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Autor:	Singh, B.
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On certain integrals involving $\tilde{\omega}_{\mu,v}(x)$

By B. Singh, Pilani (Rajasthan), India

Summary

In this paper, the author has evaluated $\tilde{\omega}_{\mu,v}(x)$ transforms of a number of functions making use of the self-reciprocal property of some functions.

1. Introduction: The function $\tilde{\omega}_{\mu,v}(x)$ has been defined as

$$\tilde{\omega}_{\mu,v}(x) = \sqrt{x} \int_0^\infty J_\mu(xt) J_v\left(\frac{1}{t}\right) \frac{dt}{t}, \quad R(\mu, v) \geq -\frac{1}{2}. \quad (1.1)$$

The integral on the right was first evaluated by Rao [4] and that it plays the role of a transform was conjectured by Watson [6]. Later, Bhatnagar [1] proved in detail that it plays the role of a transform. He also indicated the following properties of the function:

- (i) $\tilde{\omega}_{\mu,v}(x) = \tilde{\omega}_{v,\mu}(x),$
- (ii) $\tilde{\omega}_{v-1,v}(x) = J_{2v-1}(2\sqrt{x}),$
- (iii) $\tilde{\omega}_{\mu,v}(x) = O(x^{v+\frac{1}{2}}, x^{\mu+\frac{1}{2}}),$ when x is small,
 $= O(x^{-\frac{1}{4}}),$ when x is large.

The object of the present paper is to obtain certain integrals involving the function $\tilde{\omega}_{\mu,v}(x).$ The paper is in continuation of my paper [5].

2. We have [3, pp. 368, (34)]

$$(1) \quad \int_0^\infty x \sin\left(\frac{1}{2x}\right) K_0(ax) dx = \frac{\pi}{2a} J_1(\sqrt{a}) K_1(\sqrt{a}).$$

On multiplying both sides by $a^{v+\frac{1}{2}}\tilde{\omega}_{v,v}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned} & \int_0^\infty a^{v-\frac{1}{2}} J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{v,v}(ab) da \\ &= \frac{2}{\pi} \int_0^\infty a^{v+\frac{1}{2}} \tilde{\omega}_{v,v}(ab) da \int_0^\infty x \sin\left(\frac{1}{2x}\right) K_0(ax) dx \\ &= \frac{2}{\pi} \int_0^\infty x^{-v-\frac{1}{2}} \sin\left(\frac{1}{2x}\right) dx \int_0^\infty a^{v+\frac{1}{2}} K_0(a) \tilde{\omega}_{v,v}\left(\frac{ab}{x}\right) da. \end{aligned}$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(v) > -\frac{1}{2}$].

$$= \frac{2}{\pi} b^{v+\frac{1}{2}} \int_0^\infty x^{2v-1} K_0(bx) \sin\left(\frac{1}{2}x\right) dx, \text{ since } a^{v+\frac{1}{2}} K_0(a) \text{ is } R_{v,v}. *)$$

Evaluating the integral on the right hand side, we get

$$\begin{aligned} & \int_0^\infty a^{v-\frac{1}{2}} J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{v,v}(ab) da \\ &= \frac{2^{2v-1} \{\Gamma(v + \frac{1}{2})\}^2}{\pi b^{v+\frac{1}{2}}} {}_2F_1(v + \frac{1}{2}, v + \frac{1}{2}; \frac{3}{2}; -\frac{1}{4}b^{-2}), \quad R(v) > -\frac{1}{2}. \quad (2.1) \end{aligned}$$

Also from (1) we get

$$\begin{aligned} & \int_0^\infty a J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da \\ &= \frac{4}{\pi} \int_0^\infty x^2 K_0(x) dx \int_0^\infty \sin(a) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(2abx) da, \end{aligned}$$

*) If $f(x) = \int_0^\infty \tilde{\omega}_{\mu, v}(xy) f(y) dy$, then $f(x)$ is called $R_{\mu, v}$.

the change in the order of integrations is justified.

$$= \frac{4}{\pi} \int_0^\infty x^2 K_0(x) \sin(2bx) dx, \text{ since } \sin(a) \text{ is } R_{\frac{1}{2}, \frac{1}{2}}.$$

Hence we obtain

$$\int_0^\infty a J_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da = \frac{32}{\pi} b {}_2F_1(2, 2; \frac{3}{2}; -4b^2). \quad (2.2)$$

