{i,j} c_{ji}^2.$$

Also:

$$tr\;abab = tr\;cc = \sum_{j,i} c_{ij}\,c_{ji}$$

and therefore, by (31):

$$\begin{aligned} o &= 2\sum_{j,i} c_{ij}^2 + \sum_{j,i} (c_{ij}^2 + c_{ji}^2 + 2\,c_{ij}\,c_{ji}) \\ &= 2\sum_{j,i} c_{ij}^2 + \sum_{j,i} (c_{ij} + c_{ji})^2. \end{aligned}$$

This implies that for each $i, j : c_{ij} = o$. Thus $ab = o$.
For a simple demonstration of sufficiency, see Graybill (1961), Theorem 4.10. In fact, Graybill states the theorem for positive semidefinite quadratic forms, but his proof is general.
Finally, we note that Craig's theorem has been extended to non-central variates (Carpenter, 1950).

3. The Poisson case and an alternative method for calculating S_π^p

3.1. General combinatorial theorem

3.1.1. Assumptions

Here $Y_1, Y_2, \ldots$ are i.i.d. Poisson variables with mean value λ. The r-th moment of Y_1 is denoted by m_r.

$$m_r = E(Y_1^r) = \sum_{n=0}^{\infty} n^r\,e^{-\lambda}\,\lambda^n/n!.$$

The m_r are polynomials $m_r(\lambda)$ in λ:

$$m_0 = 1, m_1 = \lambda, m_2 = \lambda^2 + \lambda, m_3 = \lambda^3 + 3\,\lambda^2 + \lambda,$$

$$m_4 = \lambda^4 + 6\,\lambda^3 + 7\,\lambda^2 + \lambda, m_5 = \lambda^5 + 10\,\lambda^4 + 25\,\lambda^3 + 15\,\lambda^2 + \lambda, \ldots$$

They may be found recursively from the relation

$$m_{r+1}(\lambda) = \lambda\, m_r(\lambda) + \lambda\, m_r'(\lambda), (r = o, 1, 2, \ldots).$$

The following theorem is a deeper result than might be thought at first sight.

3.1.2. Theorem

Let the number of indices $i, j, \ldots$ be p. Then

$$E(Y_i\, Y_j \ldots) = \sum_{\pi} \lambda^{n(\pi)}\, S_\pi(i, j, \ldots), \tag{32}$$

where π runs through the partitions of p and where $n(\pi)$ is the number of terms in the sequence π.

Demonstration.

A complete proof of this result is rather lenghty. We shall only sketch the main lines. However, we shall give enough considerations to make it possible for the reader to complete the proof.

If $\varrho = \varrho(i, j, \ldots) = (r, s, \ldots)$, (32) states that for each ϱ,

$$m_r\, m_s \ldots = \sum_{\pi} \lambda^{n(\pi)}\, S_\pi^\varrho. \tag{33}$$

For the particular $\varrho = (p)$, (33) reads:

$$m_p = \sum_{\pi} \lambda^{n(\pi)}\, S_\pi^p. \tag{34}$$

It is this relation that we prove first.

To the infinite sequence $(m_0, m_1, m_2, \ldots)$ of moments, we associate the generating function

$$g(x) = \sum_{p=0}^{\infty} m_p \frac{x^p}{p!} = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{n=0}^{\infty} e^{-\lambda}\, n^p \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{p=0}^{\infty} \frac{(xn)^p}{p!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{xn} = e^{-\lambda}\, e^{\lambda e^x} = \exp(\lambda\,(e^x - 1)),$$

where the transformations are permitted by known arguments.

Now we shall develop $g(x)$ in a power series, in a different way. Then an identification of coefficients will furnish the needed result. All transformations being valid by well known arguments, we have

$$g(x) = \exp(\lambda(e^x - 1)) = \exp\left(\lambda \sum_{\alpha=1}^{\infty} \frac{x^\alpha}{\alpha!}\right) =$$
$$\sum_{\beta=0}^{\infty} \frac{\lambda^\beta}{\beta!}\left(\sum_{\alpha=1}^{\infty} \frac{x^\alpha}{\alpha!}\right)^\beta. \tag{35}$$

The coefficient of $\lambda^\beta/\beta!$ in (35) is

$$\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\dots\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \tag{36}$$

where there are β factors. In this expression, we look for the term in x^p. Such a term will only exist if $\beta \leq p$. To each partition $\pi = (a, b, \dots)$ of p in β parts will correspond a term

$$\frac{x^a}{a!}\,\frac{x^b}{b!}\cdots\frac{\beta!}{u!\,v!\dots},\ (u, v, \dots) = \varrho(a, b, \dots), \tag{37}$$

obtained by picking $x^a/a!$ in the first factor of (36), $x^b/b!$ in the second factor, ..., and then varying the choices in the different factors, taking into account the repetitions in π. (For example, if $a = b = \dots$, there is only one possible choice.) This shows that the coefficient of x^p in (36) is, using (4),

$$\beta! \sum_{(a,b,\dots)} \frac{1}{a!\,b!\dots u!\,v!\dots} = \frac{\beta!}{p!} \sum_{\pi} S_\pi^p$$

where $\pi = (a, b, \dots)$ runs through the partitions of p in β parts. Indeed, then no term can be forgotten.

