# Some Notes on Contingent Debts

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## Summary

The article deals with decreasing liens (contingent debts) as a means of allowing for increased mortality in the assurance of sub-standard lives. A practical method is proposed which enables underwriters without special actuarial training to compute the contingent debts corresponding to a given extra mortality.

## 1.

Contingent debts are gaining ever-increasing importance in the underwriting of substandard lives. As a result of the rapid progress of modern medicine, various types of risks, more particularly certain cardio-vascular diseases, which had previously been classified as totally uninsurable, are now considered for insurance. The increasing length of human life has brought about a correspondingly increased demand for life assurance among the higher age groups.

The insurance of such lives may involve very high premium rates and often it is difficult to form a reliable estimate of the magnitude of the risk during the early policy years. In many such cases a contingent debt, carefully adapted to the nature of the extra risk and sometimes combined with a comparatively low extra premium, represents the best way of providing life assurance at reasonable cost.

Contingent debts have been extensively dealt with in actuarial literature; a review of the most important work is found in Jecklin's appendix to Vol. II of Saxer's "Versicherungsmathematik" (Springer, 1958).

The present paper owes its origin to the practical needs of an underwriting service for substandard lives; these needs which, in the author's view, have not been adequately provided for in the existing literature call for a simple method enabling underwriters, especially those without specialised actuarial training, to carry out calculations of the following types:

*a)* to compute the initial amount and term of a decreasing contingent debt corresponding to a given extra premium;

*b)* to vary the initial amount and term of a contingent debt without altering the cost of the assurance;

*c)* to select the amount and term of a contingent debt so that the resulting premium corresponds to the policy-holder's capacity to pay.

To meet practical underwriting requirements, such a method must be valid for all ages at entry and it must be applicable to both endowment and whole life assurances.

## 2.

We first consider the special case of an endowment assurance for a term of $m$ years and a survival benefit of 1. The assurance is subject to a decreasing contingent debt reducing the death benefits in successive policy years to $\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m}$. The level annual premium for this assurance is:

$$\frac{1}{m} P(IA)_{x:\overline{m|}} = \frac{1}{\ddot{a}_{x:\overline{m|}}} \left[ \sum_{t=1}^{m} \frac{t}{m} \frac{C_{x+t-1}}{D_x} + \frac{D_{x+m}}{D_x} \right]$$

$$= \frac{1}{\ddot{a}_{x:\overline{m|}}} \left[ \sum_{t=1}^{m} \frac{s_{\overline{t|}}}{s_{\overline{m|}}} \frac{C_{x+t-1}}{D_x} + \frac{D_{x+m}}{D_x} \right] + \frac{1}{\ddot{a}_{x:\overline{m|}}} \sum_{t=1}^{n} \left( \frac{t}{m} - \frac{s_{\overline{t|}}}{s_{\overline{m|}}} \right) \frac{C_{x+t-1}}{D_x} .$$

It can be easily shown that

$$\frac{1}{\ddot{a}_{x:\overline{m|}}} \left[ \sum_{t=1}^{m} \frac{s_{\overline{t|}}}{s_{\overline{m|}}} \frac{C_{x+t-1}}{D_x} \quad \frac{D_{x+m}}{D_x} \right] = P_{\overline{m|}}$$

so that we can write

$$\frac{1}{m} P(IA)_{x:\overline{m|}} = P_{\overline{m|}} + r(x,m) .$$

The remainder $r(x,m)$ is obviously $\geqq 0$, since

$$\frac{t}{m} \geqq \frac{s_{\overline{t|}}}{s_{\overline{m|}}} .$$

It is also obvious that

$$r(x,m) = m(x,m)\, P^1_{x:\overline{m}|},$$

where $m(x,m)$ is some mean value of the function $g(t) = \frac{t}{m} - \frac{s_{\overline{t}|}}{s_{\overline{m}|}}$.

If we assume that $m(x,m)$ is approximately equal to the arithmetic mean of the function $g(t)$, i.e.

$$m(x,m) \sim \frac{1}{m} \sum_{t=1}^{m} g(t) = \frac{m^2-1}{12m}\, i - \frac{m^2-1}{24m}\, i^2 \ldots$$

or, ignoring terms involving $\frac{1}{m}$ and second and higher powers of $i$,

$$m(x,m) \sim \frac{mi}{12}$$

we obtain the approximation

$$\frac{1}{m} P(IA)_{x:\overline{m}|} \sim P_{\overline{m}|} + \frac{mi}{12} P^1_{x:\overline{m}|}. \tag{1}$$

As the table in Appendix I shows this approximation produces results of a high degree of accuracy.

For the purposes of everyday underwriting practice formula (1) is still too complicated. To obtain a suitable simplification we note that, for small values of $m$, the factor $\frac{mi}{12}$ is very small, whereas for values of $m$ corresponding to comparatively long periods of contingent debts (around, say, 20 years), $\frac{mi}{12}$ does not differ very materially from $\frac{1}{m}$. For example, $\frac{1}{20} = \frac{20i}{12}$ when $i = 0.03$. These considerations lead to the further approximation

$$\frac{1}{m} P(IA)_{x:\overline{m}|} \sim P_{\overline{m}|} + \frac{1}{m} P^1_{x:\overline{m}|} \tag{2}$$

which, although less accurate than (1), may still be accepted as a sufficiently close approximation (see Appendix I).

For the purposes of computation the small remainder $r(x,m)$ will be ignored, i.e. $P_{\overline{m}|}$ will be substituted for $\frac{1}{m} P(IA)_{x:\overline{m}|}$. (The technical advantages arising from this substitution will become apparent in the following paragraphs.)

For the purposes of practical underwriting decisions the error involved in substituting $P_{\overline{m}|}$ for $\frac{1}{m} P(IA)_{x:\overline{m}|}$ will often be ignored also. In those cases where the resulting error is considered too large, formula (2) shows that a suitable correction is readily available in the form of a reduction by $\frac{1}{m}$ in every successive amount of death benefit.

3.

