Zeitschrift: Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries

Herausgeber: Vereinigung Schweizerischer Versicherungsmathematiker

Band: 59 (1959)

Artikel: On some estimation problems with regard to the Poisson-distribution and the x2-minimum method

Autor: Klinken, J. van

DOI: https://doi.org/10.5169/seals-966827

On some estimation problems with regard to the Poisson-distribution and the χ^2-minimum method

By J. van Klinken, Amsterdam

Summary

In this paper the problem of estimation is dealt with. The author first discusses the χ^2-minimum method for Poisson variables. Then he explains a particular least squares principle containing the modified χ^2-minimum method as a special case. Applying this principle to discrete distributions one obtains very easily estimators for the Poisson, Binomial, and Negative Binomial case.

Introduction

In this paper we intend to make some additional remarks on the χ^2-minimum method. In most textbooks on statistics the χ^2-minimum method for estimating unknown parameters is not or rather inadequately dealt with. This is especially true for the so-called modified χ^2-minimum methods [1]. The connection with regression theory is mostly disregarded as well. An important case of regression is the one in which the random variables follow Poisson-distributions. If we want to obtain maximum likelihood estimates for the regression coefficients, we essentially have to apply a χ^2-minimum method. What we want here is stressing this connection and making some remarks on numerical solutions.

A randow variable x_t has a Poisson-distribution if

$$P_t(x) = \frac{e^{-\lambda(t)}\,(\lambda(t))^x}{x!}, \qquad x = 0, 1, 2, \ldots .$$

We consider the parameter $\lambda(t)$ as a function of the time t. In many applications $\lambda(t)$ may be represented by a polynomial $\lambda(t) = \sum_{\nu=0}^{k} \lambda_\nu t^\nu$.

The χ^2-minimum method for estimating parameters consists of the following procedure.

Suppose we have functions

$$p_i(\lambda_1, \ldots, \lambda_k) > 0, \qquad (i = 1, \ldots, h)$$

and observations

$$x_i,\ x = \sum_{i=1}^{h} x_i$$

$$\text{with } E(x_i | x) = x p_i(\lambda_1, \ldots, \lambda_k).$$

We now have to determine the estimates $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$ from

$$\sum_{i=1}^{h} \frac{(x_i - x p_i(\lambda_1, \ldots, \lambda_k))^2}{x\, p_i(\lambda_1, \ldots, \lambda_k)} = \text{minimum}.$$

We have not imposed restrictions on the distributions of the x_i. To solve this minimum problem, there are a number of approximative solutions all known as modified χ^2-minimum methods [2]. In particular if $\sum_{i=1}^{h} p_i(\lambda_1, \ldots, \lambda_k) = 1$ we may obtain a simple approximative solution. Equally, as already mentioned, if the x_i follow special distributions, e. g. Poisson-distributions, there is a simple connection with maximum likelihood estimation.

In the first section we are going to discuss the problem of determining the $\hat{\lambda}_\nu$ in relation to the χ^2-minimum method. In the second part we are treating a particular least squares method, including modified χ^2-minimum methods, which yield some estimates for the parameters of important discrete distributions. If we have the distribution $\{p_x\}$ and observations n_x which are the observed frequencies, of the value x, $x = 0, 1, 2, \ldots$, we sometimes have simple recurrence relations for En_x. As example for the Poisson-distribution we have

$$En_x = n \frac{e^{-\lambda} \lambda^x}{x!} = \frac{\lambda}{x} En_{x-1},$$

so a special form of the recurrence relation

$$En_x = \varphi(\lambda)\, f(x)\, En_{x-1}.$$

Now by using this recurrence relation we are able to indicate a few least squares methods, all giving estimates which converge in probability to the true parameter value.

