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Some Notes on the Qualifying Period in Disability Insurance

I. Actuarial Values

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Contents

Summary

A disability model is described in terms of a three-state, age-dependent, time-continuous Markov chain, which has non-select forces of transition. It is shown that the introduction of a qualifying period in disability insurance gives rise to another three-state process, called the registered process. The latter will have a select force of mortality for those of the insured who do not currently receive disability benefits, as well as a select force of disability.

Actuarial values are established for a disability annuity of 1 with a qualifying period, as well as for two auxiliary forms of disability insurance. In the former case, prospective premium reserve formulae are also given.

1. The actual disability process

§1 A. Disability insurance can be described in terms of two interacting sub-populations, called the group of active and the group of disabled persons, respectively. For the active sub-population, a force of mortality μ^a and a force of disablement ν are defined. Similarly a force of mortality μ^i and a force of recovery ϱ are defined for the disabled sub-population. Most of the remaining theory is based on these concepts (Du Pasquier, 1912/1913; Sverdrup, 1965, 1967).

These forces may appear as functions of the age x of the insured at the issue of the policy, of the duration t since issue, as well as of the duration u since last recovery (if any) in the case of μ^a and ν, and of the duration u since last disablement in the case of μ^i and ϱ. If one of these forces depends on x and t, separately, and not only on age attained $x+t$, we shall say that this force is *issue-select*. If a force genuinely depends on the duration u in the sub-population to which the insured belongs, we shall similarly call it *present-state-select*. Obviously a force may be both issue-select and present state-select.

§1 B. As above, we shall consistently let the symbol x denote the age of the insured at the issue of the policy, at which age he is active. Whenever necessary, we shall indicate this age by a subscript on the symbols used. Thus e. g. $P_x(A)$ will be taken to mean the probability that the event A will occur to a person who was x years old at issue.

Although this is in no way 'realistic', *we shall assume that all forces are non-select.* We could have covered the case of issue-select forces by simply writing the age x in brackets everywhere. This is not a particularly interesting generalization, however, and for simplicity we shall refrain from making it.

§1 C. If for $t \geqq 0$ the insured is active at age $x+t$, we shall write $S(t)=a$. Similarly, if he is disabled at that age, we shall write $S(t)=i$. Specifically, $S(0)=a$. Thus $S(t)$ indicates the sub-population to which the insured belongs at age $x+t$ if he is then alive. We shall write $S(t)=d$ if the insured dies within age $x+t$. In partial conformity with common actuarial practice we shall write

$$(1.1) \qquad {}_{\tau}p_{x+t}^{jk} = P_x\{S(t+\tau) = k \mid S(t) = j\}$$

for j and $k=a, i, d$; for $\tau \geqq 0$; and for $t \geqq 0$ when $j=a$, $t>0$ when $j=i, d$.

This actually amounts to defining an age-dependent, time-continuous Markov chain with the three states 'active' (state a), 'disabled' (state i), and 'dead' (state d). $S(t)$ is the state of the process observed at time t, the ${}_{\tau}p_{x+t}^{jk}$ represent the transition probabilities of the Markov chain, and μ^a_{x+t}, ν_{x+t}, ϱ_{x+t}, and μ^i_{x+t} represent the forces of transition of the chain at time t.

We shall call this Markov chain *the actual disability process.*

2. Forms of disability insurance to be studied

§ 2 A. Disability insurance has been studied by many authors (e.g. Du Pasquier (1912/13), Sverdrup (1967), Simonsen (1966/67)), to whom we refer for the general theory. We shall concentrate on some problems ensuing from the fact that disability benefits are generally paid only after a certain qualifying period. In the present paper we shall show how formulae for actuarial values and prospective premium reserves may be established in the presence of such a period. In a companion paper (Hoem, 1968 b), some of the corresponding estimation theory will be studied.

§ 2 B. In practice, disability benefits are graded according to the degree of disability of the insured. We have studied graded disability insurance *without* a qualifying period elsewhere (Hoem, 1968 a, examples 5.8, 7.4, and 12.24). The introduction of a qualifying period does not create interesting new problems connected with the grading. We shall therefore disregard the grading throughout the present paper. It might have been introduced by the methods previously given wherever desired.

§2 C. We shall study the following forms of single-life disability insurance to a person who is able and of age x at issue:

A. A disability annuity of 1 payable on each sojourn in the disabled state that lasts for more than a qualifying period of length $\varkappa$. Payments are made continuously from the end of the qualifying period and until the recipient dies or recovers.