We now consider the general case of an endowment assurance of term $n$, where $n$ may take any integral value from 1 to $\omega - x$. (Whole life assurances are thus automatically included in our considerations.) The assurance is subject to a decreasing contingent debt of term $m$ where

$$1 \leq m \leq n \qquad \text{when } n < \omega - x$$

and

$$1 \leq m < n \qquad \text{when } n = \omega - x.$$

Let $h$ $(0 \leq h \leq 1)$ denote that portion of the sum assured in respect of which the contingent debt is operative, i.e. $hP_{\overline{m}|}$ is the level annual premium, payable for $m$ years, for that portion of the assurance. If accented symbols indicate functions relating to the life assured $(x)$ and $P(x,n,h,m)$ denotes the level annual premium for the benefit provided, the following equation of value holds:

$$P(x,n,h,m)\, \ddot{a}'_{x:\overline{n}|} = h(P_{\overline{m}|}\, \ddot{a}'_{x:\overline{m}|} - {}_mE'_x + {}_{m|}A'_{x:\overline{n-m}|}) + (1-h)\, A'_{x:\overline{n}|}. \quad (3)$$

Equation (3) is readily transformed into

$$[P(x,n,h,m) + d]\, \ddot{a}'_{x:\overline{n}|} = 1 - h\left(1 - \frac{\ddot{a}'_{x:\overline{m}|}}{\ddot{a}_{\overline{m}|}}\right).$$

If we now define a hypothetical annuity value $\ddot{a}(x,n,h,m)$ by means of the relation

$$[P(x,n,h,m) + d]\, \ddot{a}(x,n,h,m) = 1 \quad (4)$$

we obtain the fundamental equation

$$\ddot{a}\,(x,n,h,m) = \frac{\ddot{a}'_{x:\overline{n|}}}{1-h\left(1-\dfrac{\ddot{a}'_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}}\right)} \tag{5}$$

which clearly shows the interdependence of the various factors involved in a contingent debt.

4.

Before proceeding to a detailed examination of formula (5) it will be helpful to show that the function

$$f(m) = \frac{\ddot{a}_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}}$$

decreases monotonously as $m$ increases, i.e. that

$$\frac{\ddot{a}_{x:\overline{m+1|}}}{\ddot{a}_{\overline{m+1|}}} < \frac{\ddot{a}_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}}. \tag{6}$$

It is obvious that a value $k$ $(0 \leqq k \leqq m-1)$ exists so that

$$\ddot{a}_{x:\overline{m|}} = {}_kp_x\,\ddot{a}_{\overline{m|}}.$$

Bearing in mind that ${}_kp_x > {}_mp_x$ we can write

$$\frac{\ddot{a}_{x:\overline{m+1|}}}{\ddot{a}_{\overline{m+1|}}} = \frac{{}_kp_x\,\ddot{a}_{\overline{m|}} + {}_mp_x\,v^m}{\ddot{a}_{\overline{m+1|}}} < {}_kp_x = \frac{\ddot{a}_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}}$$

which proves the proposition.

5.

Reverting to our basic equation (5), it will be noted in the first place, that $\ddot{a}(x,n,h,m)$ can vary within the limits

$$\ddot{a}'_{x:\overline{n|}} = \begin{Bmatrix} \ddot{a}\,(x,n,0,m) \\ \ddot{a}\,(x,n,h,1) \end{Bmatrix} \leqq \ddot{a}(x,n,h,m) \leqq \ddot{a}\,(x,n,1,n) = \ddot{a}_{\overline{n|}}$$

and that, in respect of $P(x,n,h,m)$, the following inequalities hold

$$P'_{x:\overline{n|}} \geqq P(x,n,h,m) \geqq P_{\overline{n|}}.$$

If $m$ remains constant while $h$ increases from 0 to 1, $\ddot{a}(x,n,h,m)$ increases from $\ddot{a}'_{x:\overline{n|}}$ to $\ddot{a}'_{x:\overline{n|}} \dfrac{\ddot{a}_{\overline{m|}}}{\ddot{a}'_{x:\overline{m|}}}$. We note, in particular, the equation

$$\ddot{a}(x,n,1,m) = \ddot{a}'_{x:\overline{n|}} \frac{\ddot{a}_{\overline{m|}}}{\ddot{a}'_{x:\overline{m|}}} \tag{7}$$

which enables a rapid calculation in the case where the full death benefit is deferred for $m$ years whilst, during the period of deferment, the benefit increases by approximately equal annual steps to its final amount.

If, on the other hand, $h$ remains constant while $m$ increases from 1 to $n$, then it follows from the proposition proved in section 4 that $\ddot{a}(x,n,h,m)$ increases monotonously from $\ddot{a}'_{x:\overline{n|}}$ to $\dfrac{\ddot{a}'_{x:\overline{n|}}}{1-h\left(1-\dfrac{\ddot{a}'_{x:\overline{n|}}}{\ddot{a}_{\overline{n|}}}\right)}$; we obtain, in particular, the equation

$$\ddot{a}(x,n,h,n) = \frac{\ddot{a}_{\overline{n|}}\,\ddot{a}'_{x:\overline{n|}}}{(1-h)\,\ddot{a}_{\overline{n|}} + h\,\ddot{a}'_{x:\overline{n|}}}. \tag{8}$$

Formula (8) which is applicable when the term of the contingent debt equals the term of the assurance is readily transformed into the more suitable form

$$P(x,n,h,n) = (1-h)\,P'_{x:\overline{n|}} + h\,P_{\overline{n|}}. \tag{9}$$

6.

If we write formula (5) in the form

$$h\left(1-\frac{\ddot{a}'_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}}\right) = 1-\frac{\ddot{a}'_{x:\overline{n|}}}{\ddot{a}(x,n,h,m)}$$

and leave the right-hand side constant, the nature of the interdependence of $h$ and $m$ becomes apparent. If $\ddot{a}(x,n,h,m)$ and $m$ are given, $h$ takes the form

$$h = \frac{1-\dfrac{\ddot{a}'_{x:\overline{n|}}}{\ddot{a}(x,n,h,m)}}{1-\dfrac{\ddot{a}'_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}}} \tag{10}$$

and if $\ddot{a}(x,n,h,m)$ and $h$ are given, $m$ is determined by means of the equation

$$f(m) = \frac{\ddot{a}'_{x:\overline{m|}}}{\ddot{a}_{\overline{m|}}} = 1-\frac{1}{h}\left(1-\frac{\ddot{a}'_{x:\overline{n|}}}{\ddot{a}(x,n,h,m)}\right). \tag{11}$$

Since $f(m)$ is a monotonously decreasing function, only one value of $m$ can satisfy equation (11). The determination of $m$ can be simplified by using prepared tables of the function $f(m)$.

