Zeitschrift: Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries

Herausgeber: Vereinigung Schweizerischer Versicherungsmathematiker

Band: 71 (1971)

Artikel: Numerical calculation of the Bohman-Esscher family of convolution-mixed negative binomial distribution functions

Autor: Seal, Hilary L.

DOI: https://doi.org/10.5169/seals-967162

Numerical Calculation of the Bohman-Esscher Family of Convolution-mixed Negative Binomial Distribution Functions

By Hilary L. Seal, Yale University

The convolution-mixed negative binomial has been found to provide at least a good first approximation to the probability distribution of the aggregate claims of a casualty insurance company during a period of a few years. Let X be the random variable representing the aggregate claims in a period during which t claims are expected, and write $N(t)$ and Y for the random variables representing the number of claims during that interval and the amount of an individual claim, respectively. The variable Y is supposed independent of $N(t)$ and its probability distribution invariate throughout the interval t. The convolution-mixed negative binomial distribution function is then (e.g., Seal, 1969)

$$P\{X \leqq x\} \equiv F(x,t) = \sum_{n=0}^{\infty} p_n(t)\, P^{n*}(x) \quad 0 \leqq x < \infty \tag{1}$$

where

$$p_n(t) = \binom{-k}{n}\left(\frac{k}{t+k}\right)^k \left(\frac{-t}{t+k}\right)^n \quad n = 0, 1, 2, \ldots \tag{2}$$

$P(\cdot)$ is the distribution function of Y and $P^{n*}(\cdot)$ is the distribution function of the sum of n individual claims given by

$$P^{n*}(y) = \begin{cases} \left.\begin{matrix} 0 & y < 0 \\ 1 & y \geqq 0 \end{matrix}\right\} & n = 0 \\ \int_0^y P^{(n-1)*}(y-z)\, dP(z) & n = 1, 2, 3, \ldots \end{cases} \tag{3}$$

We note that the mean of the probability distribution $p_n(t)$ of $N(t)$ is t, as stated, and its variance is $t(1+t/k)$. The mean and variance of the distribution of X are $\varkappa_1 = tp_1$ and $\varkappa_2 = tp_2 + t^2\, p_1^2/k$, respectively, where $p_j = \int\limits_0^\infty y^j\, dP(y)$. When $k \to \infty$ the negative binomial becomes a Poisson distribution with parameter t. The distribution function (1) has a discontinuity at $x = 0$ since $F(0,t) = \left(\frac{k}{t+k}\right)^k$.

The Bohman-Esscher Family

Bohman & Esscher (1963/64) proposed a general family of functions for $P(\cdot)$, namely

$$P(y) = \sum_{j=1}^{4} A_j\,(1-e^{-y/a_j}) + \sum_{j=1}^{10} P_j(y) \qquad 0 < y < \infty \tag{4}$$

where

$$P_j(y) = \begin{cases} 0 & y \leqq b_j - 2d \\ (y-b_j+2d)^2\, B_j/8d^2 & b_j - 2d < y \leqq b_j \\ B_j - (b_j+2d-y)^2\, B_j/8d^2 & b_j < y \leqq b_j + 2d \\ B_j & b_j + 2d < y \end{cases} \tag{5}$$

and

$$\sum_{j=1}^{4} A_j + \sum_{j=1}^{10} B_j = 1.$$

The frequency function corresponding to $P_j(\cdot)$ is an isosceles triangle of area B_j with a base of length $4d$, and this was chosen as an approximation to a spike of probability of height B_j corresponding to a claim of size b_j. The Laplace-Stieltjes transform of $P(\cdot)$ is given by

$$\pi(s) \equiv \int\limits_0^\infty e^{-sy}\, dP(y) = \sum_{j=1}^{4} \frac{A_j}{1+a_j s} + \frac{\sinh^2(sd)}{(sd)^2} \sum_{j=1}^{10} B_j\, e^{-b_j s}. \tag{7}$$

The authors provided the numerical values of the parameters obtained in four successful graduations of individual claim distributions of life, industrial and non-industrial fire, and third party automobile insurance, respectively. As is usual in such studies the mean value of Y (namely, the average claim size) was chosen as unity.

The skewest of the four distributions was that of non-industrial fire insurance claims. The least skew case of (4) is achieved when $A_1 = a_1 = 1$ and $A_j = 0$, $j \neq 1$; $B_j = 0$, all j; this is the negative exponential distribution. In this paper it is proposed to limit numerical calculations to the distributions of X obtained from these two extreme members of the Bohman-Esscher family of individual claim distributions. We note that the parameters of $P(\cdot)$ in the non-industrial fire distribution were:

A_1	=	.54584	a_1	=	.169061	B_1	=	.00129	b_1	=	56.269
A_2	=	.33021	a_2	=	.220886	B_2	=	.00030	b_2	=	79.715
A_3	=	.08113	a_3	=	1.929190	B_3	=	.00005	b_3	=	103.160
A_4	=	.04074	a_4	=	11.751260	B_4	=	.00010	b_4	=	126.606
						B_5	=	.00009	b_5	=	150.051
d	=	2.3				B_6	=	.00006	b_6	=	173.497
p_1	=	1	p_2	=	47.5854	B_7	=	.00007	b_7	=	208.665
						B_8	=	.00007	b_8	=	283.691
p_3	=	12,600.1				B_9	=	.00002	b_9	=	398.574
						B_{10}	=	.00003	b_{10}	=	628.281

