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Autor(en): **Gerber, Hans U.**

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# A Geometric Proof of Borch's Theorem

By Hans U. Gerber

1. *Introduction.* Most of the existing proofs of Borch's Theorem are of an analytical nature (see [1], [2] p.122–127, [3] p.190–196, or [4]). The purpose of this note is to show how geometric arguments can be used to prove and interpret Borch's Theorem.

Let  $Y$  be a real random variable (defined in connection with an appropriate probability space). We interpret  $Y$  as the aggregate income of  $n$  insurance companies. To fix ideas, let us assume that  $Y$  has a bounded range. A *risk exchange* is a random vector  $(Y_1, Y_2, \dots, Y_n)$ , for which

$$Y_1 + Y_2 + \dots + Y_n = Y \text{ a.s.} \quad (1)$$

and for which the regularity condition (2) below is satisfied. Intuitively,  $Y_i$  is the share of company  $i$  after the exchange. We suppose that company  $i$  is primarily interested in  $E[U_i(Y_i)]$ , where  $U_i(x)$ ,  $-\infty < x < \infty$ , is its utility function. We assume that the functions  $U_1(x), \dots, U_n(x)$  have the following properties:

- (a)  $U_i(x)$  is twice differentiable
- (b)  $U_i'(x) > 0$ ,  $-\infty < x < \infty$
- (c)  $U_i''(x) < 0$ ,  $-\infty < x < \infty$

( $i = 1, 2, \dots, n$ ). Condition (b) amounts to the trivial requirement that the utility of company  $i$  is an increasing function of its surplus. Condition (c) means that company  $i$  is a "risk averter". We are only interested in risk exchanges  $(Y_1, \dots, Y_n)$ , for which

$$E[Y_i] \text{ and } E[U_i(Y_i)] \text{ exist} \quad (2)$$

for  $i = 1, 2, \dots, n$ .

A risk exchange  $(Y_1^*, \dots, Y_n^*)$  is said to be *Pareto-optimal*, if for any risk exchange  $(Y_1, \dots, Y_n)$  the  $n$  simultaneous inequalities

$$E[U_i(Y_i)] \geq E[U_i(Y_i^*)] \quad (3)$$

are only possible if equality holds for  $i = 1, 2, \dots, n$ . A sufficient condition for Pareto-optimality is the existence of positive constants  $k_1, k_2, \dots, k_n$ , such that

$$\sum_{i=1}^n k_i \{E[U_i(Y_i)] - E[U_i(Y_i^*)]\} \leq 0 \quad (4)$$

for any risk exchange  $(Y_1, \dots, Y_n)$ . We shall see that this condition is also necessary.

We observe that for a function  $U(x)$  with properties (a) and (c) above the following inequality holds:

$$U(s) - U(t) \leq U'(t) \cdot (s - t) \quad (5)$$

for any pair of real numbers  $s, t$  (with equality holding only if  $s = t$ ). This is best seen by plotting the graph of the function  $U$  (which is concave from below) and its tangent line at the point with coordinates  $(t, U(t))$ .

2. *The surface  $F_c$ .* For any real number  $c$ , let  $F_c$  denote the following surface in  $R^n$ :

$$F_c = \{(x_1, \dots, x_n) \mid x_i = U_i(t_i), t_1 + \dots + t_n = c\}. \quad (6)$$

In this context we call  $(t_1, \dots, t_n)$  the *coordinates* of the point  $(x_1, \dots, x_n)$ . If  $(x_1, \dots, x_n) \in F_c$ ,  $x_i = U_i(t_i)$ , the normal vector at this point is parallel to the vector with components

$$U_1'(t_1)^{-1}, U_2'(t_2)^{-1}, \dots, U_n'(t_n)^{-1}. \quad (7)$$

This can be seen as follows: Let  $(y_1, \dots, y_n)$  be another point on the surface  $F_c$ , say with coordinates  $(s_1, \dots, s_n)$ . Formula (5) tells us that

$$y_i - x_i \leq U_i'(t_i) \cdot (s_i - t_i) \quad (8)$$

with equality holding only if  $s_i = t_i$ . Now we divide both sides of the above inequality by  $U_i'(t_i)$ . Since  $s_1 + \dots + s_n = t_1 + \dots + t_n = c$ , summation on both sides leads to the inequality

$$\sum_{i=1}^n U_i'(t_i)^{-1} \cdot (y_i - x_i) \leq 0, \quad (9)$$

with equality holding only if  $(x_1, \dots, x_n) = (y_1, \dots, y_n)$ . Thus the vector with components (7) is indeed parallel to the normal vector; moreover, formula (9)

shows that the tangential plane at any point “supports” the surface in a strict sense. So  $F_c$  is an unbounded, convex surface in  $R^n$ .