7.

Equation (5) can be transformed into

$$P(x,n,h,m) = P'_{x:\overline{n|}} - \frac{h}{\ddot{a}_{\overline{m|}}} \frac{\ddot{a}_{\overline{m|}} - \ddot{a}'_{x:\overline{m|}}}{\ddot{a}'_{x:\overline{n|}}} \tag{12}$$

which admits of the following simple interpretation: The premium reduction which corresponds to a contingent debt of initial amount $h$ is equivalent to the level annual premium, payable throughout the term of assurance, for an income benefit payable annually in advance for $m$ years, the annual amount of the income benefit being equal to the premium for a sinking fund policy for the sum assured $h$.

This reasoning retains its validity for certain other types of assurance. If, for example, we consider a whole life assurance with a limited term $k$ of premium payments, we can write

$$P = {}_kP'_x - \frac{h}{\ddot{a}_{\overline{m|}}} \frac{\ddot{a}_{\overline{m|}} - \ddot{a}'_{x:\overline{m|}}}{\ddot{a}'_{x:\overline{k|}}}.$$

It is apparent that $m$, the term of the contingent debt, need not be equal to the premium term; it can be shorter or longer, as occasion requires.

The practical application of formulae (5) to (12) is illustrated by the examples in Appendix II.

8.

The expressions for $\ddot{a}(x,n,h,m)$ and $P(x,n,h,m)$ derived above are exact and have general validity, irrespective of the nature of the mortality table underlying the accented premiums and annuity values. In practice, contingent debts are quoted in the case of substandard lives and it is generally accepted that a contingent debt of initial amount $\frac{\alpha}{1+\alpha}$, $(\alpha > 0)$, uniformly decreasing over the entire term of an endowment assurance of amount 1, provides adequate compensation for the constant multiplicative extra mortality $\alpha$, where abnormal mortality rates $q'_{x+t}$ are connected with normal rates $q_{x+t}$ by the relation

$$q'_{x+t} = (1+\alpha)\, q_{x+t} \qquad (t = 0, 1, 2, \ldots).$$

Assuming that this relation holds, it is possible to express $\ddot{a}(x,n,h,m)$ and $P(x,n,h,m)$ in terms of premiums and annuity values calculated at normal mortality rates, but it is necessary to point out that the relationships so obtained are approximate; moreover, those obtained below are applicable to endowment assurances only.

If, in equation (9), we so determine $h_1$ and $h_2$ that

$$P(x,n,h_1,n) = P_{x:\overline{n}|}$$

and

$$P(x,m,h_2,m) = P_{x:\overline{m}|},$$

we obtain

$$\frac{1}{1-h_1} = \frac{P'_{x:\overline{n}|}-P_{\overline{n}|}}{P_{x:\overline{n}|}-P_{\overline{n}|}}$$

and

$$\frac{1}{1-h_2} = \frac{P'_{x:\overline{m}|}-P_{\overline{m}|}}{P_{x:\overline{m}|}-P_{\overline{m}|}}.$$

If the contingent debts of initial amount $h_1$ and $h_2$ do, in fact, approximately compensate for extra mortality $\alpha$ it is permissible to write

$$h_1 \sim h_2 \sim \frac{\alpha}{1+\alpha}$$

or

$$\frac{P'_{x:\overline{n}|}-P_{\overline{n}|}}{P_{x:\overline{n}|}-P_{\overline{n}|}} \sim \frac{P'_{x:\overline{m}|}-P_{\overline{m}|}}{P_{x:\overline{m}|}-P_{\overline{m}|}} \sim 1+\alpha. \tag{13}$$

The validity of this well-known approximation is also apparent from the fact that, quite generally, $P_{x:\overline{t}|}-P_{\overline{t}|}$ may be interpreted as the mean risk premium, say $\bar{q}(x,t)$, for an endowment assurance for $t$ years. One is therefore justified in expecting the following relationships to hold:

$$\bar{q}'(x,n) \sim (1+\alpha)\,\bar{q}(x,n)$$

and

$$\bar{q}'(x,m) \sim (1+\alpha)\,\bar{q}(x,m).$$

These relationships which are equivalent to (13) are examined in Appendix III, on the basis of a modern mortality table of assured lives. It is found that, for the terms of contingent debts used in practice, formula (13) is surprisingly accurate. This formula, therefore, forms the starting point for our further deliberations.

Substituting annuity values for premiums, formula (13) can be transformed into

$$\ddot{a}'_{x:\overline{n}|} \sim \frac{\ddot{a}_{\overline{n}|}\,\ddot{a}_{x:\overline{n}|}}{(1+\alpha)\,\ddot{a}_{\overline{n}|} - \alpha\,\ddot{a}_{x:\overline{n}|}} \tag{14}$$

and a similar relationship for $\ddot{a}'_{x:\overline{m}|}$. (These approximations are also found on page 255 of Vol. II of Saxer's "Versicherungsmathematik".) Substituting these expressions for $\ddot{a}'_{x:\overline{n}|}$ and $\ddot{a}'_{x:\overline{m}|}$ in formula (5), with $h = \frac{\alpha}{1+\alpha}$, we find

$$\ddot{a}\left(x,n,\frac{\alpha}{1+\alpha},m\right) \sim \ddot{a}_{x:\overline{n}|}\,\frac{1-\dfrac{\alpha}{1+\alpha}\,\dfrac{\ddot{a}_{x:\overline{m}|}}{\ddot{a}_{\overline{m}|}}}{1-\dfrac{\alpha}{1+\alpha}\,\dfrac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{\overline{n}|}}}. \tag{15}$$

When $m = 1$ (i.e. no debt), (15) can be transformed into (14). When $m = n$, (15) reduces to

$$\ddot{a}\left(x,n,\frac{\alpha}{1+\alpha},n\right) = \ddot{a}_{x:\overline{n}|}.$$

Since, therefore, (14) is approximately correct for both limiting values of $m$, it may be expected that the formula will also produce satisfactory results for intermediate values of $m$, i.e. $1 < m < n$. Example (1) in Appendix IV confirms this expectation.

