On the Calculation of IBNR Reserves II

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1. Introductory remarks

In direct insurance as well as in reinsurance it is suitable to use a combination of individual claims handling and a blanket method to calculate IBNR-reserves: the largest and most complex cases are reserved individually whereas the remaining "unproblematic" losses are treated as a whole on the basis of experience. We will limit ourselves to this second, purely statistical problem and will use the minimum-variance-method of estimating "correct" reserves derived in [1]. As was to be expected, the covariances as the simplest measures of dependency between the burning costs of different run-off years are of paramount importance. In this connection the most complicated numerical task is the inverting of such covariance matrices. Fortunately, this is explicitly possible for the models treated in paragraphs 3, 4 and 5, so that the corresponding results become especially transparent and illustrative compared with the general case.

2. The general result (summary of [1])

For a given insurance or reinsurance portfolio we denote by $X_i^{(h)}$ the burning cost (= total of claims divided by underlying premium volume) of the underwriting year i as observed at the end of the h-th year of run-off.

Example:

Year of occurrence		Burning cost observed at the end of 1966	1967	1968	1969	1970	
1966	$i = 5$	$X_5^{(1)}$	$X_5^{(2)}$	$X_5^{(3)}$	$X_5^{(4)}$	$X_5^{(5)}$	
1967	$i = 4$		$X_4^{(1)}$	$X_4^{(2)}$	$X_4^{(3)}$	$X_4^{(4)}$	$h = 5$
1968	$i = 3$			$X_3^{(1)}$	$X_3^{(2)}$	$X_3^{(3)}$	$h = 4$
1969	$i = 2$				$X_2^{(1)}$	$X_2^{(2)}$	$h = 3$
1970	$i = 1$					$X_1^{(1)}$	$h = 2$
							$h = 1$

e.g. $X_4^{(3)} =$ burning cost of year 1967 as known at the end of 1969

Our observations over n years consist thus in a run-off triangle which we denote by

$$\triangledown X = \left\{ X_i^{(h)} \,\middle|\, i = 1, 2, \ldots n; h = 1, 2, \ldots i \right\}$$

so that the calculation of IBNR-reserves or rather the final burning cost may be formulated in the following way, e.g. for the year of occurrence no. q:

Determine an estimator $\hat{\mu}_q^{(m)}$ for the conditional expected value

$$E\left[X_q^{(m)} \,\middle|\, \triangledown X \right]$$

such that

$$E\left[\left\{ E\left[X_q^{(m)} \,\middle|\, \triangledown X \right] - \hat{\mu}_q^{(m)} \right\}^2 \right] = \text{minimum}$$

and

$$E\left[\hat{\mu}_q^{(m)} \right] = E\left[X_q^{(m)} \right].$$

(We assume that after m years all claims are settled.)

In other words: We are looking for an unbiased estimator for the unknown final burning cost $X_q^{(m)}$ of year q such that the expected quadratic error be minimized.

Under the assumptions

(i) $X_i^{(h)}$ stochastically independent of $X_{i'}^{(h')}$ for $i \neq i'$

(ii) $E\left[X_i^{(h)}\right] = e^{(h)}$ independent of i

(iii) $P_i \mathrm{Cov}\left[X_i^{(h)}, X_i^{(h')}\right] = c_{hh'}$ independent of i
where P_i = premium volume of year i

(iv) $\hat{\mu}_q^{(m)} = \sum_{i=1}^{n} \sum_{h=1}^{i} a_{ih} X_i^{(h)}$

(I.e. we confine ourselves to linear homogenous estimators.)