*Remark.* The above considerations show that the normal vectors of  $F_c$  point to the first  $2^n$ -tant. Also, for a given vector, there is at most one point in  $F_c$  at which this vector is the normal vector. Furthermore, the set of vectors for which there is a corresponding normal vector on  $F_c$  is independent of  $c$ . For example, if  $U'_i(\infty) = 0$  for  $i = 1, \dots, n$ , or if  $U'_i(-\infty) = \infty$  for  $i = 1, \dots, n$ , this set consists of the unit vectors of the open first  $2^n$ -tant.

3. *Borch's Theorem.* Let  $\kappa = (k_1, k_2, \dots, k_n)$  be a unit vector in the first  $2^n$ -tant, such that for all  $c$  (or, equivalently, for at least one  $c$ ) there is a point on  $F_c$  where  $\kappa$  is the normal vector. For such a  $\kappa$  we construct a risk exchange  $(Y_1^\kappa, Y_2^\kappa, \dots, Y_n^\kappa)$  as follows: Let  $(X_1^\kappa, X_2^\kappa, \dots, X_n^\kappa)$  denote the point on  $F_Y$  whose normal vector is  $\kappa$ . Then  $Y_i^\kappa$  is defined as the  $i$ -th coordinate of this point,  $X_i^\kappa = U_i(Y_i^\kappa)$ . The reader is invited to verify that  $Y_1^\kappa, \dots, Y_n^\kappa$  is indeed a risk exchange.

*Theorem.* (a) Risk exchanges of the form  $(Y_1^\kappa, \dots, Y_n^\kappa)$  are Pareto-optimal.

(b) If a risk exchange  $(Y_1, \dots, Y_n)$  is not of this form, there is a  $\kappa$  such that  $E[U_i(Y_i)] < E[U_i(Y_i^\kappa)]$  for  $i = 1, 2, \dots, n$ .

*Proof:* (a) Let  $(Y_1, \dots, Y_n)$  be an arbitrary risk exchange, and let  $(Y_1^\kappa, \dots, Y_n^\kappa)$  be one of the special form.

From inequality (9) we gather that

$$\sum_{i=1}^n k_i \cdot [U_i(Y_i) - U_i(Y_i^\kappa)] \leq 0 \text{ a.s.} \quad (10)$$

Taking expected values, we obtain

$$\sum_{i=1}^n k_i \cdot \{E[U_i(Y_i)] - E[U_i(Y_i^\kappa)]\} \leq 0, \quad (11)$$

which shows the Pareto-optimality of  $(Y_1^\kappa, \dots, Y_n^\kappa)$ . Furthermore, equality holds in (11) only if  $Y_i = Y_i^\kappa$  a.s.

(b) Let us introduce the surface  $F$  and the solid  $S$  in  $R^n$ :

$$F = \{(x_1, \dots, x_n) | x_i = E[U_i(Y_i^\kappa)], i = 1, \dots, n\} \quad (12)$$

$$S = \{(x_1, \dots, x_n) | x_i = E[U_i(Y_i)], i = 1, \dots, n\}. \quad (13)$$

In the first definition the variation is extended over all risk exchanges of the special form, in the second over all risk exchanges. Inequality (11) shows that  $F$  is a convex surface (unbounded of course). Similarly, one can show that  $S$  is a convex solid whose boundary is  $F$ , where  $S$  is “below”  $F$ . From the last remark in the proof of part (a) it follows that the points of  $F$  can only be generated by risk exchanges of the special form q.e.d.

*Remark.* The reader may find it helpful to draw a picture in the case  $n = 2$ . Here  $S$  is a two dimensional region that is unbounded to the south and west. On the north-east side  $S$  is bounded by the curve  $F$ .

Hans U. Gerber  
Department of Mathematics  
The University of Michigan  
Ann Arbor, MI 48104  
U.S.A.

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**Zusammenfassung**

Der Satz von Borch wird bewiesen und interpretiert anhand von rein geometrischen Betrachtungen.

**Résumé**

Le théorème de Borch est démontré et interprété par des considérations entièrement géométriques.

**Riassunto**

Il teorema di Borch è dimostrato ed interpretato di modo puramente geometrico.

**Summary**

The theorem of Borch is proved and interpreted by purely geometric arguments.