The χ^2-minimum method for Poisson-variables

Suppose we have the observations x_{t_μ}, $\mu = 1, \ldots, h$. We assume that they are independent Poisson-variables with expectations

$$E\,x_{t_\mu} = w_\mu\,\lambda(t_\mu) = w_\mu \sum_{\nu=0}^{k} \lambda_\nu\, t_\mu^\nu$$

(w_μ being a factor indicating the size of the material). To obtain maximum likelihood estimates for the λ_ν we have to maximize the likelihood

$$L = \prod_{\mu=1}^{h} P(x_{t_\mu}) = \prod_{\mu=1}^{h} \frac{e^{w_\mu \lambda(t_\mu)}\,(w_\mu\,\lambda(t_\mu))^{x_{t_\mu}}}{x_{t_\mu}!}\,.$$

We find the system

$$\frac{\partial \log L}{\partial \lambda_\nu} = \sum_{\mu=1}^{h} \frac{x_{t_\mu} - w_\mu\,\lambda(t_\mu)}{w_\mu\,\lambda(t_\mu)}\, w_\mu \frac{\partial \lambda(t_\mu)}{\partial \lambda_\nu} = \sum_{\mu=1}^{h} (x_{t_\mu} - w_\mu\,\lambda(t_\mu)) \frac{\partial \log \lambda(t_\mu)}{\partial \lambda_\nu} = 0,$$

$$\nu = 0, \ldots, k. \quad (1)$$

System (1) is rather difficult to solve, but it is easy to indicate an approximative numerical solution. We may replace the denominators $w_\mu\,\lambda(t_\mu)$ by x_{t_μ}. Hence we obtain the system

$$\sum_{\mu=1}^{h} \frac{x_{t_\mu} - w_\mu\,\lambda(t_\mu)}{x_{t_\mu}}\, w_\mu \frac{\partial \lambda(t_\mu)}{\partial \lambda_\nu} = 0, \quad \nu = 0, \ldots, k. \quad (2)$$

If $\lambda(t_\mu)$ is a polynomial $\lambda(t_\mu) = \sum_{\nu=0}^{k} \lambda_\nu\, t_\mu^\nu$ we get linear equations which in general can be solved.

Because $\frac{x_{t_\mu}}{w_\mu\,\lambda(t_\mu)} \xrightarrow{\text{pr.}} 1$ the solutions of (1) and (2) are asymptotically identical. However, there is still another reason to see the solution of (2) as an approximative solution. Let the solution of (1) be ${}_0\hat{\lambda}_0, \ldots, {}_0\hat{\lambda}_k$ and that of (2) ${}_1\hat{\lambda}_0, \ldots, {}_1\hat{\lambda}_k$. We may successively solve the system

$$\sum_{\mu=1}^{h} \frac{x_{t_\mu} - w_\mu\,\lambda(t_\mu)}{{}_\iota\hat{\lambda}(t_\mu)} \frac{\partial \lambda(t_\mu)}{\partial \lambda_\nu} = 0, \qquad v = 0, \ldots, k \quad (3)$$

in which

$${}_\iota\hat{\lambda}(t_\mu) = \sum_{\nu=0}^{k} {}_\iota\hat{\lambda}_\nu\, t_\mu^\nu$$

and the solution is ${}_{\iota+1}\hat{\lambda}_0, \ldots, {}_{\iota+1}\hat{\lambda}_k$.

If this vector converges for $l \to \infty$ the limit is ${}_0\hat{\lambda}_0, \ldots, {}_0\hat{\lambda}_k$, hence the maximum likelihood estimate. We may try to obtain an answer to this question of convergence by writing the system in the form

$$ {}_l\hat{\lambda}_\nu = \varphi_\nu({}_{l-1}\hat{\lambda}_0, \ldots, {}_{l-1}\hat{\lambda}_k), \qquad \nu = 0, \ldots, k \tag{4}$$

and checking the condition

$$\sum_{\nu=0}^{k} \left| \frac{\partial \varphi_\nu}{\partial \lambda_i} \right| < 1, \quad i = 0, \ldots, k \ [3].$$