From this discussion results that the coefficient of $x^p/p!$ in (35) is the second member of (34) and so this relation is proved, since by definition of $g(x)$, this coefficient is m_p.

The case where the sequence ϱ has more than one term can be treated similarly. For example, in the case of two terms, we consider the 2-dimensional generating function

$$g(x,y) = \sum_{r=0}^{\infty} m_r \frac{x^r}{r!} \sum_{s=0}^{\infty} m_s \frac{x^s}{s!} = \exp(\lambda(e^x - 1 + e^y - 1)) =$$
$$\sum_{\beta=0}^{\infty} \frac{\lambda^\beta}{\beta!}\left(\sum_{\alpha=1}^{\infty} \frac{x^\alpha + y^\alpha}{\alpha!}\right)^\beta \tag{38}$$

and we proof by a combinatorial argument using the theorem in 1.4.2, that the coefficient of $x^r y^s$ in (38), $(r \geq s)$, is

$$\frac{1}{r!}\frac{1}{s!}\sum_{\pi} \lambda^{n(\pi)} S_{\pi}^{\varrho},$$

where $\varrho = (r, s)$ and where π runs through the partitions of $p = r + s$.

3.2. *Mean value of a multiform*

Using (32), the mean value of a general multiform in $Y_1, Y_2, \ldots$ can easily been calculated. For example:

$$\begin{aligned} &E\Big(\sum_{i,j,k} a_{ijk} Y_i Y_j Y_k\Big) \\ &= \sum_{i,j,k} a_{ijk} \left(\lambda^3 + \lambda^2 (\delta_{ij} + \delta_{jk} + \delta_{ki}) + \lambda\, \delta_{ijk}\right) \\ &= \lambda^3 \sum_{i,j,k} a_{ijk} + \lambda^2 \sum_{i,j} (a_{iij} + a_{jii} + a_{iji}) + \lambda \sum_{i} a_{iii}. \end{aligned}$$

Then covariances of couples of forms can be calculated and implications of indepence examined. We shall not dwell on the subject here.

3.3. *Alternative method for the calculation of S_{π}^{ϱ}*

We illustrate the method in the case of all possible partitions of 5 for ϱ and π. Here we use the abbreviated notation $5', 4', \ldots, 2'', 2'$ for the partitions of 5, as is indicated in the table:

General notation	11111	2111	311	221	41	32	5
Abbreviated not.	$5'$	$4'$	$3'$	$3''$	$2'$	$2''$	$1'$

(39)

We write down relation (33) for ϱ running through the sequence of partitions (39). This gives, for the dummy partition π running trough the same sequence:

$$\begin{aligned} m_1^5 &= \lambda^5 && = \lambda^5 \\ m_2\, m_1^3 &= \lambda^5 + \lambda^4 && = \lambda^5 + \lambda^4 \\ m_3\, m_1^2 &= \lambda^5 + 3\,\lambda^4 + \lambda^3 && = \lambda^5 + \lambda^4\, S_{4'}^{3'} + \lambda^3 \\ m_2^2\, m_1 &= \lambda^5 + 2\,\lambda^4 + \lambda^3 && = \lambda^5 + \lambda^4\, S_{4'}^{3''} + \lambda^3\, S_{3'}^{3''} + \lambda^3 \end{aligned}$$

$$m_4 m_1 = \lambda^5 + 6\,\lambda^4 + \ \ 7\,\lambda^3 + \quad \lambda^2 \qquad = \lambda^5 + \lambda^4\, S^{2'}_{4'} + \lambda^3\, S^{2'}_{3'} + \lambda^3\, S^{2'}_{3''} + \lambda^2$$

$$m_3 m_2 = \lambda^5 + 4\,\lambda^4 + \ \ 4\,\lambda^3 + \quad \lambda^2 \qquad = \lambda^5 + \lambda^4\, S^{2''}_{4'} + \lambda^3\, S^{2''}_{3'} + \lambda^3\, S^{2''}_{3''} + \\ + \lambda^2\, S^{2''}_{2'} + \lambda^2$$