B. A lump sum benefit of 1 payable at the end of the qualifying period of length $\varkappa$ on each sojourn as disabled that outlasts this period.

C. A disability annuity of 1 payable like the annuity in A with the additional requirement that payments are made only upon disablements taking place during an initial period $<0, \zeta]$[1]).

All payments come to an end at age $x+m$.

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[1]) Time is reckoned from the issue of the policy.

§ 2 D. Insurance form A is the basis of the ordinary disability pensions insurance and insurance for waiver of premiums during disability. In Norway, the qualifying period length $\varkappa$ is generally 3 months.

In individual disability insurance, Norwegian insurance companies will waive premius for the qualifying period as well provided the disability outlasts this period. Although no premium is drawn at present for this benefit, it may be interesting to study the mathematics involved. This is the study of insurance form B.

In collective (single-life) disability insurance in Norway, disability pensions are not paid nor is waiver of premiums accepted for periods of disability which start during the first six months after the issue of the insurance contract. (We shall call this *the initial waiting period.*) The basis for this contract is the difference between insurance forms A and C with $\zeta = \frac{1}{2}$ (if the year is chosen as time unit).

In practice therefore, insurance form A is the basic one, and forms B and C represent adjustments. This paper is devoted to these forms of insurance.

3. The registered disability process

§ 3 A. The study of the development of the $S(t)$ of § 1 C as t increases for a specific policy, is in fact the study of a sample path of a Markov chain. The insurance company does not observe this development, however.

At any time the company will classify each of the insured in their portfolio either as a disability annuity recipient or as a non-recipient, but they will not know whether a non-recipient is actually active or wheter he is disabled but still within the qualifying period. In fact, part of the motivation for the introduction of a qualifying period is the wish to avoid having to keep track of short-term disability.

What the company observes for an insured at age $x+t$ may therefore be taken to be $S_\varkappa(t)$, where

$S_\varkappa(t) = n$ if the insured is alive but does not receive disability annuity payments (is a non-recipient),

$S_\varkappa(t) = r$ if he receives such payments (is a recipient), and

$S_\varkappa(t) = d$ if he has died (while insured) at an age below $x+t$.

Thus the company still observes sample paths of a three-state process. We shall call this the *registered (disability) process.*

What we have stated above needs one qualification: At age x, the insured must really be active. Thus, $S_\varkappa(0) = n$, $S(0) = a$, and $S_\varkappa(t) \neq r$ for $0 \leqq t < \varkappa$.

§ 3 B. Unfortunately the following theorem holds:

Theorem 3.1: If it is at all possible to observe a transition into state r of the registered process, this process is not a Markov chain over the state space $\{n, r, d\}$.

Proof: The transition probabilities of the registered process have the form $P_x\{S_\varkappa(t+\tau) = k | S_\varkappa(t) = j\}$, which is undefined for $0 \leqq t < \varkappa$ when $j = r$. Quite apart from this defect, the theorem may be proved as follows: We see that $P_x\{S_\varkappa(t+s+\tau) = r | S_\varkappa(t+s) = n \text{ and } S_\varkappa(t) = r\} = 0$ if $0 < s+\tau < \varkappa$. On the other hand $P_x\{S_\varkappa(t+s+\tau) = r | S_\varkappa(\xi) = n$ for all $\xi \in [0, t+s]\}$ must be positive for some choice of s, t and τ such that $0 < s+\tau < \varkappa$, since otherwise it will be impossible to observe any transition into state r. □

If $S_\varkappa(\tau) = n$ for all $\tau \in [0, t]$, let $T_\varkappa(t) = t$. Otherwise let $t - T_\varkappa(t)$ be the moment at which state $S_\varkappa(t)$ was last entered during the period $<0, t]$. Thus at time t, $T_\varkappa(t)$ is the duration of the present stay in the state $S_\varkappa(t)$ (*current duration*) of the registered process. If the values of $S_\varkappa(t)$ and $T_\varkappa(t)$ are known, no additional information about the development of the registered process during the period $[0, t]$ will change our probability statements about the development after time t. More precisely *the random process* $\{(S_\varkappa(t), T_\varkappa(t)) : t \geqq 0\}$ *is a Markov process over the state space* $\{n, r, d\} \times [0, \infty>$.