We add that for the negative exponential $p_j = j!$, $j = 1, 2, 3, \ldots$

In our own numerical work with the Bohman-Esscher family of distributions (4) we found it convenient to "smooth" the corners of the isosceles triangles defined by (5) by replacing these triangles by Normal distributions of area B_j, centered at b_j ($j = 1, 2, \ldots 10$) and with standard deviation $d/2$. This implies

$$P_j(y) = B_j \, \Phi\left(\frac{y - b_j}{d/2}\right) \qquad (5')$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt,$$

but the limitation of y-values to the positive axis means that we thus ignore an aggregate probability equal to $\sum_{j=1}^{10} B_j\, \Phi(-2b_j/d)$. In the case of the non-industrial fire distribution this expression is of the order of 10^{-510} and thus utterly insignificant. (However we must extend the range of y-values throughout the negative axis if (5′) is to have a simple (bilateral) Laplace-Stieltjes transform.) To this degree of approximation then (7) assumes the slightly simpler form

$$\pi(s) = \sum_{j=1}^{4} \frac{A_j}{1+a_j s} + e^{(sd)^2/8} \sum_{j=1}^{10} B_j\, e^{-b_j s}. \qquad (7')$$

The resulting distribution function (1) is continuous and differentiable for all positive x.

Methods of Calculating $F(x, t)$

A number of methods have been proposed for the approximate evaluation of (1) for given $P(.)$ on a desk calculating machine. The Bohman-Esscher paper was devoted to determining the accuracy of several of these and two further methods have since been examined numerically by Bowers (1966) and Kauppi & Ojantakanen (1969), respectively. Perhaps the most successful general method has proved to be the two- or four-term Esscher approximations (see, e.g., Seal, 1969), the former being labelled $A2$ in the paper by Bohman & Esscher. Its accuracy improves with increasing t and for larger k-values but is only impressive when $P(.)$ is the negative exponential and $k \to \infty$.

As an illustration of the relatively poor performance of the two-term approximation when $t = 100$, the smallest t-value used by Bohman & Esscher, we cite the following results from that paper for the non-industrial fire claim distribution. In only one of the twelve illustrations is accuracy achieved to the second decimal place and in four cases even one-decimal accuracy is lacking.

Values of $F(x, 100)$				
$x_0 = (x-t)/\sqrt{\varkappa_2}$	$k = 20$		$k \to \infty$	
	Exact	Esscher	Exact	Esscher
0	.6199	.7330	.6257	.7552
1	.8994	.8852	.9053	.8853
2	.96479	.93354	.96550	.93121
3	.98320	.96418	.98291	.96188
4	.99145	.98128	.99107	.97953
6	.99635	.99512	.99622	.99438

It is to be noted that this failure of approximate desk machine methods occurred for the relatively large parameter pair $k = 20$ and $t = 100$. Periods during which expected claims number less than 100 are certainly of interest and k-values of the order of unity have been found to apply in automobile insurance (see Seal, 1969). The Esscher approximations would presumably be totally inadequate in these cases. However, the widespread availability of fast computers through «time sharing» makes desk machine approximations only of theoretical interest provided a simple and accurate computer calculation of (1) can be achieved.

For values of t less than 10 it would seem that the convolutions $P^{n^*}(\cdot)$ appearing in (1) could be calculated directly by means of approximate integration. However, for larger t-values this step-by-step procedure ($n = 2, 3, 4, \ldots$) becomes increasingly time consuming. In view of the functional simplicity of the Laplace-Stieltjes transform of $F(\cdot)$, namely

$$\psi(s) \equiv \int_{0-}^{\infty} e^{-sx}\, dF(x, t) = \left[1 - \frac{t}{k}\{\pi(s) - 1\}\right]^{-k} \tag{8}$$

$$\to \exp\left[t\{\pi(s) - 1\}\right] \qquad \text{when } k \to \infty$$

where $\pi(s)$ is given by (7) or (7′), the most obvious computer approach to the evaluation of (1) would be to invert (8) by standard formulas and proceed to the quadrature of the resulting integral. In fact this was the method used by Bohman & Esscher to produce their «exact» values with which to compare the various desk machine approximations.

Purpose of this Paper

The quadrature formula used by these authors was based on an ingenious method (Bohman, 1963) of bracketing the true value of an integral over an infinite range when, of necessity, the range was made finite for the numerical calculations. Unfortunately the results of Bohman & Esscher showed that the upper and lower approximations to (1) were impractically far apart and the mid-range value was adopted as the true value of the integral. In this paper we are proposing for general use a simpler quadrature formula and to illustrate it numerically we will apply it to the two «extreme» Bohman-Esscher $P(\cdot)$ distributions mentioned above. We will make calculations for the smallest and largest k values used by the joint authors, namely $k = 20$ and $k = \infty$, respectively, and also for the "difficult" k-value of unity. Besides illustrating the method for $t = 100$, Bohman & Esscher's smallest value, we will examine what simplification occurs when t is ten times as large, namely $t = 1000$, and what difficulties arise when t is as small as 10. Furthermore, we will illustrate the calculations for values of X both below and above the mean instead of limiting ourselves to the larger values as Bohman & Esscher did. Actually all our computational difficulties occurred for parameter and variable values not considered by those authors. Finally, we mention that while our objective has been to produce correct five-decimal values of $F(x,t)$ we have only achieved three or four decimal accuracy in certain cases.