Again, on using the result [3, pp.369, (35)]

$$\int_0^\infty x \cos\left(\frac{1}{2x}\right) K_0(ax) dx = -\frac{\pi}{2a} Y_1(\sqrt{a}) K_1(\sqrt{a}),$$

we get by a method similar to the above

$$\int_0^\infty a^{v-\frac{1}{2}} Y_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{v, v}(ab) da = -\frac{2^{2v-1} \{ \Gamma(v) \}^2}{\pi b^{v-\frac{1}{2}}} {}_2F_1(v, v; \frac{1}{2}; -\frac{1}{4}b^{-2}), \\ R(v) > 0; \quad (2.3)$$

and

$$\int_0^\infty a Y_1(\sqrt{a}) K_1(\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da = \frac{2(8b^2-1)}{(1+4b^2)^{5/2}}. \quad (2.4)$$

3. We have [3, pp.344, (34)]

$$\int_1^\infty (x^2-1)^{-\frac{1}{2}} J_v(ax) dx = -\frac{\pi}{2} J_{\frac{1}{2}v}(\frac{1}{2}a) Y_{\frac{1}{2}v}(\frac{1}{2}a), \quad R(v) > -1.$$

On multiplying both sides by $a^\lambda \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\int_0^\infty a^\lambda J_{\frac{1}{2}v}(\frac{1}{2}a) Y_{\frac{1}{2}v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\ = -\frac{2}{\pi} \int_1^\infty (x^2-1)^{-\frac{1}{2}} dx \int_0^\infty a^\lambda J_v(ax) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da.$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(\lambda + v) \geq 0$, $R(v - \lambda) \geq -1$, $R(\lambda) < -\frac{1}{4}$]

$$\begin{aligned}
 &= -\frac{2}{\pi} y^\lambda \int_0^1 x^{2\lambda} (1-x^2)^{-\frac{1}{2}} J_v(xy) dx, \text{ since } a^\lambda J_v(a) \text{ is } R_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}, [1] \\
 &= -\frac{y^{v+\lambda} \Gamma(\lambda + \frac{1}{2}v + \frac{1}{2})}{\sqrt{\pi} 2^v \Gamma(\lambda + \frac{1}{2}v + 1) \Gamma(v+1)} {}_1F_2(\lambda + \frac{1}{2}v + \frac{1}{2}; \lambda + \frac{1}{2}v + 1, v+1; -y^2); \\
 &\quad R(2\lambda + v) > -1.
 \end{aligned}$$

{The integral on the right has been evaluated with the help of [3, pp. 26, (34)]}.

Hence we obtain

$$\begin{aligned}
 &\int_0^\infty a^\lambda J_{\frac{1}{2}v}(\frac{1}{2}a) Y_{\frac{1}{2}v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\
 &= -\frac{y^{v+\lambda} \Gamma(\lambda + \frac{1}{2}v + \frac{1}{2})}{\sqrt{\pi} 2^v \Gamma(\lambda + \frac{1}{2}v + 1) \Gamma(v+1)} {}_1F_2(\lambda + \frac{1}{2}v + \frac{1}{2}; \lambda + \frac{1}{2}v + 1, v+1; -\frac{1}{4}y^2); \\
 &\quad R(\lambda + v) \geq 0, R(v - \lambda) \geq -1, R(2\lambda + v) > -1, R(\lambda) < \frac{3}{4}. \quad (3.1)
 \end{aligned}$$

4. We have [3, pp. 385, (12)]

$$\int_0^\infty \cos\left(\frac{2a}{x}\right) [I_0(x) - L_0(x)] \frac{dx}{x} = 2J_0(2\sqrt{a}) K_0(2\sqrt{a}).$$

Multiplying both sides by $\tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned}
 &2 \int_0^\infty J_0(2\sqrt{a}) K_0(2\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da \\
 &= \int_0^\infty \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da \int_0^\infty \cos\left(\frac{2a}{x}\right) [I_0(x) - L_0(x)] \frac{dx}{x} \\
 &= \int_0^\infty [I_0(x) - L_0(x)] \frac{dx}{x} \int_0^\infty \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) \cos\left(\frac{2a}{x}\right) da,
 \end{aligned}$$

on changing the order of integrations, which is justified.

$$\begin{aligned}
 &= \frac{1}{2} \int_0^\infty [I_0(x) - L_0(x)] \cos(\frac{1}{2}bx) dx, \text{ since } \cos(x) \text{ is } R_{-\frac{1}{2}, -\frac{1}{2}}, [1] \\
 &= \frac{1}{\pi} \int_0^\infty \sin(u) K_0(\frac{1}{2}bu) du, \\
 &= \frac{1}{\pi} \left(1 + \frac{b^2}{4}\right)^{-\frac{1}{2}} \log\left(\frac{2 + \sqrt{b^2 + 4}}{b}\right), \text{ on using [2, pp, 105, (46)]}.
 \end{aligned}$$