## 9.

The comparative simplicity of formula (15) is due to the fact that a particularly suitable value, namely $\frac{\alpha}{1+\alpha}$, has been selected for $h$. This value is doubtless of special practical importance, but it would be wrong to assume that serviceable formulae cannot be obtained for other values of $h$.

If, for example, we use (12) as our basic formula and write it in the form

$$P(x,n,h,m) = P'_{x:\overline{n}|} - h\,\frac{\ddot{a}'_{x:\overline{m}|}}{\ddot{a}'_{x:\overline{n}|}}\left(P'_{x:\overline{m}|} - P_{\overline{m}|}\right)$$

we obtain, having regard to (13)

$$P(x,n,h,m) \sim P_{x:\overline{n}|} + \alpha\left(P_{x:\overline{n}|} - P_{\overline{n}|}\right) - (1+\alpha)\,\frac{h}{\ddot{a}_{\overline{m}|}}\,\frac{\ddot{a}_{\overline{m}|} - \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{n}|}}\,\frac{\ddot{a}'_{x:\overline{m}|}}{\ddot{a}'_{x:\overline{n}|}}\,\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}.$$

Dealing with the quotient $\frac{\ddot{a}'_{x:\overline{m}|}}{\ddot{a}'_{x:\overline{n}|}}$ in terms of formula (14), we finally obtain

$$P(x,n,h,m) \sim P_{x:\overline{n}|} + \alpha(P_{x:\overline{n}|} - P_{\overline{n}|}) - (1+\alpha)\frac{h}{\ddot{a}_{\overline{m}|}}\frac{\ddot{a}_{\overline{m}|} - \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{n}|}}\frac{1 - \frac{\alpha}{1+\alpha}\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{\overline{n}|}}}{1 - \frac{\alpha}{1+\alpha}\frac{\ddot{a}_{x:\overline{m}|}}{\ddot{a}_{\overline{m}|}}}. \quad (16)$$

Example 2 in Appendix IV illustrates the application of formula (16). It will be noted that, despite the unavoidably large number of terms, sufficiently accurate results can be obtained.

Two more formulae, both readily deduced from (16), may be mentioned without comment:

$$\ddot{a}(x,n,1,m) \sim \ddot{a}_{x:\overline{n}|}\frac{\ddot{a}_{\overline{m}|}}{\ddot{a}_{x:\overline{m}|}}\frac{1 - \frac{\alpha}{1+\alpha}\frac{\ddot{a}_{x:\overline{m}|}}{\ddot{a}_{\overline{m}|}}}{1 - \frac{\alpha}{1+\alpha}\frac{\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{\overline{n}|}}} \quad (17)$$

which corresponds to (7) and

$$P(x,n,h,n) \sim P_{x:\overline{n}|} + [\alpha - h(1+\alpha)]\,(P_{x:\overline{n}|} - P_{\overline{n}|}) \quad (18)$$

which corresponds to (9).

## 10.

We revert to the statement in section 8 that a contingent debt of initial amount $\frac{\alpha}{1+\alpha}$, uniformly decreasing to zero over the entire term of an endowment assurance for the amount 1, is generally thought to provide adequate compensation for the constant multiplicative extra mortality $\alpha$. A measure of precision can be imparted to this statement and, incidentally, a connection be established with the theory of extra premiums, in the following manner:

In view of (13) $P_{x:\overline{n}|}$ satisfies an equation of the form

$$P_{x:\overline{n}|} + \varepsilon = \frac{1}{1+\alpha}P'_{x:\overline{n}|} + \frac{\alpha}{1+\alpha}P_{\overline{n}|}$$

where $P_{x:\overline{n}|}$ is calculated on the basis of the normal $q_{x+t}$, $P'_{x:\overline{n}|}$ on the basis of

$$q'_{x+t} = (1+\alpha)\,q_{x+t} \qquad (t = 0, 1, 2, \ldots)$$

and $\varepsilon$ is a small positive or negative quantity or may be 0.

In terms of formula (9)

$$P\left(x, n, \frac{\alpha}{1+\alpha}, n\right) = \frac{1}{1+\alpha} P'_{x:\overline{n}|} + \frac{\alpha}{1+\alpha} P_{\overline{n}|}$$

so that we can write

$$P\left(x, n. \frac{\alpha}{1+\alpha}, n\right) - \varepsilon = P_{x:\overline{n}|}. \tag{19}$$

On the other hand, the extra premium $P'_{x:\overline{n}|} - P_{x:\overline{n}|}$ satisfies the equation

$$P'_{x:\overline{n}|} - P_{x:\overline{n}|} - (1+\alpha)\,\varepsilon = \alpha(P_{x:\overline{n}|} - P_{\overline{n}|}). \tag{20}$$

Equations (19) and (20) show that when $\varepsilon = 0$, the extra mortality can be fully and accurately allowed for *either* by an extra premium amounting to $\alpha(P_{x:\overline{n}|} - P_{\overline{n}|})$, without adjustment in death benefits, *or* by a series of contingent debts amounting to $\frac{\alpha}{1+\alpha}\left(1 - \frac{s_{\overline{t}|}}{s_{\overline{n}|}}\right)$ $(t = 1, 2, \ldots)$ without increase in the premium. If $\varepsilon \neq 0$ the quotation of the extra premium $\alpha(P_{x:\overline{n}|} - P_{\overline{n}|})$ as well as the quotation of the contingent debts $\frac{\alpha}{1+\alpha}\left(1 - \frac{s_{\overline{t}|}}{s_{\overline{n}|}}\right)$ involve errors, the error in the case of the extra premium amounting to $(1+\alpha)$-times the error involved in the case of the contingent debts. If, due to the error, the premium is understated (overstated) in the one case it is also understated (overstated) in the other case.