Inversion Formula

We understand $F(x,t)$ to extend to the negative x-axis by means of $F(x,t) = 0$, $x < 0$, and define $\varphi(u)$, the characteristic function of X, by

$$\psi(-iu) \equiv \varphi(u) = \int_{-\infty}^{\infty} e^{iux}\, dF(x,t)$$

$$= \left[1 - \frac{t}{k}\{\pi(-iu) - 1\}\right]^{-k} \equiv A(u)\, e^{iB(u)} \qquad (9)$$

where (7′) shows that (9) implies that

$$A(u) = \left[\left\{ \frac{tw(u)}{k} \right\}^2 + \left\{ 1 + \frac{t\{1-v(u)\}}{k} \right\}^2 \right]^{-k/2} \tag{10}$$

$$\to e^{-t\{1-v(u)\}} \qquad \text{when } k \to \infty$$

and

$$B(u) = k \arctan \left[\frac{tw(u)}{k + t\{1-v(u)\}} \right] \tag{11}$$

$$\to tw(u) \text{ when } k \to \infty$$

where

$$v(u) = \sum_{j=1}^{4} \frac{A_j}{1 + a_j^2 u^2} + e^{-u^2 d^2/8} \sum_{j=1}^{10} B_j \cos(b_j u) \tag{12}$$

and

$$w(u) = \sum_{j=1}^{4} \frac{A_j a_j u}{1 + a_j^2 u^2} + e^{-u^2 d^2/8} \sum_{j=1}^{10} B_j \sin(b_j u) . \tag{13}$$

We note that $v(-u) = v(u)$, $w(-u) = -w(u)$, $v(0) = w'(0) = 1$ and $w(0) = v'(0) = 0$.

For later use we have

$$B'(u) = \{A(u)\}^{2/k} \left[\left\{ 1 + \frac{t\{1-v(u)\}}{k} \right\} tw'(u) + \frac{t^2 w(u) v'(u)}{k} \right] \to tw'(u)$$

when $k \to \infty$. In the particular case where $P(y) = 1 - e^{-y}$ it is easily verified that

$$B'(u) = \{A(u)\}^{2/k} \frac{t}{(1+u^2)^3} \left\{ 1 - \frac{t}{k} u^2 - \left(1 + \frac{t}{k} \right) u^4 \right\} \qquad B'(0) = t .$$

This function changes from positive to negative when u is the single real positive root of the equation $(k+t)u^4 + tu^2 - k = 0$, namely $u = +(1 + t/k)^{-\frac{1}{2}}$. In general $B'(u)$ becomes permanently negative for some sufficiently large value of u.

The Lévy inversion formula for $F(\cdot, t)$ may be written (Moran, 1968)

$$F(a+h, t) - F(a, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iua} - e^{-iu\overline{a+h}}}{iu} \varphi(u)\, du$$

where $a+h$ and a are continuity points *of* $F(\cdot, t)$. Writing $a = x$ and letting $h \to \infty$ we obtain the alternative form valid in our case for $x > 0$

$$F(x, t) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}}{iu} \varphi(u)\, du$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} u^{-1} A(u) \sin\{xu - B(u)\}\, du. \tag{14}$$

Since $F(0-, t) = 0$ an alternative form would be

$$F(x, t) = \frac{2}{\pi} \int_{0}^{\infty} u^{-1} A(u) \sin(xu) \cos\{B(u)\}\, du.$$

Before we apply a quadrature formula to (14) we must investigate the error introduced by replacing the upper, infinite limit of the integral by T, where T is a suitably chosen finite number.

The Introduced Error

Both the non-negative functions $v(u)$ and $w(u)$, defined by (12) and (13), respectively, tend towards zero for sufficiently large u-values. Except when the B_j's are zero strict monotonicity is not attained but, broadly, $v(\cdot)$ decreases from unity to zero while $w(\cdot)$ increases from zero to a maximum (less than unity) and then decreases to zero. When $k \to \infty$ (the Poisson case) $A(u)$ is a strictly decreasing function with $A(0) = 1$, and for both types of $P(\cdot)$ considered here $A(u)$ appears to be monotonic downwards for $k \geqq 1$. Furthermore $\lim_{u \to \infty} A(u) = (1 + t/k)^{-k}$ and we may expect a substantial simplification in our quadrature procedures when k and t are large enough for this limit to be essentially zero. The non-negative function $B(\cdot)$ increases (with small oscillations that dampen down) from zero to a maximum from which it decreases to become zero for large values of u.

Consider the error introduced into (14) by replacing the infinite limit by $T < \infty$. This error is

$$\frac{1}{\pi} \int_{T}^{\infty} u^{-1} A(u) \sin\{xu - B(u)\} du =$$

$$= \frac{1}{\pi} \int_{Tx - B(T)}^{\infty} \frac{A(\{y + B(u)\}/x)}{1 + B(u)/y} \cdot \frac{1}{1 - B'(u)/x} \cdot \frac{\sin y}{y} dy . \qquad (15)$$

In the latter integral we have introduced the monotonic transformation $ux - B(u) = y$ having supposed that $B(\cdot)$ has passed its largest local maximum so that $B'(u)$ has become permanently negative. The positive function preceding $\sin y/y$ in the integral on the right of (15) then has a decreasing denominator and a decreasing numerator. A double application of Bonnet's form of the second mean value theorem enables (15) to be written

$$\frac{A(\zeta)}{\pi} \int_{\xi_1}^{\xi_2} \frac{\sin y}{y} dy = \frac{A(\zeta)}{\pi} \{si(\xi_1) - si(\xi_2)\}$$

$$Tx - B(T) \leqq \xi_1 < \xi_2 < \infty , \qquad \zeta x = \xi_1 + B(u_{\xi_1})$$

where $si(z)$ is a sine integral (Gautschi & Cahill, 1968)

$$si(z) = \int_{z}^{\infty} \frac{\sin y}{y} dy \qquad si(0) = \pi/2$$

and has been extensively tabulated. The absolute value of the introduced error is thus less than

$$\frac{2A(\mathrm{T})}{\pi}\,|si(\xi_0)|$$

where $\xi_0<\xi_1$ is the first argument preceding ξ_1 at which $si(z)$ assumes a local maximum or minimum, namely at a multiple of π. Now ξ_1 is not known and to use for ξ_0 the multiple of π that is less than $Tx—B(T)$ could be unnecessarily stringent. In what follows we have accordingly used $A(T)$ as our error criterion when $A(T)$ is very small and in other cases the approximation

$$\frac{A(T)}{\pi}\,|si\big(Tx-B(T)\big)| \tag{16}$$

where $Tx-B(T) = m\pi$.