The general solution is given (cf [1]) by the two equations

$$\hat{\mu}_q^{(m)} = \alpha \sum_{i=1}^{m} P_i \left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{X}_i\right) + \left(\underline{c}_{mq}, \mathfrak{C}_q^{-1} \underline{X}_q\right) \tag{1}$$

$$e^{(m)} = \alpha \sum_{i=1}^{m} P_i \left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{e}_i\right) + \left(\underline{c}_{mq}, \mathfrak{C}_q^{-1} \underline{e}_q\right) \tag{2}$$

where the following vector and matrix notation has been used

$$\underline{e}_i = \begin{pmatrix} e^{(1)} \\ e^{(2)} \\ \cdot \\ \cdot \\ e^{(i)} \end{pmatrix}, \quad \mathfrak{C}_i = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1i} \\ c_{21} & c_{22} & \cdots & c_{2i} \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ c_{i1} & c_{i2} & \cdots & c_{ii} \end{bmatrix}, \quad \underline{X}_i = \begin{pmatrix} X_i^{(i)} \\ X_i^{(i)} \\ \cdot \\ \cdot \\ X_i^{(i)} \end{pmatrix}, \quad \underline{c}_{mi} = \begin{pmatrix} c_{m1} \\ c_{m2} \\ \cdot \\ \cdot \\ c_{mi} \end{pmatrix},$$

$\left(\mathfrak{C}_i^{-1}\right.$ = inverse of the covariance matrix $\mathfrak{C}_i$ and $(\underline{a},\underline{b})$ = inner product of $\underline{a}$ and $\left.\underline{b}\right)$

When the expected values $\underline{e}_m$ and the covariances $\mathfrak{C}_m$ are known first (2) may be solved for the Lagrange multiplier α and secondly (1) for the estimator $\hat{\mu}_q^{(m)}$. Incidentally, equation (2) is nothing more than the condition of unbiasedness applied to equation (1).

3. Application to a multiplicative run-off model

We now investigate how this general method can be applied to the "classical" case in which the burning cost of run-off year $h+1$ is the one of year h multiplied by a stochastic factor. So we assume

$$X_i^{(h+1)} = \Lambda_{h+1} X_i^{(h)}, \quad h = 1, 2, 3, \ldots$$

with $$E\left[\Lambda_h\right] = \lambda_h \quad \text{and} \quad \operatorname{Var}\left[\Lambda_h\right] = \varrho_h^2 .$$

We note (for $P_i = 1$)

$$e^{(h)} = E\left[X_i^{(h)}\right] = x_h \quad \text{and} \quad c_{hh} = \operatorname{Var}\left[X_i^{(h)}\right] = c_h$$

dropping the lower subscript i of $X_i^{(h)}$ so that

$$E\left[X^{(h+1)}\right] = \lambda_{h+1} E\left[X^{(h)}\right]$$

and assuming Λ and X to be independent of each other, we obtain for $h \geqq k$

$$E\left[X^{(k)} X^{(h)}\right] = E\left[E\left[X^{(k)} X^{(h)} \mid X^{(h-1)}\right]\right] = E\left[E\left[\Lambda_h X^{(h-1)} X^{(k)} \mid X^{(h-1)}\right]\right]$$
$$= \lambda_h E\left[X^{(h-1)} X^{(k)}\right]$$

This means

$$E\left[X^{(h)}\right] = \prod_{r=k+1}^{h} \lambda_r \cdot E\left[X^{(k)}\right] = x_1 \prod_{r=2}^{h} \lambda_r$$

and $$E\left[X^{(h)} X^{(k)}\right] = \prod_{r=k+1}^{h} \lambda_r \cdot E\left[X^{(k)} X^{(k)}\right].$$

We also have

$$E\left[X^{(k)}X^{(k)}\right]=E\left[E\left[X^{(k)}X^{(k)}\,|\,X^{(k-1)}\right]\right]=E\left[E\left[A_k^2X^{(k-1)}X^{(k-1)}\right]\right]$$

$$=(\lambda_k^2+\varrho_k^2)E\left[X^{(k-1)}X^{(k-1)}\right].$$

We thus may compute the c_k recursively by

$$c_k=\operatorname{Var}\left[x^{(k)}\right]=c_{k-1}\left(\lambda_k^2+\varrho_k^2\right)+x_{k-1}^2\,\varrho_k^2$$

and note that for $h\geqq k$

$$c_{hk}=c_k\prod_{r=k+1}^{h}\lambda_r,\quad\left(\prod_{r=k+1}^{k}\lambda_r=1\text{ by definition}\right)$$