Transition probabilities of this Markov process will be generated by probabilities of the form

$$^\varkappa F_{jk}(x;\, s,\, t;\, u,\, w) \;=\; P_x\{S_\varkappa(t) = k, \quad T_\varkappa(t) \leqq w | S_\varkappa(s) = j, \quad T_\varkappa(s) = u\}$$

for $0 \leqq s \leqq t$. Since $^\varkappa F_{jk}(x; s, t; u, \infty) = P_x\{S_\varkappa(t) = k | S_\varkappa(s) = j,\ T_\varkappa(s) = u\}$, forces of transition would be defined as

$$(3.2) \qquad {}^\varkappa\mu_{jk}(x, s, u) = \lim_{\triangle s \downarrow 0} {}^\varkappa F_{jk}(x; s, s+\triangle s; u, \infty)/\triangle s \qquad \text{for } k \neq j,$$

provided such limits exist. As indicated by the notation, these forces of transition may depend on current duration u as well as on time s. Thus they may be present-state-select.

We shall devote the next paragraph to considerations on the forces of transition in (3.2).

§ 3 C. (*i*) For $j=r$ or $j=d$, existence of the limits in (3.2) poses no problem. In fact ${}^{\varkappa}\mu_{rn}(x,s,u) = \varrho_{x+s}$ and ${}^{\varkappa}\mu_{rd}(x,s,u) = \mu^{i}_{x+s}$ for all $u \geqq 0$ and $\varkappa \geqq 0$, and ${}^{\varkappa}\mu_{dn}(s,u) \equiv {}^{\varkappa}\mu_{dr}(s,u) \equiv 0$.

(ii) Similarly

$$
\begin{aligned}
&{}^{\varkappa}F_{nd}(x;s,s+\triangle s;u,\infty) \\
&= \sum_{k=1}^{2} P_x\{S_\varkappa(s+\triangle s) = d \mid S_\varkappa(s) = n,\ T_\varkappa(s) = u, S(s) = k\}. \\
&\qquad P_x\{S(s) = k \mid S_\varkappa(s) = n,\ T_\varkappa(s) = u\} \\
&= \sum_{k=1}^{2} \mu^{k}_{x+s} P_x\{S(s) = k \mid S_\varkappa(s) = n,\ T_\varkappa(s) = u\} \triangle s + o(\triangle s),
\end{aligned}
$$

with $k=1$ equivalent to $k=a$ and $k=2$ equivalent to $k=i$. Thus

$$
(3.3) \qquad {}^{\varkappa}\mu_{nd}(x,s,u) = \sum_{k=1}^{2} \mu^{\varkappa}_{x+s} P_x\{S(s) = k \mid S_\varkappa(s) = n,\ T_\varkappa(s) = u\},
$$

so this force of mortality exists and is probably genuinely present-state-select. It probably also depends on $\varkappa$. (Moen and Trier, 1960.)

(iii) There is finally the case $j=n$, $k=r$. We introduce

$$
\begin{aligned}
(3.4) \qquad {}^{\varkappa}q^{k}_{x}(s,t) &= P_x\{S_\varkappa(\tau) = n \ \text{ for } \ s \leqq \tau \leqq s+t \ \text{ and} \\
&\qquad S(s+t) = k \mid S(s) = a\} \ \text{ for } \ k = a, i; \ \text{ and}
\end{aligned}
$$

$$
(3.5) \qquad {}^{\varkappa}q_{x}(s,t) = {}^{\varkappa}q^{n}_{x}(s,t) + {}^{\varkappa}q^{r}_{x}(s,t)\,.
$$

For $s \geqq u \geqq \varkappa$, we then have ${}^{\varkappa}F_{nr}(x;s,s+\triangle s;u\cdot\infty)$

$$
\begin{aligned}
&= P_x\{S_\varkappa(s+\triangle s) = r \mid S(s-u) = a,\ S_\varkappa(\tau) = n \ \text{ for } \ s-u \leqq \tau \leqq s\} \\
&= P_x\{S_\varkappa(s+\triangle s) = r, S_\varkappa(\tau) = \text{n} \ \text{ for } \ s-u \leqq \tau \leqq s \mid S(s-u) = a\} \\
&\ / \ {}^{\varkappa}q_x(s-u,u) = \{{}^{\varkappa}q^{a}_{x}(s-u,u-\varkappa)\,\nu_{x+s-\varkappa} \triangle s\, {}_{\varkappa}\bar{p}^{i}_{x+s-\varkappa} + o(\triangle s)\} \ / \ {}^{\varkappa}q_x(s-u,u),
\end{aligned}
$$

where

$$
{}_{\tau}\bar{p}^{i}_{x+s} = \exp\{-\int_{0}^{\tau} (\varrho_{x+s+t} + \mu_{x+s+t})\, dt\}\,.
$$