Actually the T-value obtained by the foregoing procedure can be replaced by a smaller quantity when t and k are large. This is because the function $A(\cdot)$ has become very small long before $B'(\cdot)$ is permanently negative. Writing T' for the value of u at which $A(\cdot)$ is suitably small we may utilize T' instead of T when

$$\frac{A(T')}{\pi}\left|\int_{T'}^{T}\frac{\sin\{xu-B(u)\}}{u}\,du\right| < \frac{A(T')}{\pi}\int_{T'}^{T}\frac{du}{u} = \frac{A(T')}{\pi}\,ln(T/T') \tag{17}$$

is sufficiently small to be neglected. Since we are seeking five decimal accuracy we must find T' from the relation

$$A(T')\log(T/T') < 5\times 10^{-6}\pi\log e = 6.8\times 10^{-6}. \tag{18}$$

On the other hand when $(1+t/k)^{-k}$ is not essentially zero we have to make use of (16). The following Table 1 provides some auxiliary values which will assist us in the determination of T. For values of u well in excess of unity $B(u)$ is relatively small compared to u and, in view of the constancy of $A(u)$, it is desirable to choose T to be an even multiple of π.

Table 1

k	t	$10^6(1+t/k)^{-k}$	$5\pi/10^6(1+t/k)^{-k}$	z	$si(\pi z)$
1	10	90909	.000173	2	.152645
	100	9901	.001587	4	.078634
	1000	999	.015724	6	.052762
				8	.039665
20	10	301	.052233	10	.031757
∞	10	45	.345991	12	.026489
				14	.022713
				16	.019879
				18	.017673
				20	.015907
				22	.014463

Calculation of T

Coincidentally the range $k \geqq 20$ and $t \geqq 100$ which was chosen by Bohman & Esscher for their numerical calculations leads to limiting values of $A(\cdot)$ that are close to zero. Table 2 shows some values of $A(\cdot)$ and $B'(\cdot)$ for the two k-values at the ends of the Bohman-Esscher range and for our two selected t-values, namely 100 and 1000. In each case we have indicated by T a value of u after which $B'(u)$ is permanently negative. For the non-industrial fire distribution this T-value has been determined rather roughly by inspection of the computed values of $B'(\cdot)$. The T'-values were calculated in each case by using (18). They are all very small in comparison with the Bohman-Esscher choice of $T = 90\pi/\sqrt{\varkappa_2}$.

Turning to the smaller values of k and t we observe that the expression on the right of (16) is rather sensitive to the size of x which can range from unity to several multiples of $\sqrt{\varkappa_2}$ in excess of t. In fact the quantity $5\pi/10^6(1+t/k)^{-k}$ shown in Table 1 represents the largest value of $si(Tx-B(T))$ that will result in an introduced error not affecting the fifth decimal place. We conclude that three decimal

Table 2

		Non-industrial Fire				Negative Exponential			
t	k	$\sqrt{\varkappa_2}$	u	$10^6 A(u)$	$B'(u)$	$\sqrt{\varkappa_2}$	u	$10^6 A(u)$	$B'(u)$
100	20	72.5158	.70	16	3.110	26.4575	.21	114	27.22
			.71	14	3.463		.22	67	24.68
			.72	13	2.193		.23	39	22.32
			$T'=$ .73	12	2.294		$T'=$ .24	23	20.13
			.74	11	2.439		.25	14	18.10
			...	...	...		...	...	...
			2.4	5×10^{-4}	0.017		$T=$ .408	2×10^{-2}	0.000
			$T=$2.6	2×10^{-4}	–0.127				
100	∞	68.9822	.76	10	15.375	14.1421	.33	54	72.47
			.77	9	16.224		.34	32	71.06
			.78	9	19.609		.35	18	69.64
			$T'=$ .79	8	21.838		$T'=$ .36	10	68.21
			.80	7	19.622		.37	6	66.78
			...	...	...		...	...	...
			5.14	$<10^{-5}$	0.040		$T=$1.0	$<10^{-5}$	0.000
			$T=$5.16	$<10^{-5}$	–0.050				
1000	20	312.387	.029	10	152.6	228.035	.026	39	349.8
			.030	7	138.0		.027	24	332.2
			.031	5	121.2		.028	15	315.6
			$T'=$ .032	3	104.6		$T'=$ .029	9	300.0
			.033	2	90.7		.030	6	285.3
			...	...	...		...	...	...
			1.88	2×10^{-20}	0.060		$T=$ .1400	8×10^{-12}	0.000
			$T=$1.90	2×10^{-20}	–0.597				
1000	∞	218.141	.040	18	533.4	44.7214	.107	12	966.3
			.041	12	523.5		.108	10	965.7
			.042	8	508.1		.109	8	965.1
			$T'=$ .043	6	490.2		$T'=$ .110	6	964.4
			.044	4	473.6		.111	5	963.8
			...	...	...		...	...	...
			5.14	$<10^{-5}$	0.397		$T=$1.0	$<10^{-5}$	0.000
			$T=$5.16	$<10^{-5}$	–0.498				