The inverse of such a covariance matrix is given explicitly by means of the following

Lemma:

The inverse of a symmetric matrix of the type

$$c_{hk}=c_k\prod_{r=k+1}^{h}\lambda_r,\quad 1\leqq k\leqq h\leqq m$$

is – if it exists – defined by

$$c_{kk}^{-1}=\frac{1}{c_k}\left(\frac{c_{k-1}\lambda_k^2}{c_k-c_{k-1}\lambda_k^2}+\frac{c_{k+1}}{c_{k+1}-c_k\lambda_{k+1}^2}\right)\text{ for }k=1,2,\ldots m-1\ (c_0=0)$$

$$c_{mm}^{-1}=\frac{1}{c_m-c_{m-1}\lambda_m^2},\quad c_{k,k+1}^{-1}=c_{k+1,k}^{-1}=\frac{-\lambda_{k+1}}{c_{k+1}-c_k\lambda_{k+1}^2},\quad c_{hk}^{-1}=0\text{ else}$$

which is an 0-matrix except for the main diagonal and the two adjacent diagonals.

Proof: for $h>j$ we have

$$\sum_{k=1}^{m}c_{hk}c_{kj}^{-1}=c_{h,j-1}c_{j-1,j}^{-1}+c_{hj}c_{jj}^{-1}+c_{h,j+1}c_{j,j+1}^{-1}=c_{j-1}\prod_{r=j}^{h}\lambda_r\left(\frac{-\lambda_j}{c_j-c_{j-1}\lambda_j^2}\right)$$

$$+c_j\prod_{r=j+1}^{h}\lambda_r\left(\frac{c_{j-1}\lambda_j^2}{c_j-c_{j-1}\lambda_j^2}+\frac{c_{j+1}}{c_{j+1}-c_j\lambda_{j+1}^2}\right)\cdot\frac{1}{c_j}\quad+c_{j+1}\prod_{r=j+2}^{h}r\left(\frac{-\lambda_{j+1}}{c_{j+1}-c\lambda_{j+1}^2}\right)=0$$

but for $h=j\ c_{h,j+1}=c_{j+1,j}=c_j\lambda_{j+1}$ and thus

$$\sum_{k-1}^{m} c_{jk}\, c_{kj}^{-1}=c_{j-1}\lambda_j\,\frac{-\lambda_j}{c_j-c_{j-1}\lambda_j^2}+\frac{c_{j-1}\,\lambda_j^2}{c_j-c_{j-1}\lambda_j^2}+\frac{c_{j+1}}{c_{j+1}-c_j\lambda_{j+1}^2}+c_j\,\lambda_{j+1}\frac{-\lambda_{j+1}}{c_{j+1}-c_j\lambda_{j+1}^2}=1 \text{ qed.}$$

Let us return to equation (2)

$$e^{(m)}=\alpha\sum_{i-1}^{m}P_i\left(\underline{e}_i,\mathfrak{C}_i^{-1}\underline{e}_i\right)+\left(\underline{c}_{mq},\mathfrak{C}_q^{-1}\underline{e}_q\right).$$