Since obviously ${}^{\varkappa}F_{nr}(x; s, s+\triangle s; u, \infty) = 0$ if $u < u+\triangle s < \varkappa$, we therefore get

$$(3.7) \qquad {}^{\varkappa}\mu_{nr}(x, s, u) = \begin{cases} \dfrac{{}^{\varkappa}q_x^a(s-u, u-\varkappa)}{{}^{\varkappa}q_x(s-u, u)} \nu_{x+s-\varkappa}\, {}_{\varkappa}\bar{p}^{i}_{x+s-\varkappa} & \text{if } \varkappa \leqq u \leqq s, \\ 0 \text{ otherwise}. \end{cases}$$

Presumably ${}^{\varkappa}q_x^a(s-u, u-\varkappa)$ will converge to ${}^{\varkappa}q_x^a(s-\varkappa, 0) = 1$ as $u \downarrow \varkappa$. Similarly ${}^{\varkappa}q_x(s-u, u)$ will probably converge to ${}^{\varkappa}q_x(s-\varkappa, \varkappa)$. If this is correct, ${}^{\varkappa}\mu_{nr}(x, s, \varkappa+) = \nu_{x+s-\varkappa}\, {}_{\varkappa}\bar{p}^{i}_{x+s-\varkappa} / {}^{\varkappa}q_x(s-\varkappa, \varkappa)$ for $s > \varkappa$, while ${}^{\varkappa}\mu_{nr}(x, s, \varkappa-) = 0$. *We must therefore expect the force of disablement in the registered process to be discontinuous at* $u = \varkappa$.

For this as well as for other reasons, this force will probably depend strongly on both u and $\varkappa$, as has also been observed empirically (Stolz, 1930; Moen and Trier, 1960; Romer, 1964, p. 27).

4. Benefit form A: The disability annuity with a qualifying period

§ 4 A. In analogy with (1.1) we define

${}^{\varkappa}_{s}p^{ak}_{x+t} = P_x\{S_\varkappa(t+s) = k | S(t) = a\}$ and

${}^{\varkappa}_{s}p^{rk}_{x+t} = P_x\{S_\varkappa(t+s) = k | S_\varkappa(t) = r\}$ for $k = n, r,$ and d.

We shall also need ${}^{\varkappa}_{s}p^{ir}_{x+t}(\tau) = P_x\{S_\varkappa(t+s) = r | S(t) = i \text{ and } T(t) = \tau\}$. Obviously ${}^{\varkappa}_{s}p^{ir}_{x+t}(\tau) = {}^{\varkappa}_{s}p^{rr}_{x+t}$ for $\tau \geqq \varkappa$. If we let all ${}_{s}p^{jk}_{x+t} = 0$ for $s < 0$, we easily see that

$${}^{\varkappa}_{s}p^{jn}_{x+t} = {}_{s}p^{ja}_{x+t} + \int\limits_{s-\varkappa}^{s} {}_{\xi}p^{ja}_{x+t}\, \nu_{x+t+\xi}\, {}_{s-\xi}\bar{p}^{i}_{x+t+\xi}\, d\xi$$

for $j = a, r$, with ${}_{0}p^{ra}_{x+t} = {}_{0}p^{ia}_{x+t}$. If $\delta(x) = 1$ for $x \geqq 0$, $\delta(x) = 0$ for $x < 0$,

we also get ${}^{\varkappa}_{s}p^{an}_{x+t} = \{1-\delta(s-\varkappa)\}\ (1 - {}_{s}p^{ad}_{x+t}) + {}_{s-\varkappa}p^{aa}_{x+t}(1 - {}_{\varkappa}p^{ad}_{x+t+s-\varkappa})$,

$${}^{\varkappa}_{s}p^{rn}_{x+t} = \{1-\delta(s-\varkappa)\} \int\limits_{0}^{s} {}_{\xi}\bar{p}^{i}_{x+t}\varrho_{x+t+\xi}(1 - {}_{s-\xi}p^{ad}_{x+t+\xi})\, d\xi + {}_{s-\varkappa}p^{ia}_{x+t}(1 - {}_{\varkappa}p^{ad}_{x+t+s-\varkappa}),$$