Thus we obtain

$$\int_0^\infty J_0(2\sqrt{a}) K_0(2\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da = \frac{1}{\pi} (b^2 + 4)^{-\frac{1}{2}} \log\left[\frac{2 + (b^2 + 4)^{\frac{1}{2}}}{b}\right]. \quad (4.1)$$

By a similar method we can prove that

$$\int_0^\infty J_0(2\sqrt{a}) K_0(2\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da = \frac{1}{\pi} (b^2 + 4)^{-\frac{1}{2}} \log\left[\frac{b + (b^2 + 4)^{\frac{1}{2}}}{2}\right]. \quad (4.2)$$

5. We know that [3, pp.347, (52)]

$$\int_0^\infty \cos\left(\frac{a}{2x}\right) [\sin(x) Y_0(x) - \cos(x) J_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}), \quad a > 0.$$

Multiplying both sides by $\tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab)$ and integrating with respect to a between the limits $(0, \infty)$, we get

$$\begin{aligned}
 I &= \int_0^\infty J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da \\
 &= \frac{1}{\pi} \int_0^\infty [\sin(x) Y_0(x) - \cos(x) J_0(x)] \frac{dx}{x} \int_0^\infty \cos\left(\frac{a}{2x}\right) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da.
 \end{aligned}$$

(The change in the order of integrations on the right hand side is justified).

Hence we obtain

$$I = \frac{2}{\pi} \int_0^\infty [\sin(x) Y_0(x) - \cos(x) J_0(x)] \cos(2bx) dx, \text{ since } \cos(x) \text{ is } R_{-\frac{1}{2}, -\frac{1}{2}}. [1]$$

Evaluating the right hand side with the help of [2, pp.103, (31), and pp.46, (19)] we have

$$\begin{aligned} \int_0^\infty J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{-\frac{1}{2}, -\frac{1}{2}}(ab) da &= -\frac{1}{2\pi} (b - b^2)^{-\frac{1}{2}} \\ &\quad + \frac{1}{\pi^2} [(b + b^2)^{-\frac{1}{2}} \log \{(1 + 2b) - 2\sqrt{b + b^2}\} + (b - b^2)^{-\frac{1}{2}} \sin^{-1}(1 - 2b)], \\ &\qquad\qquad\qquad 0 < b < 1. \end{aligned} \quad (5.1)$$

$$\begin{aligned} &= \frac{1}{\pi^2} [(b + b^2)^{-\frac{1}{2}} \log \{(2b + 1) - 2\sqrt{b + b^2}\}] \\ &\quad - \frac{1}{\pi^2} [(b^2 - b)^{-\frac{1}{2}} \log \{(2b - 1) - 2\sqrt{b^2 - b}\}], \quad b > 1. \end{aligned} \quad (5.2)$$

6. On using the result [3, pp.346, (51)]

$$\int_0^\infty [\sin(x) J_0(x) + \cos(x) Y_0(x)] \sin\left(\frac{a}{2x}\right) \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}),$$

we get

$$\begin{aligned} &\int_0^\infty J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da \\ &= \frac{1}{\pi} \int_0^\infty [\sin(x) J_0(x) + \cos(x) Y_0(x)] \frac{dx}{x} \int_0^\infty \sin\left(\frac{a}{2x}\right) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da. \end{aligned}$$

[The change in the order of integrations on the right hand side is justified.]

$$\begin{aligned} &= \frac{2}{\pi} \int_0^\infty [\sin(x) J_0(x) + \cos(x) Y_0(x)] \sin(2bx) dx, \\ &\qquad\qquad\qquad \text{since } \sin(x) \text{ is } R_{\frac{1}{2}, \frac{1}{2}}. [1] \end{aligned}$$

On evaluating the integrals on the right hand side, we get

$$\begin{aligned} & \int_0^\infty J_0(\sqrt{a}) Y_0(\sqrt{a}) \tilde{\omega}_{\frac{1}{2}, \frac{1}{2}}(ab) da \\ &= \frac{1}{2\pi} (b - b^2)^{-\frac{1}{2}} + \frac{1}{\pi^2} (b + b^2)^{-\frac{1}{2}} \log [(2b+1) - 2\sqrt{b+b^2}] \\ & \quad - \frac{1}{\pi^2} (b - b^2)^{-\frac{1}{2}} \sin^{-1}(1-2b), \quad 0 < b < \frac{1}{2}. \end{aligned} \quad (6.1)$$

$$\begin{aligned} &= \frac{1}{2\pi} (b - b^2)^{-\frac{1}{2}} + \frac{1}{\pi^2} (b + b^2)^{-\frac{1}{2}} \log [(2b+1) - 2(b+b^2)^{\frac{1}{2}}] \\ & \quad + \frac{1}{\pi^2} (b - b^2)^{-\frac{1}{2}} \sin^{-1}(2b-1), \quad \frac{1}{2} < b < 1. \end{aligned} \quad (6.2)$$

$$\begin{aligned} &= \frac{1}{\pi^2} (b + b^2)^{-\frac{1}{2}} \log [(2b+1) - 2(b+b^2)^{\frac{1}{2}}] \\ & \quad + \frac{1}{\pi^2} (b^2 - b)^{-\frac{1}{2}} \log [(2b-1) - 2(b^2 - b)^{\frac{1}{2}}], \quad b > 1. \end{aligned} \quad (6.3)$$