In our special case we get

$$\underline{e}_q=\begin{pmatrix}\mu\\ \mu\lambda_2\\ \mu\lambda_2\lambda_3\\ \vdots\\ \mu\prod_{r=2}^{q}\lambda_r\end{pmatrix},\quad \underline{c}_{mq}=\begin{pmatrix}c_{11}\prod_{r=2}^{m}\lambda_r\\ c_{22}\prod_{r=3}^{m}\lambda_r\\ c_{33}\prod_{r=4}^{m}\lambda_r\\ \vdots\\ c_{qq}\prod_{r=q+1}^{m}\lambda_r\end{pmatrix},\quad \mathfrak{C}_q^{-1}=\begin{bmatrix}c_{11}^{-1} & c_{12}^{-1} & 0 & 0 & \cdots & 0\\ c_{12}^{-1} & c_{22}^{-1} & c_{23}^{-1} & 0 & \cdots & 0\\ 0 & c_{23}^{-1} & c_{33}^{-1} & c_{34}^{-1} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & c_{qq}^{-1}\end{bmatrix}$$

and furthermore for each q-dimensional vector $\underline{a}$

$$\left(\underline{c}_{mq},\ \mathfrak{C}_q^{-1}\underline{a}\right)=\sum_{s=1}^{q}c_{ms}\sum_{h=1}^{q}c_{sh}^{-1}\,a_h=\sum_{h=1}^{q}a_h\sum_{s=1}^{q}c_{ms}\,c_{sh}^{-1}$$

where
$$\underline{c}_{mq}=\prod_{r=q+1}^{m}\lambda_r\cdot\underline{c}_{qq},\ \left(\underline{c}_{qq}=q\text{th column of }\mathfrak{C}_q\right)$$

so
$$\left(\underline{c}_{mq},\mathfrak{C}_q^{-1}\underline{a}\right)=a_q\prod_{r=q+1}^{m}\lambda_r$$

in particular this means for equation (2)

$$\left(\underline{c}_{mq},\mathfrak{C}_q^{-1}\underline{e}_q\right)=e^{(q)}\prod_{r=q+1}^{m}\lambda_r=e^{(m)}$$

from which $\alpha=0$ and thus, according to (1)

$$\hat{\mu}_q^m=\left(\underline{c}_{mq},\mathfrak{C}_q^{-1}\underline{X}_q\right)=X_q^{(q)}\cdot\prod_{r=q+1}^{m}\lambda_r=E\left[X_q^{(m)}\,\middle|\,\triangledown X\right].$$

As we see our rule for estimating the final value of multiplicatively developing burning costs is identical with the well-known procedure of estimating "IBNR factors λ_r" based on the IBNR-triangle whose product is to be multiplied with the most recent $X_q^{(q)}$.

Finally, it should be noted that in this simple model the linear estimator $\hat{\mu}_q^{(m)}$ is equal to

$$E\left[X_q^{(m)} \middle| \nabla X\right]$$

which is to be estimated, provided the expected values x_q and the covariances c_{-mq} are known.

4. Multiplicative development with further additive variation

In the more general case of

$$X^{(h+1)} = \Lambda_{(h+1)} X^{(h)} + Y_{h+1} \text{ für } h=1, 2, \ldots$$

with Λ_{h+1}, X and Y_{h+1} mutually independent, we denote (for premium volume $P = 1$):

$$E\left[X^{(h)}\right] = x_h, \quad \mathrm{Var}\left[X^{(h)}\right] = c_{hh} = c_h$$

$$E\left[\Lambda_h\right] = \lambda_h, \quad \mathrm{Var}\left[\Lambda_h\right] = \varrho_h^2 \quad \text{and}$$

$$E\left[Y_h\right] = y_h, \quad \mathrm{Var}\left[Y_h\right] = \theta_h^2.$$

We find the expected values

$$E\left[X^{(h+1)}\right] = \lambda_{h+1} E\left[X^{(h)}\right] + y_{h+1}$$

or

$$x_h = e^{(h)} = E\left[X^{(h)}\right] = \sum_{s=1}^{h} y_s \prod_{r=s+1}^{h} \lambda_r$$

whereas for the covariances – if $k \leqq h$, we still have

$$c_{hk} = c_k \prod_{r=k+1}^{h} \lambda_r$$

since

$$\operatorname{Cov}\left[X^{(h)}, X^{(k)}\right] = \operatorname{Cov}\left[\Lambda_h X^{(h-1)} + Y_h, X^{(k)}\right] = \operatorname{Cov}\left[\Lambda_h X^{(h-1)}, X^{(k)}\right]$$
$$= \lambda_h \operatorname{Cov}\left[X^{(h-1)}, X^{(k)}\right] \qquad \text{etc.}$$