For the sake of simplicity we assume $\lambda(t) = \lambda_0 + \lambda_1 t$, so a linear function. System (1) is now

$$\left\{ \begin{aligned} &\sum_{\mu=1}^{h} \frac{(x_{t_\mu} - (\lambda_0 + \lambda_1 t_\mu))\, t_\mu}{\lambda_0 + \lambda_1 t_\mu} = 0 \\ &\sum_{\mu=1}^{h} \frac{(x_{t_\mu} - (\lambda_0 + \lambda_1 t_\mu))}{\lambda_0 + \lambda_1 t_\mu} = 0, \end{aligned} \right.$$

or

$$\left\{ \begin{aligned} &\sum_{\mu=1}^{h} \frac{x_{t_\mu} t_\mu}{\lambda_0 + \lambda_1 t_\mu} = \lambda_0 \sum_{\mu=1}^{h} \frac{t_\mu}{\lambda_0 + \lambda_1 t_\mu} + \lambda_1 \sum_{\mu=1}^{h} \frac{t_\mu^2}{\lambda_0 + \lambda_1 t_\mu} \\ &\sum_{\mu=1}^{h} \frac{x_{t_\mu}}{\lambda_0 + \lambda_1 t_\mu} = \lambda_0 \sum_{\mu=1}^{h} \frac{1}{\lambda_0 + \lambda_1 t_\mu} + \lambda_1 \sum_{\mu=1}^{h} \frac{t_\mu}{\lambda_0 + \lambda_1 t_\mu}. \end{aligned} \right.$$

For this we write

$$\begin{aligned} s_{1,0} &= \lambda_0 s_{1,1} + \lambda_1 s_{1,2}, \\ s_{2,0} &= \lambda_0 s_{2,1} + \lambda_1 s_{2,2}. \end{aligned} \tag{5}$$

in which the $s_{i,j}$ are the sums $\sum_{\mu=1}^{h} \frac{x_{t_\mu} t_\mu}{\lambda_0 + \lambda_1 t_\mu}$ etc.

We can solve this system for the unknown λ_0 and λ_1, in the meantime considering the $s_{i,j}$ as coefficients. This leads to

$$\begin{aligned} \lambda_0 &= \psi_0(s_{i,j}) = \varphi_0(\lambda_0, \lambda_1), \\ \lambda_1 &= \psi_1(s_{i,j}) = \varphi_1(\lambda_0, \lambda_1), \end{aligned}$$

in which the φ_0 and φ_1 are expressions in the sums mentioned above.

But it is easy to see that the iteration

$$\left\{\begin{aligned} &\sum_{\mu=1}^{h} \frac{(x_{t_\mu} - ({}_{\iota+1}\hat{\lambda}_0 + {}_{\iota+1}\hat{\lambda}_1 t_\mu))\, t_\mu}{{}_\iota\hat{\lambda}_0 + {}_\iota\hat{\lambda}_1 t_\mu} = 0, \\ &\sum_{\mu=1}^{h} \frac{(x_{t_\mu} - ({}_{\iota+1}\hat{\lambda}_0 + {}_{\iota+1}\hat{\lambda}_1 t_\mu))}{{}_\iota\hat{\lambda}_0 + {}_\iota\hat{\lambda}_1 t_\mu} = 0, \end{aligned}\right. \tag{6}$$

is identical with

$$\begin{aligned} {}_{\iota+1}\hat{\lambda}_0 &= \varphi_0({}_\iota\hat{\lambda}_0, {}_\iota\hat{\lambda}_1), \\ {}_{\iota+1}\hat{\lambda}_1 &= \varphi_1({}_\iota\hat{\lambda}_0, {}_\iota\hat{\lambda}_1). \end{aligned} \tag{7}$$