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## Appendix I

The following table, based on A1924–29 Ult. mortality with $i = 0.03$, illustrates the approximations

$$\frac{1}{m} P(IA)_{x:\overline{m}|} \sim P_{\overline{m}|} + \frac{mi}{12} P^{1}_{x:\overline{m}|} \qquad (1)$$

and

$$\frac{1}{m} P(IA)_{x:\overline{m}|} \sim P_{\overline{m}|} + \frac{1}{m} P^{1}_{x:\overline{m}|}, \qquad (2)$$

N. B. $$\frac{1}{m} P(IA)_{x:\overline{m}|} = \frac{R_x - R_{x+m} - m M_{x+m} + m D_{x+m}}{m (N_x - N_{x+m})}$$

| $m$ | $1000\, P_{\overline{m}|}$ | $x$ | $\frac{1000}{m} P(IA)_{x:\overline{m}|}$ | $1000\left(P_{\overline{m}|} + \frac{mi}{12} P^{1}_{x:\overline{m}|}\right)$ | $1000\left(P_{\overline{m}|} + \frac{1}{m} P^{1}_{x:\overline{m}|}\right)$ |
|---|---|---|---|---|---|
| 10 | 84.7 | 20 | 84.7 | 84.8 | 84.9 |
| | | 30 | 84.8 | 84.8 | 85.0 |
| | | 40 | 84.8 | 84.8 | 85.2 |
| | | 50 | 84.9 | 85.0 | 85.8 |
| | | 60 | 85.4 | 85.4 | 87.6 |
| | | 70 | 86.5 | 86.6 | 92.2 |
| 15 | 52.2 | 20 | 52.3 | 52.3 | 52.4 |
| | | 30 | 52.3 | 52.3 | 52.4 |
| | | 40 | 52.4 | 52.4 | 52.6 |
| | | 50 | 52.7 | 52.7 | 53.2 |
| | | 60 | 53.5 | 53.6 | 54.6 |
| | | 70 | 55.4 | 55.4 | 57.8 |
| 20 | 36.1 | 20 | 36.2 | 36.2 | 36.2 |
| | | 30 | 36.3 | 36.3 | 36.3 |
| | | 40 | 36.5 | 36.5 | 36.5 |
| | | 50 | 37.0 | 37.0 | 37.0 |
| | | 60 | 38.3 | 38.3 | 38.3 |
| | | 65 | 39.4 | 39.3 | 39.3 |
| 25 | 26.6 | 20 | 26.8 | 26.8 | 26.7 |
| | | 30 | 26.9 | 26.9 | 26.8 |
| | | 40 | 27.2 | 27.2 | 27.0 |
| | | 50 | 28.0 | 28.0 | 27.5 |
| | | 60 | 29.8 | 29.7 | 28.6 |
| 30 | 20.4 | 20 | 20.6 | 20.4 | 20.5 |
| | | 30 | 20.8 | 20.5 | 20.6 |
| | | 40 | 21.2 | 21.0 | 20.8 |
| | | 50 | 22.4 | 22.1 | 21.3 |
| | | 55 | 23.4 | 23.0 | 21.6 |

## Appendix II

The numerical examples shown below are based on A1949–52 Ultimate mortality with interest at 2½%. The normal mortality rates and single life commutation functions are found in "A1949–52 Tables for Assured Lives–Volume I" published by the Cambridge University Press in 1957. Columns of $l_x$, $D_x$, $N_x$ and $M_x$ with $q_x = 2 \times$ normal, $3 \times$ normal, $4 \times$ normal and $6 \times$ normal have been published in Volume 25, Part 5 (1958) of the "Transactions of the Faculty of Actuaries".

*Example 1*

The following data are given:

$$x = 40; \qquad n = 25; \qquad 1+\alpha = 2.$$

The relevant premiums and annuity values are as follows:

$$\ddot{a}_{\overline{25}|} = 18.885 \qquad P_{40:\overline{25}|} = 31.59\,‰$$

$$\ddot{a}_{40:\overline{25}|} = 17.862 \qquad P'_{40:\overline{25}|} = 34.62\,‰$$

$$\ddot{a}'_{40:\overline{25}|} = 16.945$$

Extra premium, without contingent debt

$$P'_{40:\overline{25}|} - P_{40:\overline{25}|} = 3.03\,‰$$

*Question:* What is the initial amount $h$ of the decreasing debt, extending over the whole term of the assurance, which is exactly equivalent to the above extra premium of 3.03‰?

*Answer:* From

$$\ddot{a}(40, 25, h, 25) = \ddot{a}_{40:\overline{25}|} = 17.862$$

and equation (10) it follows that

$$h = \frac{1 - \dfrac{16.945}{17.862}}{1 - \dfrac{16.945}{18.885}} = 500\,‰.$$

*Comment:* The result is obviously correct, but it must be borne in mind that the successive death benefits, per 1,000 sum assured, corresponding to this solution, are $500\left(1 + \dfrac{s_{\overline{1}|}}{s_{\overline{25}|}}\right)$, $500\left(1 + \dfrac{s_{\overline{2}|}}{s_{\overline{25}|}}\right)$, etc. If the death benefit is to increase by equal annual increments, exact values for the initial debt $h_1$ or, alternatively, $h_2$ are found by solving the equations

$$P_{40:\overline{25}|} = h_1 \tfrac{1}{25} P'(IA)_{40:\overline{25}|} + (1-h_1)\, P'_{40:\overline{25}|}$$

and

$$P_{40:\overline{25}|} = h_2 \tfrac{1}{25} [P'(IA)_{40:\overline{25}|} - P^1_{40:\overline{25}|}] + (1-h_2)\, P'_{40:\overline{25}|}.$$

We find $h_1 = 0.57$ and $h_2 = 0.51$. The successive annual amounts of cover $^0/_{00}$ corresponding to these solutions are, in the case of $h_1$ $430 + \frac{570}{25}$; $430 + 2 \cdot \frac{570}{25}$; etc., and, in the case of $h_2$, 490; $490 + \frac{510}{25}$; $490 + 2 \cdot \frac{510}{25}$; etc.