Therefore, the matrix $\mathfrak{C}_m$ is also of the type

$$c_{hk} = c_k \prod_{r=k+1}^{h} \lambda_r, \quad 1 \leqq k \leqq h \leqq m$$

and again its inverse can be calculated explicitly. We only have to bear in mind that now

$$c_k = \operatorname{Var}\left[X^{(k)}\right] = \operatorname{Var}\left[\Lambda_k X^{(k-1)} + Y_k\right] = c_{k-1}\left(\lambda_k^2 + \varrho_k^2\right) + x_{k-1}^2 \varrho_k^2 + \theta_k^2$$

compared with the preceding case in which $\theta_k^2 = 0$
we thus get

$$\hat{\mu}_q^{(m)} = X_q^{(q)} \cdot \prod_{r=q+1}^{m} \lambda_r + \frac{\sum_{i=1}^{m} P_i\left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{X}_i\right)}{\sum_{i=1}^{m} P_i\left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{e}_i\right)} \cdot \sum_{s=q+1}^{m} y_s \prod_{r=s+1}^{m} \lambda_r$$

which as we see coincides with the future burning costs

$E\left[X_q^{(m)} \mid \triangledown X\right]$ to be expected for year q only if

$$\sum_{i=1}^{m} P_i\left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{X}_i\right) = \sum_{i=1}^{m} P_i\left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{e}_i\right),$$

but otherwise, of course, still is an unbiased minimum variance estimator. Intuitively, however, one would have expected

$$\hat{\hat{\mu}}_q^{(m)} = X_q^{(q)} \cdot \prod_{r=q+1}^{m} \lambda_r + \sum_{s=q+1}^{m} y_s \prod_{r=s+1}^{m} \lambda_r$$

to be a better estimator in this special case than our $\hat{\mu}_q^{(m)}$.

This is in fact the case, since

$$\hat{\hat{\mu}}_q^{(m)} = E\left[X_q^{(m)} \middle| \triangleleft X\right]$$

but one should not forget that $\hat{\hat{\mu}}_q^{(m)}$ does not have the initially postulated homogenous linear form

$$\hat{\mu}_q^{(m)} = \sum_{i=1}^{n} \sum_{h=1}^{i} \alpha_{ih} X_i^{(h)}$$

but contains an "unforeseen" constant term

$$\alpha_0 = \sum_{s=q+1}^{m} y_s \prod_{r=s+1}^{m} \lambda_r$$

instead.

5. The purely additive model

If in the above case we assume $\Lambda \equiv 1$, we get a purely additive model where

$$E\left[X^{(h)}\right] = x_h = \sum_{s=1}^{h} y_s \quad \text{and} \quad c_{hk} = c_k \quad \text{for} \quad 1 \leqq k \leqq h \leqq m$$

and the c_k given recursively by

$$c_k = c_{k-1} + \theta_k^2, \quad \text{i.e.} \quad c_k = \sum_{s=1}^{k} \theta_s^2.$$

So, for the inverse of the covariance matrix we obtain

$$c_{kk}^{-1} = \frac{1}{c_k}\left(\frac{c_{k-1}}{\theta_h^2} + \frac{c_{k+1}}{\theta_{k+1}^2}\right) = \frac{1}{\theta_h^2} + \frac{1}{\theta_{k+1}^2} \quad \text{for } k = 1, 2, \ldots m-1.$$

$$c_{mm}^{-1} = \frac{1}{\theta_m^2} \quad \text{and} \quad c_{k,k+1}^{-1} = c_{k+1,k}^{-1} = \frac{-1}{\theta_{k+1}^2}$$

therefore

$$\left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{X}_i\right) = \sum_{r=1}^{i} \frac{y_r Y_i^{(r)}}{\theta_r^2} \quad \text{and} \quad \left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{e}_i\right) = \sum_{r=1}^{i} \frac{y_r^2}{\theta_r^2}$$

where $\quad Y_i^{(r)} = X_i^{(r)} - X_i^{(r-1)}, \; r = 1, 2, \ldots i, \; X_i^{(0)} = 0.$