Therefore, in order to investigate the convergence of the process, we may try to determine whether

$$\left|\frac{\partial \varphi_0}{\partial \lambda_0}\right| + \left|\frac{\partial \varphi_1}{\partial \lambda_0}\right| < 1, \quad \left|\frac{\partial \varphi_0}{\partial \lambda_1}\right| + \left|\frac{\partial \varphi_1}{\partial \lambda_1}\right| < 1.$$

These partial derivatives are rational expressions of the sums

$$\sum_{\mu=1}^{h} \frac{x_{t_\mu} t_\mu^2}{(\lambda_0 + \lambda_1 t_\mu)^2} \quad \text{etc.}$$

If we afterwards replace in the denominators $\lambda_0 + \lambda_1 t_\mu$ by x_{t_μ}, we obtain approximative values; hence we may perhaps check the condition. Similar results can be obtained if $\lambda(t)$ is not a linear function, but a polynomial of higher degree.

Now we return again to the system (1)

$$\sum_{\mu=1}^{h} (x_{t_\mu} - w_\mu \lambda(t_\mu)) \frac{\partial \log \lambda(t_\mu)}{\partial \lambda_\nu} = 0, \qquad \nu = 0, \ldots, k.$$

If we apply the χ^2-minimum method we obtain

$$\sum_{\mu=1}^{h} \frac{(x_{t_\mu} - w_\mu \lambda(t_\mu))^2}{w_\mu \lambda(t_\mu)} = \text{minimum}, \tag{8}$$

or

$$\sum_{\mu=1}^{h} (x_{t_\mu} - w_\mu \lambda(t_\mu)) \frac{\partial \log \lambda(t_\mu)}{\partial \lambda_\nu} + \frac{1}{2} \sum_{\mu=1}^{h} \frac{(x_{t_\mu} - w_\mu \lambda(t_\mu))^2}{w_\mu \lambda(t_\mu)} \frac{\partial \log \lambda(t_\mu)}{\partial \lambda_\nu} = 0,$$
$$\nu = 0, \ldots, k.$$

The first sum of this equation (8) put equal to zero coincides with the equation of the maximum likelihood estimation (1).

Hence the neglection of the second terms

$$\frac{1}{2}\sum_{\mu=1}^{h}\frac{(x_{t_\mu}-w_\mu\lambda(t_\mu))^2}{w_\mu\lambda(t_\mu)}\frac{\partial\log\lambda(t_\mu)}{\partial\lambda_\nu}$$

leads to maximum likelihood estimation. Following Cramer, we shall call this procedure the modified χ^2-minimum method. The equations of the modified χ^2-minimum method are often easier to be solved. However, in our case they remain difficult. As a second approximation therefore we solve the linear system (2).

$$\sum_{\mu=1}^{h}\frac{(x_{t_\mu}-w_\mu\lambda(t_\mu))}{x_{t_\mu}}\,w_\mu\,\frac{\partial\lambda(t_\mu)}{\partial\lambda_\nu}=0,\quad \nu=0,\ldots,k\,.$$

This method is also often mentioned as the χ^2-minimum method [4].

At last we shall make a remark on the condition

$$\sum_{i=1}^{h}p_i(\lambda_0,\ldots,\lambda_k)=1\,.$$

If this condition is fulfilled, the modified χ^2-minimum method gives maximum likelihood estimates for $\lambda_0,\ldots,\lambda_k$. In our case we have to identify $w_\mu\lambda(t_\mu)$ with $xp_i(\lambda_0,\ldots,\lambda_k)$, but we do not have necessarily that $\sum_{\mu=1}^{h}w_\mu\lambda(t_\mu)$ is a constant and consequently system (1) does not reduce to

$$\sum_{\mu=1}^{h}x_{t_\mu}\frac{\partial\log\lambda(t_\mu)}{\partial\lambda_\nu}=0,\quad \nu=0,\ldots,k\ [5].$$