${}^{\varkappa}_{s}p^{ar}_{x+t} = {}_{s-\varkappa}p^{ai}_{x+t}\, {}_{\varkappa}\bar{p}^{i}_{x+t+s-\varkappa}$,

$$ {}^{\varkappa}_{s}p^{rr}_{x+t} = \{1-\delta(s-\varkappa)\}\, {}_{s}\bar{p}^{i}_{x+t} + {}_{s-\varkappa}p^{ii}_{x+t}\, {}_{\varkappa}\bar{p}^{i}_{x+t+s-\varkappa}, \text{ and} $$

$$ {}^{\varkappa}_{s}p^{ir}_{x+t}(\tau) = {}_{s}\bar{p}^{i}_{x+t}\, \delta(s+\tau-\varkappa) + \int_{0}^{s-\varkappa} {}_{\xi}p^{ia}_{x+t}\, \nu_{x+t+\xi}\, {}_{s-\xi}\bar{p}^{i}_{x+t+\xi}\, d\xi $$

for $0 \leqq \tau$. All of these functions are integrable in s and t.

§4 B. The actuarial value of our disability annuity with a qualifying period equals

$$ (4.1) \qquad {}^{\varkappa}\bar{a}^{ar}_{x:\overline{m}|} = \int_{\varkappa}^{m} v^{t}\, {}^{\varkappa}_{t}p^{ar}_{x}\, dt\,, $$

and the value of the corresponding «activity» annuity equals

$$ (4.2) \qquad {}^{\varkappa}\bar{a}^{an}_{x:\overline{m}|} = \bar{a}^{a}_{x:\overline{m}|} - {}^{\varkappa}\bar{a}^{ar}_{x:\overline{m}|} = \int_{0}^{m} v^{t}\, {}^{\varkappa}_{t}p^{an}_{x}\, dt\,. $$

Proof of the integral formulae: We observe that

$$ P_{x}\{S_{\varkappa}(t) = r \text{ and } S_{\varkappa}(t+s) = r\} = {}^{\varkappa}_{t}p^{ar}_{x}\, {}^{\varkappa}_{s}p^{rr}_{x+t}\,, $$

which is an integrable function of s and t. The rest of the proof of (4.1) is quite similar to that of theorem 12.3 of Hoem (1968a). The integral formula in (4.2) then follows from the fact that ${}^{\varkappa}_{t}p^{an}_{x} = {}_{t}p^{a}_{x} - {}^{\varkappa}_{t}p^{ar}_{x}$. □

We introduce

$$ {}_{k|}\bar{a}^{\bar{i}\bar{i}}_{x:\overline{m}|} = \int_{k}^{m} v^{t}\, {}_{t}\bar{p}^{i}_{x}\, dt\,, $$

which for $m > k$ is the actuarial value of a disability annuity of 1 payable between the ages $x+k$ and $x+m$ to a person who is an invalid at age x, provided the period of invalidity is not interrupted by recovery or death, in wich case all payments are discontinued. Some easy manipulations then give

$$ (4.3) \qquad {}^{\varkappa}\bar{a}^{ar}_{x:\overline{m}|} = \int_{0}^{m-\varkappa} {}_{t}p^{aa}_{x}\, \nu_{x+t}\, v^{t}\, {}_{\varkappa|}\bar{a}^{\bar{i}\bar{i}}_{x+t:\overline{m-t}|}\, dt\,. $$

§4 C. We also introduce

$$ {}^{\varkappa}\bar{a}^{rn}_{x+t:\overline{m|}} = \int_0^m v^s\, {}^{\varkappa}_{s}p^{rn}_{x+t}\, ds, \quad {}^{\varkappa}\bar{a}^{rr}_{x+t:\overline{m|}} = \int_0^m v^s\, {}^{\varkappa}_{s}p^{rr}_{x+t}\, ds, \quad \text{and} $$

$$ {}^{\varkappa}\bar{a}^{ir}_{x+t:\overline{m|}}(\tau) = \int_0^m v^s\, {}^{\varkappa}_{s}p^{ir}_{x+t}(\tau)\, ds, $$