The following table shows the complete sets of death benefits which correspond to the three initial amounts of contingent debt (it will be remembered that each of these sets of benefit is secured by the same annual premium, i.e. 31.59):

| Policy Year $t$ | Death benefit during policy year per 1000 sum assured | | | |
|---|---|---|---|---|
| | $500\left(1 + \dfrac{s_{\overline{t}|}}{s_{\overline{25}|}}\right)$ | $430 + \dfrac{570}{25}\,t$ | $490 + \dfrac{510}{25}(t-1)$ | $500 + 20(t-1)$ |
| (1) | (2) | (3) | (4) | (5) |
| 1 | 515 | 453 | 490 | 500 |
| 2 | 530 | 476 | 510 | 520 |
| 3 | 545 | 498 | 531 | 540 |
| 4 | 561 | 521 | 551 | 560 |
| 5 | 577 | 544 | 572 | 580 |
| 6 | 594 | 567 | 592 | 600 |
| 7 | 610 | 590 | 612 | 620 |
| 8 | 628 | 612 | 633 | 640 |
| 9 | 646 | 635 | 653 | 660 |
| 10 | 664 | 658 | 674 | 680 |
| 11 | 683 | 681 | 694 | 700 |
| 12 | 702 | 704 | 714 | 720 |
| 13 | 722 | 726 | 735 | 740 |
| 14 | 742 | 749 | 755 | 760 |
| 15 | 762 | 772 | 776 | 780 |
| 16 | 784 | 795 | 796 | 800 |
| 17 | 805 | 818 | 816 | 820 |
| 18 | 828 | 840 | 837 | 840 |
| 19 | 851 | 863 | 857 | 860 |
| 20 | 874 | 886 | 878 | 880 |
| 21 | 898 | 909 | 898 | 900 |
| 22 | 922 | 932 | 918 | 920 |
| 23 | 948 | 954 | 939 | 940 |
| 24 | 974 | 977 | 959 | 960 |
| 25 | 1000 | 1000 | 980 | 980 |

In the present case the adoption of the second set of figures, i.e. those corresponding to $h_1 = 0.57$, hardly commends itself. The obvious practical solution is to fix the initial debt at 500 and the annual decrement of the debt as 20. The resulting amounts of death benefit are shown in col. (5) of the above table. The remaining error has the effect of overstating the average death benefit by not more than 1% or understating the annual premium by about 0.3%.

*Example 2*

The basic data are the same as in Example 1.

*Question:* What is the amount of the extra premium which, together with a decreasing debt of an initial amount of 500‰ of the sum assured and a term of 15 years, provides full compensation for the extra mortality?

*Answer:* Since $\ddot{a}_{\overline{15|}} = 12.691$ and $\ddot{a}'_{40:\overline{15|}} = 12.165$, we have

$$\ddot{a}(40, 25, 0.5, 15) = \frac{16.945}{1-0.5\left(1-\dfrac{12.165}{12.691}\right)} = 17.303$$

and

$$P(40, 25, 0.5, 15) = 33.40\text{‰}.$$

The required extra premium consequently amounts to

$$P(40, 25, 0.5, 15) - P_{40:\overline{25|}} = (33.40 - 31.59)\text{‰} = 1.81\text{‰}.$$

*Comment:* The payment of $P(40, 25, 0.5, 15) = 33.40$‰ secures death benefits amounting to $500\left(1+\dfrac{s_{\overline{1|}}}{s_{\overline{15|}}}\right)$, $500\left(1+\dfrac{s_{\overline{2|}}}{s_{\overline{15|}}}\right)$, etc. In practice these amounts will be adjusted to either $\frac{16}{15}\cdot 500$; $\frac{17}{15}\cdot 500$; etc. or to $\frac{15}{15}\cdot 500$; $\frac{16}{15}\cdot 500$; etc., the exact premiums for these benefits being, respectively,

$$\frac{1000}{\ddot{a}'_{40:\overline{25|}}}\left[\tfrac{1}{30}(IA')^{1}_{40:\overline{15|}} + \tfrac{1}{2}A'^{1}_{40:\overline{15|}} + {}_{15|}A'_{40:\overline{10|}}\right] = 33.49$$

or

$$\frac{1000}{\ddot{a}'_{40:\overline{25|}}}\left[\tfrac{1}{30}(IA')^{1}_{40:\overline{15|}} + \tfrac{7}{15}A'^{1}_{40:\overline{15|}} + {}_{15|}A'_{40:\overline{10|}}\right] = 33.28.$$

The following specimen figures illustrate the position:

| Policy Year $t$ | Death Benefit (Maximum: 1000) | | |
|---|---|---|---|
| | $500\left(1+\frac{s_{\overline{t}\rvert}}{s_{\overline{15}\rvert}}\right)$ | $500\left(1+\frac{t}{15}\right)$ | $500\left(1+\frac{t-1}{15}\right)$ |
| | Ann. Pr. = 33.40 | Ann. Pr. = 33.49 | Ann. Pr. = 33.28 |
| (1) | (2) | (3) | (4) |
| 1 | 528 | 533 | 500 |
| 5 | 647 | 667 | 633 |
| 10 | 812 | 833 | 800 |
| 14 | 961 | 967 | 933 |
| 15 | 1000 | 1000 | 967 |
| 16 and over | 1000 | 1000 | 1000 |

The differences are small and it may well be left to the underwriter to decide whether the death benefits are to be determined in accordance with column (3) or (4).

*Example 3*

The basic data are again the same as in Example 1.

*Question:* Assuming the extra premium of $1.81^0/_{00}$ as determined in Example 2 remains unchanged, but $h$ is reduced to 0.4, what is the term $m$ of the appropriate decreasing debt?