7. We have the result [3, pp.333, (7)]

$$\int_0^1 x^{-v} (1-x^2)^{-v-\frac{1}{2}} J_v(ax) dx = \sqrt{\pi} 2^{-v-1} a^v \Gamma(\frac{1}{2}-v) J_v(\frac{1}{2}a) J_{-v}(\frac{1}{2}a), \quad R(v) < \frac{1}{2}.$$

Therefore we get

$$\begin{aligned} & \int_0^\infty a^{v+\lambda} J_v(\frac{1}{2}a) J_{-v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\ &= \frac{2^{v+1}}{\sqrt{\pi} \Gamma(\frac{1}{2}-v)} \int_0^\infty a^\lambda \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \int_0^1 (1-x^2)^{-v-\frac{1}{2}} J_v(ax) \frac{dx}{x^v} \\ &= \frac{2^{v+1}}{\sqrt{\pi} \Gamma(\frac{1}{2}-v)} \int_0^1 x^{-v-\lambda-1} (1-x^2)^{-v-\frac{1}{2}} dx \int_0^\infty a^\lambda J_v(a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}\left(\frac{ay}{x}\right) da. \end{aligned}$$

[The change in the order of integrations is permissible by Fubini's theorem provided $R(v) > -\frac{1}{2}$, $R(\lambda) < -\frac{1}{4}$, $R(v+\lambda) > -\frac{1}{2}$, $R(v) < \frac{1}{2}$.]

$$\begin{aligned} &= \frac{2^{v+1} y^\lambda}{\sqrt{\pi} \Gamma(\frac{1}{2}-v)} \int_0^1 x^{-v-2\lambda-1} (1-x^2)^{-v-\frac{1}{2}} J_v\left(\frac{y}{x}\right) dx, \quad \text{since } a^\lambda J_v(a) \text{ is} \\ & \quad R_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}. \end{aligned}$$

Evaluating the integral on the right hand side with the help of [3, pp.205, (36)], we get

$$\begin{aligned} & \int_0^\infty a^{v+\lambda} J_v(\frac{1}{2}a) J_{-v}(\frac{1}{2}a) \tilde{\omega}_{v-\lambda+\frac{1}{2}, v+\lambda-\frac{1}{2}}(ay) da \\ &= \frac{2^{4v+2\lambda-1}}{\sqrt{\pi} y^{3v+\lambda-1}} G_{13}^{20} \left(\frac{y^2}{4} \middle| \begin{matrix} 0 \\ v - \frac{1}{2}, 2v + \lambda - \frac{1}{2}, v + \lambda - \frac{1}{2} \end{matrix} \right), \\ R(2\lambda + v) < \frac{3}{2}, \quad R(v + \frac{1}{2}) > 0, \quad \frac{1}{4} > R(v + \lambda) \geq 0, \quad R(v) < \frac{1}{2}. \end{aligned} \quad (7.1)$$

If we take $\lambda = -2v$, we obtain

$$\int_0^\infty a^{-v} J_v(\frac{1}{2}a) J_{-v}(\frac{1}{2}a) \tilde{\omega}_{3v+\frac{1}{2}, -v-\frac{1}{2}}(ay) da = -y^{-v} J_v(\frac{1}{2}y) Y_v(\frac{1}{2}y). \quad (7.2)$$

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Zusammenfassung

In der vorliegenden Arbeit berechnet der Autor $\tilde{\omega}_{\mu,v}(x)$ -Transformationen verschiedener Funktionen. Dabei benutzt er wesentlich die Tatsache, dass sich gewisse Funktionen unter dieser Transformation selbst reproduzieren.

Résumé

Dans le présent travail, l'auteur soumet diverses fonctions à la transformation $\tilde{\omega}_{\mu,v}(x)$. Il y utilise principalement le fait que certaines fonctions se reproduisent par cette transformation.

Riassunto

Nel presente lavoro l'autore calcola $\tilde{\omega}_{\mu,v}(x)$ -trasformazioni di differenti funzioni. Egli impiega in modo essenziale il fatto che certe funzioni sottomesse à questa trasformazione si riproducono da se stesse.