Table 3: Approximate values of T satisfying $Tx - B(T) = m\pi$

	Non-industrial Fire					Negative Exponential				
x	$k=1$ $t=10$ $m=18$	$k=1$ $t=100$ $m=20$	$k=1$ $t=1000$ $m=20$	$k=20$ $t=10$ $m=6$	$k=\infty$ $t=10$ $m=2$	$k=1$ $t=10$ $m=18$	$k=1$ $t=100$ $m=20$	$k=1$ $t=1000$ $m=20$	$k=20$ $t=10$ $m=6$	$k=\infty$ $t=10$ $m=2$
1	56.6	62.9	62.8	20.3	10.0	56.6	62.85	62.83	19.2	7.6
$t/2$	11.7	2.16	1.08	6.9	5.7	11.4	1.77	.94	5.1	3.8
t	6.2	1.56	1.02	5.3	5.1	5.8	1.28	.90	3.63	3.4
$t+\sqrt{\varkappa_2}$	2.4	1.21	.99	4.1	4.6	3.0	1.06	.88	3.20	3.3
$t+3\sqrt{\varkappa_2}$	1.4	1.06	.97	3.7	4.4	1.8	.95	.87	2.89	3.2
$t+5\sqrt{\varkappa_2}$	1.1	1.02	.96	3.6	4.4	1.5	.92	.87	2.75	3.1
Values of $\sqrt{\varkappa_2}$	23.9970	121.485	1023.52	21.9284	21.8141	10.9545	100.995	1001.00	5.000	4.47214

accuracy is obtained for the first case ($k = 1$, $t = 10$) when $Tx - B(T) = 18\pi$, and that four decimal accuracy is achieved in the second case ($k = 1$, $t = 100$) when $Tx - B(T) = 20\pi$. The approximate values of $Tx - B(T)$ for fifth decimal accuracy in the last three cases are 20π, 6π and 2π, respectively. Table 3 shows the approximate values of T obtained in each of these five cases for selected values of x. It will be observed that the smaller the value of x the longer the range of integration and thus, other things being equal, the grater the difficulty of achieving a given degree of accuracy in the approximate quadrature.

Computation of $F(x, t)$

The "brute force" approach to the calulation of the integral in (14) over a finite interval $(0, T)$ is to divide the range of integration into a large number N of panels of equal with δ and apply the trapezoidal rule to each of them. The result is

$$\int_0^T u^{-1} A(u) \sin\{xu - B(u)\}\, du \cong \tag{19}$$

$$\cong \delta \left[\frac{x-t}{2} + \sum_{j=1}^{N-1} (j\delta)^{-1} A(j\delta) \sin\{j\delta x - B(j\delta)\} + \frac{1}{2} T^{-1} A(T) \sin\{Tx - B(T)\} \right]$$

with $N\delta = T$.

An advantage of the trapezoidal rule is that it can be combined very effectively with the procedure of "extrapolation to the limit" (Henrici, 1964). We initially choose $\delta = T$ and effect the quadrature by calculating two ordinates; then change δ to $T/2$ and repeat the quadrature by calculating only one new ordinate. When δ is changed to $T/4$ two further new ordinates are required. Repeated halving of δ thus produces a series of approximations to the integral with a minimum of computational effort. If A_{j0} represents the result of the trapezoidal rule applied with a panel width $(\frac{1}{2})^j T$, $j = 0, 1, 2, \ldots,$ it can be shown that save in exceptional circumstances (Henrici, *loc. cit.*) the series A_{jl}, $l = 1, 2, 3, \ldots$ is an improvement on the corresponding A_{j0} where

$$A_{jl} = \frac{4^l A_{j,\, l-1} - A_{j-1,\, l-1}}{4^l - 1} \qquad l = 1, 2, \ldots j. \tag{20}$$

Table 4: Values of $F(x, t)$

$x_0 = (x-t)/\sqrt{\varkappa_2}$

		Non-industrial Fire						Negative Exponential					
t	k	x_0			x_0			x_0			x_0		
100	20	−5	—		1	.89943*	256	−5	—		1	.84311*,	16
		−3	—		3	.98320*,	1024	−3	.00001,	32	3	.99493*,	16
		−1	.03635,	512	5	.99474,	256	−1	.15621,	16	5	.99995,	32
100	∞	−5	—		1	.90533*,	256	−5	.00000,	32	1	.84163*,	8
		−3	—		3	.98291*,	256	−3	.00037,	16	3	.99718*,	16
		−1	.02334,	512	5	.99436,	1024	−1	.15833,	16	5	.99999,	16
1000	20	−5	—		1	.84983*,	32	−5	—		1	.84309*,	16
		−3	.00000,	64	3	.99107*,	64	−3	.00002,	32	3	.99508*,	16
		−1	.14559,	32	5	.99968,	32	−1	.15625,	16	5	.99996,	512
1000	∞	−5	—		1	.86359*,	32	−5	.00000,	128	1	.84137*,	16
		−3	.00000,	64	3	.98599*,	32	−3	.00098,	16	3	.99823*,	16
		−1	.12553,	32	5	.99911,	32	−1	.15863,	1024	5	1.00000,	16

* These values agree with Bohman and Esscher to within a unit in their final decimal place.

Although the trapezoidal approximation is known to be periodic in x (Hamming, 1962) the 'dropout' region near $x = 2\delta/\pi$, so clearly illustrated by Tuck (1967), implies that $N = xT/2\pi$ and the successive doubling of N can quickly remove the approximation from this danger zone.

Table 4 shows the results of applying the trapezoidal rule to (14) for $x_0 = (x-t)/\sqrt{\varkappa_2} = -5, -3, -1, 1, 3, 5$ and the „large" k and t values for which the appropriate T-values were given in Table 2. In each case the value of $F(\cdot)$ is followed by the value of N which provided the same five-decimal result as quadrature with half that number of panels without further change through 1024 panels. (Occasional anomalies arise because of a 5 in the sixth decimal place. Thus, for example, when $t = 1000$ and $k = \infty$, the distribution fuction in the negative exponential case with $x_0 = -1$ is .1586254 when $N = 8$, .1586243 when $N = 16$ and .1586254 when $N = 1{,}024$.) Except for the non-industrial fire case with $k = 20$ and $t = 100$ the number of ordinates necessary for five decimal accuracy is relatively small and in every case well below the 1081 chosen by Bohman & Esscher for all their illustrations.