*Answer:* From equation (11) it follows that

$$\frac{\ddot{a}'_{40:\overline{m}\rvert}}{\ddot{a}_{\overline{m}\rvert}} = 1 - \frac{1}{0.4}\left(1 - \frac{16.945}{17.303}\right) = 0.9482.$$

Since $\frac{\ddot{a}'_{40:\overline{16}\rvert}}{\ddot{a}_{\overline{16}\rvert}} = 0.9534$ and $\frac{\ddot{a}'_{40:\overline{17}\rvert}}{\ddot{a}_{\overline{17}\rvert}} = 0.9488,$

we choose $m = 17$.

*Comment:* The premium providing for death benefits of $600 + \frac{400}{17}(t-1)$ when $t \leq 17$, and 1,000 when $t > 17$, is

$$\frac{1000}{\ddot{a}'_{40:\overline{25}\rvert}} \left[\tfrac{0.4}{17}(IA')^1_{40:\overline{17}\rvert} + \left(0.6 - \tfrac{0.4}{17}\right) A'^1_{40:\overline{17}\rvert} + {}_{17\rvert}A'_{40:\overline{8}\rvert}\right] = 33.34.$$

If the death benefit for each of the first 17 years is increased by $\frac{400}{17} = 23.50\,^0/_{00}$ the premium is increased to $33.52\,^0/_{00}$. Again it is immaterial which of the two sets of linearly decreasing contingent debts is selected.

*Example 4*

The basic data are as in the previous examples.

*Question:* What is the shortest (integral) term $m$ of a decreasing contingent debt which, together with an extra premium of $1.81\,^0/_{00}$, will provide full compensation for the extra mortality?

*Answer:* Using equation (7), we write

$$\ddot{a}(40, 25, 1, m) = 17.303 = 16.945\,\frac{\ddot{a}_{\overline{m}|}}{\ddot{a}'_{40:\overline{m}|}}\,,$$

i. e.

$$\frac{\ddot{a}'_{40:\overline{m}|}}{\ddot{a}_{\overline{m}|}} = 0.9793\,.$$

From

$$\frac{\ddot{a}'_{40:\overline{9}|}}{\ddot{a}_{\overline{9}|}} = 0.9813 > 0.9793 > 0.9780 = \frac{\ddot{a}'_{40:\overline{10}|}}{\ddot{a}_{\overline{10}|}}$$

it follows that $m$ will be taken as 10. The corresponding value of $h$ is found from equation (10), namely

$$h = \frac{1 - 0.9793}{1 - 0.9780} = 0.94\,.$$

*Comment:* In practice, $h$ will probably not be calculated exactly, the initial death benefit being taken as, say, $100\,^0/_{00}$, namely sum assured divided by the term of the debt. Calculation shows that

$$\frac{1000}{\ddot{a}'_{40:\overline{25}|}}\left[0.1(IA')^{1}_{40:\overline{10}|} + {}_{10|}A'_{40:\overline{15}|}\right] = 33.40$$

so that, in this case, due to the short term of the debt, the obvious approximation yields an accurate result.

*Example 5*

In order to illustrate the application of our method in the case of a whole life assurance effected at an advanced age, we assume that the following data are given:

$$x = 60; \qquad n = \omega - 60; \qquad 1 + \alpha = 3\,.$$

The relevant premiums and annuity values are as follows:

$$\ddot{a}_{60} = 13.508 \qquad P_{60} = 49.64\,\text{‰}$$
$$\ddot{a}'_{60} = 8.363 \qquad P'_{60} = 95.18\,\text{‰}$$

Extra premium, without contingent debt

$$= 45.54\,\text{‰}$$

*Question:* To what extent is the extra premium of 45.54‰ reduced if part of the extra mortality is dealt with by a contingent debt, with $h = 0.667$ and $m = 15$?

*Answer:* From formula (5) we obtain (since $\ddot{a}_{\overline{15|}} = 12.691$ and $\ddot{a}'_{60:\overline{15|}} = 7.957$)

$$\ddot{a}(60, \omega-60, 0.667, 15) = \frac{8.363}{1-0.667\left(1-\frac{7.957}{12.691}\right)} = 11.133$$

and, consequently,

$$P(60, \omega-60, 0.667, 15) = 65.43\,\text{‰}.$$

The extra premium is, therefore, reduced to

$$(65.43 - 49.64)\,\text{‰} = 15.79\,\text{‰}.$$

*Comment:* The annual premium of 65.43 secures death benefits per 1000 sum assured, of $333\left(1+2\frac{s_{\overline{1|}}}{s_{\overline{15|}}}\right)$; $333\left(1+2\frac{s_{\overline{2|}}}{s_{\overline{15|}}}\right)$; etc., e.g. 370 in the first year, 658 in the eighth year and 1,000 in the fifteenth and subsequent years. On the basis of equal annual increments in death benefits, the same premium (65.43) secures death benefits amounting to $291 + 47.3\,t$ for $t \leq 15$ and 1,000 thereafter, e.g. 338 in the 1st year, 669 in the eighth year and 1,000 in the 15th and subsequent years. In practice the death benefits for the first 15 years will probably be determined as $333 + 44\,(t-1)$ or slightly higher; whatever error arises in a final determination of benefits on these lines is certainly negligible.

## Appendix III

The following table illustrates formula

$$\frac{P'_{x:\overline{n|}} - P_{\overline{n|}}}{P_{x:\overline{n|}} - P_{\overline{n|}}} \sim 1 + \alpha \tag{13}$$

where $P_{x:\overline{n}|}$ is calculated on the basis of the normal $q_{x+t}$ ($t = 0, 1, 2, \ldots$) of the A49–52 Ult. table whilst $P'_{x:\overline{n}|}$ is calculated on the basis of increased mortality rates $q'_{x+t} = (1+\alpha)\, q_{x+t}$ where $(1+\alpha) = 2, 3, 4$ and 6. Rate of interest $2\frac{1}{2}\%$.