However, the $\hat\lambda_\nu$ are still maximum likelihood estimates. There is a reason for this. It is easy to prove that the conditional distribution of the x_{t_μ} under the condition $x=\sum_{\mu=1}^{h}x_{t_\mu}$ fixed, is a multinomial distribution. We have

$$P\left[x_{t_1},\ldots,x_{t_h}\,\middle|\,x\right]=x!\prod_{\mu=1}^{h}\frac{\tilde\lambda_\mu^{x_{t_\mu}}}{x_{t_\mu}!}\tag{9}$$

in which

$$\tilde\lambda_\mu=\frac{w_\mu\lambda(t_\mu)}{\sum_{\mu=1}^{h}w_\mu\lambda(t_\mu)}\quad[6].$$

Hence we may perhaps try to obtain maximum likelihood estimates from

$$\sum_{\mu=1}^{h} \frac{(x_{t_\mu} - x\tilde{\lambda}_\mu)^2}{x\tilde{\lambda}_\mu} = \text{minimum}$$

or from

$$\sum_{\mu=1}^{h} \frac{x_{t_\mu} - x\tilde{\lambda}_\mu}{\tilde{\lambda}_\mu} \frac{\partial \tilde{\lambda}_\mu}{\partial \lambda_\nu} = \sum_{\mu=1}^{h} x_{t_\mu} \frac{\partial \log \tilde{\lambda}_\mu}{\partial \lambda_\nu} = 0, \quad \nu = 0, \ldots, k. \quad (10)$$

However, $\lambda(t_\mu) = \sum_{\nu=0}^{h} \lambda_\nu t_\mu^\nu$ and therefore we obtain only an estimate for $\lambda_0 : \lambda_1 : \ldots : \lambda_k$. To determine $\hat{\lambda}_0, \ldots, \hat{\lambda}_k$ completely, we have to make the additional condition $x = \sum_{\mu=1}^{h} w_\mu \lambda(t_\mu)$ but then

$$\sum_{\mu=1}^{h} \frac{x_{t_\mu} - x\tilde{\lambda}_\mu}{\tilde{\lambda}_\mu} \frac{\partial \tilde{\lambda}_\mu}{\partial \lambda_\nu} \equiv \sum_{\mu=1}^{h} \frac{x_{t_\mu} - w_\mu \lambda(t_\mu)}{w_\mu \lambda(t_\mu)} w_\mu \frac{\partial \lambda(t_\mu)}{\partial \lambda_\nu}, \quad (11)$$

$$\nu = 0, \ldots, k.$$

The by-condition $x = \sum_{\mu=1}^{h} w_\mu \lambda(t_\mu)$ is a natural one, because $\dfrac{x}{\sum_{\mu=1}^{h} w_\mu \lambda_t(\mu)} \xrightarrow{\text{pr.}} 1$, besides we need it to obtain maximum likelihood estimates. From the foregoing it appears that we have essentially the same situation as in the case of a multinomial distribution, that means in the case $\sum_{\iota=1}^{h} p_\iota = 1$. The condition $x = \sum_{\mu=1}^{h} x_{t_\mu}$ fixed can be removed. Moreover we saw that, if x is stochastic, we obtain the same equations for the maximum likelihood estimates of $\lambda_0, \ldots, \lambda_k$.

Particular least squares methods for the estimation of the parameters of some important discrete distributions

We here have in mind the Poisson-, the binomial- and the negative binomial-distribution. Suppose we have the observations n_x, $x = 0, 1, \ldots$; $n = \sum_x n_x$, indicating the observed frequency of the value x, $x = 0, 1, 2, \ldots$. For En_x there is a recurrence relation $En_x = \varphi(\lambda) f(k) En_{x-1}$, in which λ is the parameter of the distribution.