with obvious interpretations as actuarial values. (Proofs are similar to that of §4 B.) Formulae similar to (4.3) are easily established. For example

$$ {}^{\varkappa}\bar{a}^{rr}_{x+t:\overline{m|}} = \bar{\bar{a}}^{ii}_{x+t:\overline{m|}} + \int_0^{m-\varkappa} {}_{s}p^{ia}_{x+t}\, \nu_{x+t+s}\, v^s\, {}_{\varkappa|}\bar{\bar{a}}^{ii}_{x+t+s:\overline{m-s|}}\, ds, $$

where $\bar{\bar{a}}^{ii}_{x+t:\overline{m|}} = {}_{0|}\bar{\bar{a}}^{ii}_{x+t:\overline{m|}}$. Furthermore,

$$ {}^{\varkappa}\bar{a}^{ir}_{x+t:\overline{m|}}(\tau) = {}_{\varkappa-\tau}\bar{p}^{i}_{x+t}\, v^{\varkappa-\tau}\, {}^{\varkappa}\bar{a}^{rr}_{x+t+\varkappa-\tau:\overline{m-\varkappa+\tau|}} + \int_0^{\varkappa-\tau} {}_{\xi}\bar{p}^{i}_{x+t}\, \varrho_{x+t+\xi}\, v^{\xi}\, {}^{\varkappa}\bar{a}^{ar}_{x+t+\xi:\overline{m-\xi|}}\, d\xi . $$

§4 D. Assume that a continuous net premium π will be paid by any non-recipient in the ages up to $x+k$, with $k \leqq m$. Then obviously

$$ \pi = {}^{\varkappa}\bar{a}^{ar}_{x:\overline{m|}} \big/ {}^{\varkappa}\bar{a}^{an}_{x:\overline{k|}} . $$

§4 E. We now turn to the question of finding formulae for the net prospective premium reserve of the disability annuity. At time t one of the following cases pertain

(*i*) We may have $S(t) = a$.
(*ii*) The insured may have been disabled for a period of length $\tau < \varkappa$, $\tau < t$.
(*iii*) We may observe $S_{\varkappa}(t) = r$.
(*iv*) The insured may be dead.

Case (*iv*) is without real interest and will be left aside.

In case (*i*) the net prospective premium reserve will be

$$ {}_{t}V^{a}_{x} = {}^{\varkappa}\bar{a}^{ar}_{x+t:\overline{m-t|}} - \pi\, {}^{\varkappa}\bar{a}^{an}_{x+t,\overline{k-t|}} \tag{4.4} $$

with ${}^{\varkappa}\bar{a}^{jl}_{x:\overline{m|}} = 0$ for $m \leqq 0$.

In case (*ii*) the net prospective premium reserve may be written as

$$ {}_tV_x^i(\tau) = {}^{\varkappa}\bar{a}^{ir}_{x+t:\overline{m-t|}}(\tau) - \pi\left\{\bar{a}^{ii}_{x+t:\overline{k-t|}} + \bar{a}^{ia}_{x+t:\overline{k-t|}} - {}^{\varkappa}\bar{a}^{ir}_{x+t:\overline{k-t|}}(\tau)\right\} . $$

For an insured in state n the net prospective premium reserve therefore equals

$$ {}_tV_x^n = \left\{ {}_tp_x^{aa}\, {}_tV_x^a + \int_0^{\varkappa} {}_{t-\tau}p_x^{aa}\, \nu_{x+t-\tau}\, {}_{\tau}\bar{p}^i_{x+t-\tau}\, {}_tV_x^i(\tau)\, d\tau \right\} / {}^{\varkappa}_{t}p_x^{an} . $$

For an insured in state r, it equals

$$ {}_tV_x^r = {}^{\varkappa}\bar{a}^{rr}_{x+t:\overline{m-t|}} - \pi\, {}^{\varkappa}\bar{a}^{rn}_{x+t:\overline{k-t|}} . $$

§ 4 F. Whereas ${}_tV_x^r$ is fairly easily calculated, the expression for ${}_tV_x^n$ is ugly. Instead of operating with separate premium reserves for the registered recipients and non-recipients, the insurer may therefore prefer the same premium reserve

$$ {}_tV_x = {}^{\varkappa}_{t}p_x^{an}\, {}_tV_x^n + {}^{\varkappa}_{t}p_x^{ar}\, {}_tV_x^r $$

for each insured who is alive at time t. With obvious notation

$$ {}_tV_x v^t (1 - {}_tp_x^{ad}) = {}_{t|m-t}\bar{a}_x^{ar} - \pi\, {}_{t|k-t}\bar{a}_x^{an} . $$

5. Disability benefits of forms *B* and *C*

§ 5 A. Our treatement of insurance forms B and C will be quite brief. No premium reserve formulae will be given.