$$m_5 \quad = \lambda^5 + 10\,\lambda^4 + 25\,\lambda^3 + 15\,\lambda^2 + \lambda = \lambda^5 + \lambda^4\, S^{1'}_{4'} + \lambda^3\, S^{1'}_{3'} + \lambda^3\, S^{1'}_{3''} + \\ + \lambda^2\, S^{1'}_{2'} + \lambda^2\, S^{1'}_{2''} + \lambda.$$

where the not appearing S^{ϱ}_{π} are equal to 1 (those, not indicated, on the diagonal and the coefficients of λ^5) or 0 (upper triangle). Since these relations are true for each λ, an identification of coefficients gives the value of most of the S^{ϱ}_{π} in this case. In the exceptional cases we find that

$$7 = S^{2'}_{3'} + S^{2'}_{3''},\ 4 = S^{2''}_{3'} + S^{2''}_{3''},\ 25 = S^{1'}_{3'} + S^{1''}_{3''},\ 15 = S^{1'}_{2'} + S^{1'}_{2''}.$$

Here the theorem of 1.4.2 must be used. In such cases, the savest way is to calculate all not yet known S^{ϱ}_{π} by that theorem and to use then the previously obtained relations as a check. Actually, one has

$$S^{2'}_{3'} = 4,\ S^{2'}_{3''} = 3,\ S^{2''}_{3'} = 1,\ S^{2''}_{3''} = 3,\ S^{1'}_{3'} = 10,\ S^{1'}_{3''} = 15,\ S^{1'}_{2'} = 5,\ S^{1'}_{2''} = 10.$$

For a greater number of indices, the number of exceptional cases grows fastly. Of course, other distributions than the poissonnian might furnish supplementary relations.

4. Further developments

By the argument used in the demonstration of 3.1.2, we have the following theorem:

Theorem.
Let $X_1, X_2, \ldots$ be i.i.d. random variables with moment generating function $g(t) = E(e^{tX_1})$ supposed to exist for $t \in (-\varepsilon, +\varepsilon), \varepsilon > o$.
If $\log g(t) = c_1 t + c_2 t^2 + c_3 t^3 + \ldots$, then

$$E(X_i X_j \ldots) = \sum_{\pi} c_a\, a!\, c_b\, b! \ldots S_{\pi}(i, j, \ldots)$$

where $\pi = (a, b, \ldots)$ runs through the partitions of p, the number of indices $i, j, \ldots$
With the aid of this general theorem, the expression for $E(X_i X_j \ldots)$ can be written down for a lot of common distributions. The normal and poissonnian cases considered previously are particular cases of this general result.

Since the theorem introduces cumulants, a bridge between the method developed in this paper and Fisher's k-statistics certainly exists.
Finally, we mention that, under a quite general condition on the c_π in the expression

$$E(X_i X_j \ldots) = \sum_{\pi} c_\pi S_\pi (i, j, \ldots)$$

(where, as usual, π runs through the partitions of p, the number of indices), it can be proved that if this expression is valid for all p and equal indices $i, j, \ldots,$ then it is valid in full generality. We shall not dwell on this result, that simplifies drastically the demonstration of the theorem in 3.1.2 and that of its just mentioned generalisation.

Floriaen De Vijlder
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Zusammenfassung

Es werden elementarsymmetrische Funktionen mehrerer Indizes definiert, die als verallgemeinerte Kroneckersymbole aufgefasst und für den Beweis eines Hauptsatzes über allgemein symmetrische Funktionen benutzt werden. Ein Satz von Craig über quadratische Formen normalverteilter Variablen ergibt sich als Folgerung aus diesem Hauptsatz.

Résumé

Des fonctions élémentaires symétriques de plusieurs indices sont définies et peuvent être interprétées comme des symboles de Kronecker généralisés. A l'aide de ces fonctions un théorème fondamental sur les fonctions symétriques est démontré qui, appliqué sur des formes quadratiques de variables normales donne un théorème de Craig.

Riassunto

Dapprima vengono definite delle funzioni elementari e simmetriche come simboli, generalizzati, di Kronecker. Poi con l'aiuto di queste funzioni viene dimostrato un teorema fondamentale sulle funzioni simmetriche generali del quale, applicato su forme quadrate di variabili normali, risulta un teorema di Craig.

Summary

The author proves a main theorem on symmetric functions on the basis of a generalization of the Kronecker symbol. Craigs theorem on quadratic forms of normal variables results as an application of the above main theorem.