It seems possible that trapezoidal quadrature may always be preferable to the two-term Esscher formula for desk machine calculations. For example, in the non-industrial fire case with $k = 20$, $t = 100$ and $x_0 = 3$, trapezoidal quadrature with 64 panels has already produced the result .9831 which is a substantial improvement over the two-term Esscher value given earlier.

Finally we apply the values of T provided in Table 3 to produce Table 5. Anticipating the necessity of using a larger number of panels than the 1024 required in three exceptional illustrations in Table 4, all computer runs were made with (up to) 4096 panels. Once again the number of panels shown in Table 5 is that which provided the same result to the number of decimals shown as quadrature with half that number of panels. Exceptionally the result for 4096 panels was accepted if it differed from that for 2048 panels by no more than one unit in the last decimal place retained. In one case (non-industrial fire with $t = 10$, $k = 20$, $x = 1$) it was necessary to use (20) and this is shown by affixing an asterisk to 4096.

When $k = 1$ and x is as small as unity trapezoidal quadrature begins to break down as t increases in size. This is illustrated by applying

Table 5: Values of $F(x, t)$

		Non-industrial Fire						Negative Exponential					
t	k	x			x			x			x		
10	∞	1	.07090,	4096	$t+\sqrt{\varkappa_2}$	.93424,	1024	1	.00208,	128	$t+\sqrt{\varkappa_2}$	.84384,	32
		$t/2$	.58035,	2048	$t+3\sqrt{\varkappa_2}$	.98864,	1024	$t/2$	.11980,	64	$t+3\sqrt{\varkappa_2}$	.99308,	32
		t	.75450,	2048	$t+5\sqrt{\varkappa_2}$	.99524,	1024	t	.54489,	32	$t+5\sqrt{\varkappa_2}$	.99987,	64
10	20	1	.09545,	4096*	$t+\sqrt{\varkappa_2}$	.93369,	1024	1	.00567,	4096	$t+\sqrt{\varkappa_2}$	.84463,	4096
		$t/2$	.58174,	2048	$t+3\sqrt{\varkappa_2}$	.98850,	1024	$t/2$	.15309,	64	$t+3\sqrt{\varkappa_2}$	.99220,	32
		t	.75334,	2048	$t+5\sqrt{\varkappa_2}$	.99520,	1024	t	.55089,	256	$t+5\sqrt{\varkappa_2}$	.99981,	128
10	1	1	.344,	2048	$t+\sqrt{\varkappa_2}$	.926,	256	1	.169,	2048	$t+\sqrt{\varkappa_2}$	.864,	64
		$t/2$	.632,	2048	$t+3\sqrt{\varkappa_2}$	.986,	256	$t/2$	.422,	512	$t+3\sqrt{\varkappa_2}$	.981,	64
		t	.755,	2048	$t+5\sqrt{\varkappa_2}$	.995,	2048	t	.634,	512	$t+5\sqrt{\varkappa_2}$	.999,	64
100	1	1	.0456,	16384*	$t+\sqrt{\varkappa_2}$	.8774,	512	1	.0193,	16384*	$t+\sqrt{\varkappa_2}$	.8647,	512
		$t/2$	.4530,	2048	$t+3\sqrt{\varkappa_2}$	.9802,	512	$t/2$	.3965,	1024	$t+3\sqrt{\varkappa_2}$	.9817,	256
		t	.6552,	1024	$t+5\sqrt{\varkappa_2}$	.9954,	4096	t	.6321,	512	$t+5\sqrt{\varkappa_2}$	.9975,	256
1000	1	1	failure,	16384	$t+\sqrt{\varkappa_2}$	.86469,	4096	1	failure,	16384	$t+\sqrt{\varkappa_2}$	.86466,	4096
		$t/2$	.40222,	4096	$t+3\sqrt{\varkappa_2}$	.98161,	4096	$t/2$	.39377,	4096	$t+3\sqrt{\varkappa_2}$	.98168,	4096
		t	.63290.	4096	$t+5\sqrt{\varkappa_2}$	.99750,	2048	t	.63212,	4096	$t+5\sqrt{\varkappa_2}$	.99752,	2048

(20) to the non-industrial fire case with $t = 100$, $k = 1$ and $x = 1$ as follows:

j	$l=0$	1	2	3	4
10	−.550995				
11	−.134022	.004969			
12	.014124	.063506	.067408		
13	.043280	.052999	.052299	.052059	
14	.045436	.046155	.045698	.045593	.045568

The supposed improved results for $j = 13$ (8192 panels) are disappointing and the negative values for 1024 and 2048 panels very unsatisfactory. It is clear that we must seek another method of calculating $F(x, T)$ for very small x when t is large and k is of the order of unity.

Other Quadrature Formulas

There is a wide choice of quadrature formulas (Davis & Rabinowitz, 1967). One possibility is to retain the general framework of the simple trapezoidal (19) and to modify it to conform with various objectives. If the standard procedures of Fourier analysis are applied to a function $F(x,t)$ assumed to be zero outside the interval $(0,\omega)$ the result is formula (14) with the integral replaced by the right hand expression of (19) with $\delta = 2\pi/\omega$, N arbitrary, the last term in brackets suppressed and the first term replaced by $(\omega/2-t)/2$. A modification of this to secure faster convergence (Lanczos, 1956) is to multiply the jth term in the summation by $\sin(\pi j\delta/T)/(\pi j\delta/T)$.