| $x$ | $n$ | $\dfrac{P'_{x:\overline{n}|} - P_{\overline{n}|}}{P_{x:\overline{n}|} - P_{\overline{n}|}}$ | | | |
|---|---|---|---|---|---|
| | | $1+\alpha = 2$ | $1+\alpha = 3$ | $1+\alpha = 4$ | $1+\alpha = 6$ |
| 20 | 5 | 2.0 | 3.0 | 4.0 | 6.0 |
| | 10 | 2.0 | 3.0 | 4.0 | 6.0 |
| | 15 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 20 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 25 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 30 | 2.0 | 3.0 | 4.0 | 6.1 |
| 30 | 5 | 2.0 | 3.0 | 4.0 | 6.0 |
| | 10 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 15 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 20 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 25 | 2.0 | 3.0 | 4.0 | 6.0 |
| | 30 | 2.0 | 3.0 | 4.0 | 5.9 |
| 40 | 5 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 10 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 15 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 20 | 2.0 | 3.0 | 4.0 | 6.1 |
| | 25 | 2.0 | 3.0 | 4.0 | 5.9 |
| | 30 | 2.0 | 2.9 | 3.9 | 5.6 |
| 50 | 5 | 2.0 | 3.0 | 4.1 | 6.2 |
| | 10 | 2.0 | 3.1 | 4.1 | 6.3 |
| | 15 | 2.0 | 3.1 | 4.2 | 6.4 |
| | 20 | 2.0 | 3.1 | 4.1 | 6.2 |
| | 25 | 2.0 | 3.0 | 3.9 | 5.8 |
| | 30 | 1.9 | 2.8 | 3.6 | 5.1 |
| 60 | 5 | 2.0 | 3.1 | 4.2 | 6.5 |
| | 10 | 2.1 | 3.2 | 4.4 | 6.8 |
| | 15 | 2.1 | 3.2 | 4.3 | 6.7 |
| | 20 | 2.0 | 3.1 | 4.1 | 6.0 |
| | 25 | 2.0 | 2.9 | 3.7 | — |
| | 30 | 1.9 | 2.6 | — | — |
| 70 | 5 | 2.1 | 3.3 | 4.6 | 7.4 |
| | 10 | 2.1 | 3.4 | 4.7 | 7.4 |
| | 15 | 2.1 | 3.2 | 4.4 | — |
| | 20 | 2.0 | 3.0 | 3.9 | — |

## Appendix IV

The following examples illustrating formulae (15) and (16) are again based on A1949–52 Ult. mortality with interest at 2½%.

### *Example 1*

The following data are given:

$$x = 45; \qquad n = 25; \qquad 1+\alpha = 2.$$

*Question:* Calculate $\ddot{a}(45, 25, 0.5, 15)$ by formulae (5) and (15) and compare the results.

*Answer:* The relevant annuity values are:

$$\ddot{a}_{\overline{15|}} = 12.691 \qquad \ddot{a}_{45:\overline{25|}} = 17.192$$
$$\ddot{a}_{\overline{25|}} = 18.885 \qquad \ddot{a}'_{45:\overline{15|}} = 11.778$$
$$\ddot{a}_{45:\overline{15|}} = 12.221 \qquad \ddot{a}'_{45:\overline{25|}} = 15.778.$$

In terms of formula (5)

$$\ddot{a}(45, 25, 0.5, 15) = \frac{15.775}{1-0.5\left(1-\dfrac{11.778}{12.691}\right)} = 16.36.$$

In terms of formula (15)

$$\ddot{a}(45, 25, 0.5, 15) \sim 17.192 \frac{1-0.5\dfrac{12.221}{12.691}}{1-0.5\dfrac{17.192}{18.885}} = 16.39.$$

The approximate value is about 0.2% higher than the exact value.

### *Example 2*

The following data are given:

$$x = 50; \qquad n = 20; \qquad 1+\alpha = 4.$$

*Question:* Calculate $\ddot{a}(50, 20, 0.5, 12)$ by formulae (5) and (16) and compare the results.

*Answer:* The relevant annuity values and premiums are:

$$\ddot{a}_{\overline{12}|} = 10.514 \qquad \ddot{a}'_{50:\overline{12}|} = 8.800$$
$$\ddot{a}_{\overline{20}|} = 15.979 \qquad \ddot{a}'_{50:\overline{20}|} = 11.026$$
$$\ddot{a}_{50:\overline{12}|} = 10.043 \qquad P_{\overline{20}|} = 0.038{,}19$$
$$\ddot{a}_{50:\overline{20}|} = 14.406 \qquad P_{50:\overline{20}|} = 0.045{,}02.$$

In terms of formula (5)

$$\ddot{a}(50, 20, 0.5, 12) = \frac{11.026}{1-0.5\left(1-\dfrac{8.800}{10.514}\right)} = 12.00.$$

In terms of formula (16)

$$P(50, 20, 0.5, 12) \sim 0.045{,}02 + 3(0.045{,}02 - 0.038{,}19) -$$

$$-4 \cdot \tfrac{1}{2} \cdot \frac{1}{10.514} \frac{10.514 - 10.043}{14.406} \frac{1-\tfrac{3}{4}\dfrac{14.406}{15.979}}{1-\tfrac{3}{4}\dfrac{10.043}{10.514}} = 0.058{,}41.$$

Consequently, $\ddot{a}(50, 20, 0.5, 12) \sim 12.08.$

The approximate exceeds the exact value by 0.7%.

## Zusammenfassung

Die Arbeit befasst sich mit der Staffelung der Versicherungssumme beim Tarifierungsdienst erhöhter Risiken. Es werden einfache rechnerische Verfahren angegeben, welche auch dem mathematisch weniger geschulten Tarifikator gestatten, bei gegebener Übersterblichkeit mit richtiger Staffelungsansetzung zu operieren.

## Résumé

L'étude examine la question de l'échelonnement des prestations dans le service de tarification des risques aggravés. Des procédés de calculs simples sont indiqués qui permettent aussi au tarificateur de formation mathématique moins poussée de trouver un échelonnement correct pour une surmortalité donnée.

## Riassunto

Lo studio esamine la questione dello scaglionamento delle prestazioni assicurate nei casi di rischi tarati, indicando dei metodi di calcolo semplici che permettono anche ad un tarificatore di formazione matematica meno profonda di operare con uno scaglionamento corretto corrispondente alla sopramortalità indicata.