Finally, we abbreviate

$$P_{.}^{(r)} = \sum_{i=r}^{m} P_i \quad \text{and} \quad Y_{.}^{(r)} = \frac{1}{P_{.}^{(r)}} \sum_{i=r}^{m} P_i Y_i^{(r)}$$

and by analogy

$$\left(\sum_{i=1}^{m} P_i \left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{e}_i \right) \right) = \sum_{i=1}^{m} \frac{y_r^2}{\theta_r^2} P_{.}^{(r)}.$$

so that

$$\sum_{i=1}^{m} P_i \left(\underline{e}_i, \mathfrak{C}_i^{-1} \underline{X}_i \right) = \sum_{i=1}^{m} = P_i \sum_{r=1}^{i} \frac{y_r Y_i^{(r)}}{\theta_r^2} = \sum_{r=1}^{m} \frac{y_r Y_{.}^{(r)}}{\theta_r^2} P_{.}^{(r)}$$

In the purely additive case the optimal estimation is therefore given by

$$\hat{\mu}_q^{(m)} = X_q^{(q)} + \frac{\sum_{r=1}^{m} \frac{y_r Y_{.}^{(r)}}{\theta_r^2} P_{.}^{(r)}}{\sum_{r=1}^{m} \frac{y_r^2}{\theta_r^2} P_{.}^{(r)}} \sum_{s=q+1}^{m} y_s .$$

It can now be seen clearly how the IBNR-reserve $\hat{\mu}_q^{(m)} - X_q^{(w)}$ depends on the observed increments $Y_{.}^{(r)}$ in the r-th years of run-off:
If e.g. they exceed on average the corresponding expected values y_r, our IBNR estimation will be higher than the "true" value

$$R_q^{(m)} = \sum_{s=q+1}^{m} y_s .$$

In the above mentioned calculation the weights

$\frac{y_r P_{.}^{(r)}}{\theta_r^2}$ = expected value times reciprocal variance of $Y_{.}^{(r)}$ are to be used.
In practice the true (but unknown) parameters y_r and θ_r^2 must always be replaced by estimators $\hat{y}_r$ and $\hat{\theta}_r^2$. An interesting special case is

$$\hat{y}_r = Y_{.}^{(r)}$$

i.e. the mean r-th increment y_r is estimated on the basis of the individual experience of the risk category considered. In this situation the result for the purely additive model becomes

$$\hat{\mu}_q^{(m)} = X_q^{(q)} + \sum_{s=q+1}^{m} Y_{.}^{(s)}$$

irrespective of the estimation of the variances θ_r^2, again in accordance with the intuitively expected rule saying: IBNR reserve in % of the underlying premium volume of year q = sum of the hitherto observed increments from run-off year $q+1$ onward.

6. A numerical example

In practice of course the parameters $\underline{e}_m$ and $\mathfrak{C}_m$ are unknowns. We must estimate them on the basis of the development triangle, either directly under the hypothesis of one of the above described models or more generally we assume the correlations ϱ_{hk} between $X_i^{(h)}$ and $X_i^{(k)}$ for given k ($h \leqq k$) non-decreasing with h. (This corresponds to intuition: the burning cost of year h depends at least as much on the value in year k as the burning cost of year h–1). Such a general statement leads in the main to the methods described in [2].
In the following illustrative example, however, let us for the sake of simplicity leave aside this rather problematic estimation of monotonous correlations and instead use the true parameters $\underline{e}_m$ which – as mentioned above – in practice are unknown.
Let us assume

$$X^{(1)} = Y_1 = \begin{cases} 1 \\ 2 \end{cases} \text{with probability} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$$