§ 5 B. The actuarial value of the benefits of insurance form B equals

$$ (5.1) \qquad \int_0^{m-\varkappa} {}_tp_x^{aa}\, \nu_{x+t}\, {}_{\varkappa}\bar{p}^i_{x+t}\, v^{t+\varkappa}\, dt . $$

§ 5 C. For disability insurance form C the actuarial value at age x of the benefits payable in the interval $< x,\ x+\zeta]$ equals ${}^{\varkappa}\bar{a}^{ar}_{x:\overline{\zeta|}}$.

If the insured is active at age $x+\zeta$, he receives no benefits at ages over $x+\zeta$.

If the insured is an invalid at age $x+\zeta$, he must have become disabled at some age between x and $x+\zeta$, and have remained continuously disabled until age $x+\zeta$. Therefore, the actuarial value of the benefits he will receive after age $x+\zeta$ equals

$$\int_0^{\zeta-\varkappa} {}_tp_x^{aa}\,\nu_{x+t}\,v^t\,{}_{\zeta-t|}\bar{\ddot{a}}^{ii}_{x+t:\overline{m-t|}}\,dt + \int_{\zeta-\varkappa}^{\zeta} {}_tp_x^{aa}\,\nu_{x+t}\,v^t\,{}_{\varkappa|}\bar{\ddot{a}}^{ii}_{x+t:\overline{m-t|}}\,dt\,.$$

Combining these results, we conclude that the actuarial value of the benefits of insurance form C equals

$$\int_0^{\zeta-\varkappa} {}_tp_x^{aa}\,\nu_{x+t}\,v^t\,{}_{\zeta-t|}\bar{\ddot{a}}^{ii}_{x+t:\overline{m-t|}}\,dt + \int_0^{\zeta} {}_tp_x^{aa}\,\nu_{x+t}\,v^t\,{}_{\varkappa|}\bar{\ddot{a}}^{ii}_{x+t:\overline{m-t|}}\,dt\,.$$

§5 D. It follows that the actuarial value of a disability annuity of 1 payable with an initial waiting period of length ζ and a qualifying period of length $\varkappa$, equals

$$\bar{a}^{ar}_{x:\overline{m|}}(\zeta,\varkappa) = \int_\zeta^{m-\varkappa} {}_tp_x^{aa}\,\nu_{x+t}\,v^t\,{}_{\varkappa|}\bar{\ddot{a}}^{ii}_{x+t:\overline{m-t|}}\,dt - \int_0^{\zeta-\varkappa} {}_tp_x^{aa}\,\nu_{x+t}\,v^t\,{}_{\zeta-t|}\bar{\ddot{a}}^{ii}_{x+t:\overline{m-t|}}\,dt\,.$$

If $\zeta<\varkappa$, the latter integral is replaced by 0.

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Zusammenfassung

In einem Modell wird der Verlauf der Invalidität durch eine altersabhängige, zeitlich stetige Markovkette mit drei Zuständen beschrieben. Es wird insbesondere gezeigt, dass bei Einführung einer Karenzzeit ein neuer Prozess entsteht, der ebenfalls drei Zustände aufweist. Dieser wird mit «registered process» bezeichnet.

Dazu werden die versicherungstechnischen Barwerte für eine Invalidenpension mit Karenzzeit sowie für zwei Hilfsformen abgeleitet. Im ersten Fall werden auch die Formeln für die Berechnung des prospektiven Deckungskapitals angegeben.

Résumé

Un modèle d'invalidité, comprenant trois états, est décrit comme une chaîne markovienne continue, dépendant de l'âge de l'assuré. L'auteur montre en plus que l'introduction d'un délai de carence dans l'assurance-invalidité crée un nouveau processus qu'il appelle «registered process». Celui-ci a également trois états.

Il établit les valeurs actuarielles pour une pension d'invalidité avec délai de carence et pour deux formes auxiliaires de l'assurance-invalidité. Dans le premier cas, l'auteur indique aussi les formules pour le calcul de la réserve prospective.

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