A formally similar procedure was used by Bohman and Esscher (*loc. cit.*) who, in effect, multiplied each term of (19) by the discontinuity factor $(1-j\delta/T)\cos(\pi j\delta/T) + \frac{1}{\pi}\sin(\pi j\delta/T)$. Another modification of (19), named after its originator Filon (Tuck, 1967), is to multiply the whole expression on the right of (19) by $\{\sin(x\delta/2)/(x\delta/2)\}^2$. All the multipliers mentioned above are fairly close to unity throughout most of the range $(0,T)$. Our experiments with them suggest that they do not produce any worthwhile improvements over the straightforward use of (19).

Alternative Method for Small x

An integral equation for $F(x,t)$ can be derived as follows. By differentiating (8) we obtain

$$\left(1+\frac{k}{t}\right)\psi'(s) = k\,\psi(s)\pi'(s)+\psi'(s)\,\pi(s)\,. \tag{21}$$

Now

$$\int_0^\infty e^{-sx}\,xF(x,t)\,dx = -\frac{x}{s}F(x,t)\,e^{-sx}\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-sx}F(x,t)\,dx+$$

$$+\frac{1}{s}\int_0^\infty e^{-sx}\,xdF(x,t) = \frac{\psi(s)}{s^2}-\frac{\psi'(s)}{s}$$

Multiplying (21) by $-1/s$ and adding $\psi(s)/s^2$ whenever $-\psi'(s)/s$ occurs we obtain after appropriate adjustment to preserve equality

$$\left(1+\frac{k}{t}\right)\left\{\frac{\psi(s)}{s^2}-\frac{\psi'(s)}{s}\right\} = \left(1+\frac{k}{t}\right)\frac{\psi(s)}{s^2}+k\frac{\psi(s)}{s}\left\{-\pi'(s)\right\}+$$

$$+\left\{\frac{\psi(s)}{s^2}-\frac{\psi'(s)}{s}\right\}\pi(s)-\frac{\psi(s)}{s}\cdot\frac{\pi(s)}{s}\,.$$

Writing this in terms of Laplace transforms

$$\left(1+\frac{k}{t}\right)\mathfrak{L}\{xF(x,t)\} = \left(1+\frac{k}{t}\right)\mathfrak{L}\left\{\int_0^x F(y,t)\,dy\right\}+k\,\mathfrak{L}\{F(x,t)*xp(x)\}+$$

$$+\,\mathfrak{L}\{xF(x,t)*p(x)\}-\mathfrak{L}\{F(x,t)*P(x)\}$$

where $p(x)=P'(x)$ and the convolution notation is standard. This relation may now be inverted to yield

$$\left(1+\frac{k}{t}\right)xF(x,t) = \int_0^x F(y,t)\left\{1+\frac{k}{t}+(k\,\overline{x-y}+y)\,p(x-y)-P(x-y)\right\}dy$$

$$= \int_0^x K(x,y)\,F(y,t)\,dy \tag{22}$$

where $K(x, y) = 1 + \frac{k}{t} + (k\,\overline{x-y} + y)\, p(x-y) - P(x-y)\,.$

A similar type of equation in terms of density functions has been given for the case $k \to \infty$ by Plackett (1969) but (22) itself appears to be new.

The integral equation (22) may be solved approximately by means of repeated trapezoidal quadrature. Choose $h = x/N$ suitably small and (22) gives

$$\left(1+\frac{k}{t}\right)hF(h,t) = \int_0^h K(h,y)F(y,t)\,dy \cong \frac{h}{2}\{K(h,0)F(0,t)+K(h,h)F(h,t)\}$$

and on solving for $F(h,t)$

$$F(h,t) \cong \left[\left(1+\frac{k}{t}\right)h - \frac{h}{2}K(h,h)\right]^{-1}\frac{h}{2}K(h,0)\left(\frac{k}{t+k}\right)^k. \qquad (23)$$

Similarly $(n = 2, 3, \ldots N)$

$$F(nh,t) \cong$$

$$\left[\left(1+\frac{k}{t}\right)nh - \frac{h}{2}K(nh,nh)\right]^{-1}\left\{\frac{h}{2}K(nh,0)\left(\frac{k}{t+k}\right)^k + h\sum_{j=1}^{n-1}K(nh,jh)\,F(jh,t)\right\}. \quad (24)$$

The relations (23) and (24) thus produce approximate numerical values of $F(h,t), F(2h,t), F(3h,t), \ldots F(Nh,t) = F(x,t)$ *seriatim*. The procedure may then be repeated with an h-value one-half its previous size, and so on until the resulting set of values of $F(x,t)$ converge. It will be observed, however, that each series requires fresh computations and that if the final series is to contain $N = 2^m$ terms it will have been based on $2^{m-1}(2^m+1)$ calculations of $K(\cdot,\cdot)$ implying a total of $\sum_{j=1}^{m} 2^{j-1}(2^j+1)$ $= (2\times 4^m + 3\times 2^m - 5)/3$ such calculations. This feature makes the method very costly and it is not to be recommended except as a last resort.

The two values of $F(x,t)$ designated "failure" in Table 5 were calculated by the foregoing procedure with $m = 7$. The successive values of $F(1, 1000)$ are shown in Table 6 commencing with $N = 8$ (so that the initial $h = 1/8$) [1]). It would seem that five-decimal accuracy has been achieved by using extrapolation to the limit, namely (20).

[1]) Computer execution time for the non-industrial fire case was about two minutes on an IBM 7094 which was only 17 seconds less than that required for the application of (19) with 16,384 terms!

Table 6: Approximations to $10^6 F(1,1000)$

N	Non-ndustrial Fire	Negative Exponential
8	9049	3735
16	5479	2473
32	4893	2187
64	4751	2083
128	4714	2038
256	4703	2017
(20)	4700	2009

References

Bohman, H. (1963): "To compute the distribution function when the characteristic function is known." *Skand. Aktu. Tidskr. 46* 41–46.