$X^{(2)} = \Lambda_2 X^{(1)} + Y_2$, $X^{(1)}, \Lambda_2, Y_2$ stochastically independent

Λ_2 and Y_2 having the same distribution as Y_1,

$$X^{(3)} = \Lambda_3 X^{(2)} + Y_3$$

where $\Lambda_3, Y_3 = \begin{cases} 1 \\ 2 \end{cases} \text{with probability} \begin{matrix} 2/3 \\ 1/3 \end{matrix}$

and generally

$$X^{(n)} = \Lambda_n X^{(n-1)} + Y_n$$

with $\Lambda_n, Y_n = \begin{cases} 1 \\ 3 \end{cases} \text{with probability} \begin{matrix} (n-1)/n \\ 1/n \end{matrix}$

i.e.: $$\lambda_h = y_h = \frac{n-1}{n} + \frac{2}{n} = \frac{n+1}{n}$$

and $$\varrho_h^2 = \theta_h^2 = \left(1 - \frac{n+1}{n}\right)^2 \cdot \frac{n-1}{n} + \left(2 - \frac{n+1}{n}\right)^2 \cdot \frac{1}{n} = \frac{(n-1)}{n^2}$$

We thus get the expected values

	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$
x_h	1,50	3,75	6,33	9,17	12,20	15,40

and for $h \geqq k$ the following covariances c_{hk}

	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$
$k = 1$	0,25	0,38	0,5	0,63	0,75	0,88
$k = 2$		1,38	1,83	2,29	2,75	3,21
$k = 3$			5,79	7,24	8,69	10,14
$k = 4$				16,76	20,11	23,46
$k = 5$					37,73	44,02
$k = 6$						72,17

Finally, simulation based on these probabilities leads to the following run-off triangle $\hat{R}_q = \hat{\mu}_q^{(6)} - X_q^{(q)}$

year of occurrence	burning costs observed at the end of year					
	6	5	4	3	2	1
$i = 6$	1	4	9	10	11	12
$i = 5$		2	6	7	9	9
$i = 4$			1	3	5	6
$i = 3$				1	4	5
$i = 2$					1	4
$i = 1$						2

Using the theoretically correct expected values and covariances we get the IBNR-reserves

	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
$\hat{R}_q$	15,30	12,09	8,12	5,00	4,36

in contrast to the exact reserves

	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
R_q	15,15	11,98	8,07	4,97	4,33

References:

[1] *E. Straub*, On the Calculation of IBNR Reserves, IBNR – the prize-winning papers in the Boleslaw-Monic-Fund Competition 1971, Nederlandse Reassurantie Groep N. V., Amsterdam 1972.

[2] *R. Barlow*, *D. Bartholomew*, *A. Bremner* and *D. Brunk*, Statistical Inference under Order Restrictions, John Wiley, 1972.

Zusammenfassung

Unter gewissen, nicht allzu einschränkenden, Voraussetzungen an das Bildungsgesetz von Abwicklungsstatistiken lässt sich die zugrunde liegende Kovarianzmatrix explizit invertieren und damit die erwartungstreue Minimum-Quadrat-Schätzung der Spätschadenreserven in geschlossener Form angeben.

Summary

We show that it is possible to find a relatively simple expression for the statistical IBNR-reserves if the run-off pattern of claims reserves can be described by a certain rather general type of mathematical model.

Résumé

Sous condition que le dépouillement des taux de sinistres puisse être décrit par un modèle mathématique d'un type assez général, on peut démontrer qu'il est possible d'obtenir une formule assez simple pour l'estimation statistique des réserves IBNR.

Riassunto

A condizione che l'evoluzione delle riserve sinistri possa essere rappresentata da un modello matematico d'un certo tipo abbastanza generale, si dimostra la possibilità di ottenere una formula semplice per le riserve statistiche, detta IBNR.