Bohman, H. & Esscher, F. (1963/64): "Studies in risk theory with numerical illustrations concerning distribution functions and stop loss premiums." *Skand. Aktu. Tidskr. 46* 173–225, *47* 1–40.

Bowers, N.L., Jr. (1966): "Expansion of probability density functions as a sum of gamma densities with applications in risk theory." *Trans. Soc. Actu. 18* 125–147.

Davis, P.J. & Rabinowitz, P. (1967): *Numerical Integration.* Blaisdell, Waltham, Mass.

Gautschi, W. & Cahill, W.F. (1968): "Exponential integral and related functions." *Handbook of Mathematical Functions,* Eds. M. Abramowitz & I.A. Stegun. National Bureau of Standards, Washington.

Hamming, R.W. (1962): *Numerical Methods for Scientists and Engineers.* McGraw Hill, New York.

Henrici, P. (1964): *Elements of Numerical Analysis.* Wiley, New York.

Kauppi, L. & Ojantakanen, P. (1969): "Approximations of the generalized Poisson function." *Astin Bull. 5* 213–226.

Lanczos, C (1956): *Applied Analysis.* Prentice-Hall, Englewood Cliffs, N. J.

Plackett, R.L. (1969): "Stochastic models of capital investment." *J. Roy. Statist. Soc., B 31* 1–28.

Seal, H.L. (1969): *Stochastic Theory of a Risk Business.* Wiley, New York.

Tuck, E.O. (1967): "A simple 'Filon-Trapezoidal' rule." *Math. Computation 21* 239–241.

Zusammenfassung

Es gibt verhältnismässig wenige Beispiele in der versicherungsmathematischen und statistischen Literatur über die Berechnung einer Verteilungsfunktion auf ihrem ganzen Definitionsbereich durch Inversion ihrer charakteristischen Funktion oder Laplace-Transformierten. Eine Ausnahme bildet die von Bohman und Esscher (Skand. Aktu. Tidskr. 1963/64) benutzte originelle Inversions-Formel, um die Verteilungsfunktion der «convolution-mixed negative binomial» mit relativ grossem Mittelwert und negativem Binomialindex zu berechnen. Tabelle 4 dieser Arbeit zeigt, dass die detaillierte Analyse des numerischen Verhaltens der charakteristischen Funktion uns erlaubt, die einfache Trapezquadratur anzuwenden, mit sehr viel weniger als den 1081 Termen, welche Bohman und Esscher benötigt haben. Es wird weiter gezeigt, dass eine Ausdehnung des Verfahrens auf kleinere Mittelwerte und Binomialindizes möglich ist, wenn man ein beträchtliches Anwachsen der Anzahl der in der Quadratur benutzten Ordinaten in Kauf nimmt. Eine andere Berechnungsmethode wurde für sehr kleine Werte der Zufallsvariablen entwickelt.

Summary

There are relatively few examples in the actuarial and statistical literature of the calculation of a distribution fuction over its whole range by inversion of its characteristic function or Laplace transform. An exception is the use by Bohman & Esscher in the 1963/64 Scandinavian actuarial journal of an original inversion formula to calculate the distribution function of the convolution-mixed negative binomial with relatively large mean value and negative binomial index. Table 4 of this paper shows that detailed analysis of the numerical behavior of the characteristic function permits the use of simple trapezoidal quadrature with far fewer than the 1081 terms used by Bohman & Esscher. Extension of the procedure to smaller mean values and binomial indices is shown to be possible at the expense of a substantial increase in the number of ordinates used in the quadrature. An alternative method of computation is developed for very small values of the random variable.

Résumé

Il y a relativement peu d'exemples dans la littérature des assurances et des statistiques du calcul d'une fonction de répartition sur tout son domaine de définition par inversion de sa fonction caractéristique ou de sa transformée de Laplace. Une exception est l'emploi par Bohman et Esscher (Skand. Aktu. Tidskr. 1963/64) d'une formule d'inversion originale pour calculer la fonction de répartition de la «convolution-mixed negative binomial» avec une espérance relativement grande et un grand indice binômial négatif. Le tableau 4 de cet article montre que l'analyse détaillée du comportement numérique de la fonction caractéristique permet l'usage de la règle de quadrature trapezoïdale simple avec beaucoup moins que les 1081 termes utilisés par Bohman et Esscher. En outre, il est montré que l'extension du procédé à des valeurs plus petites de l'espérance et des indices binômiaux est possible mais entraîne un accroissement substantiel des nombres des ordonnés utilisés dans la quadrature. Une autre méthode de calcul est développée pour de très petites valeurs de la variable aléatoire.

Riassunto

Ci sono relativamente pochi esempi nella letteratura matematica assicurativa e statistica sul calcolo d'una funzione di distribuzione in tutto il suo campo di definizione mediante inversione della sua funzione caratteristica o mediante la sua trasformata di Laplace. Un'eccezione è l'uso fatto da Bohman e Esscher (Skand. Aktu. Tidskr. 1963/64) d'una formula originale d'inversione per calcolare la funzione distributiva della «convolution-mixed negative binomial» con una media relativamente grande e un grand'indice binomiale negativo. La tavola 4 di quest'articolo mostra che l'analisi dettagliata del comportamento numerico della funzione caratteristica permette l'uso della semplice integrazione trapezoidale con molto meno dei 1081 termi usati da Bohman e Esscher. Si dimostra anche possibile l'applicazione del metodo per valori minori della media e degli indici binomiali, questo implicando tuttavia un sostanziale aumento del numero delle ordinate usate nel integrazione. Un altro metodo di calcolo viene sviluppato per dei valori molto piccoli della variabile casuale.