For the Poisson-distribution we have

$$E n_x = n \frac{e^{-\lambda} \lambda^x}{x!} = \frac{\lambda}{x} E n_{x-1}$$

hence $\varphi(\lambda) = \lambda$ and $f(x) = \dfrac{1}{x}$. In the case of a binomial distribution we obtain

$$E n_x = n \binom{N}{x} \lambda^x (1-\lambda)^{N-x} = \frac{N+x-1}{x} \, \frac{\lambda}{1-\lambda} E n_{x-1}$$

$$\text{hence } \varphi(\lambda) = \frac{\lambda}{1-\lambda} \text{ and } f(x) = \frac{N+x-1}{x}.$$

At last for the negative binomial distribution we get

$$E n_x = n \binom{r+x-1}{x} \lambda^r (1-\lambda)^x = (1-\lambda) \frac{r+x-1}{x} E n_{x-1},$$

$$\text{so } \varphi(\lambda) = 1-\lambda \text{ and } f(x) = \frac{r+x-1}{x}.$$

We now apply the substitution $E n_x \longrightarrow \varphi(\lambda) f(x) n_{x-1}$ in the different least squares method formula's including the χ^2-minimum method and solve for $\varphi(\lambda)$.

In this way we obtain the following table:

$$1 \sum_x (n_x - E n_n)^2 = \min. \to \sum_x (n_x - \varphi(\lambda) f(x) n_{x-1})^2 = \min. \to {}_1\hat{\varphi}(\lambda) = \frac{\sum_x f(x) n_x n_{x-1}}{\sum_x f^2(x) n_{x-1}^2}$$

$$2 \sum_x \frac{(n_x - E n_x)^2}{E n_x} = \min. \to \sum_x \frac{(n_x - \varphi(\lambda) f(x) n_{x-1})^2}{\varphi(\lambda) f(x) n_{x-1}} = \min. \to {}_2\hat{\varphi}(\lambda) = \sqrt{\frac{\sum_x \dfrac{n_x^2}{f(x) n_{x-1}}}{\sum_x f(x) n_{x-1}}}$$

$$3 \sum_x \frac{n_x - E n_x}{E n_x} \, \frac{d E n_x}{d\lambda} = 0 \to \sum_x (n_x - \varphi(\lambda) f(x) n_{x-1}) \frac{\varphi'(\lambda)}{\varphi(\lambda)} = 0 \to {}_3\hat{\varphi}(\lambda) = \frac{\sum_x n_x}{\sum_x f(x) n_{x-1}}$$

$$4 \sum_x \frac{(n_x - E n_x)^2}{n_x} = \min. \to \sum_x \frac{(n_x - \varphi(\lambda) f(x) n_{x-1})^2}{n_x} = \min. \to {}_4\hat{\varphi}(\lambda) = \frac{\sum_x f(x) n_{x-1}}{\sum_x \dfrac{f^2(x) n_{x-1}^2}{n_x}}$$

In the left column 1 is the common least squares method, 2, 3 and 4 are the χ^2-minimum and the modified χ^2-minimum method. We can easily prove that all the $\hat{\varphi}(\lambda)$ converge in probability to the true value $\varphi(\lambda)$ by applying the theorem of Slutsky [7]. For this we have to replace in the expressions for $\hat{\varphi}(\lambda)$ the frequencies $\frac{n_x}{n}$ by $\frac{E n_x}{n}$. (We here apply the Theorem of Slutsky for the Poisson-distribution in a somewhat different situation. We do not have a fixed number of variables but an unending sequence n_x. However because

$E \frac{n_x}{n} = \frac{e^{-\lambda}\lambda^x}{x!} \longrightarrow 0$ if $x \longrightarrow \infty$ the theorem may still be applied).

For instance we have

$$_2\hat{\varphi}(\lambda) \xrightarrow{\text{pr.}} \sqrt{\frac{\sum_x \frac{E^2 n_x}{\frac{1}{\varphi(\lambda)} E n_x}}{\sum_x \frac{1}{\varphi(\lambda)} E n_x}} = \varphi(\lambda).$$

By substituting for $\varphi(\lambda)$ and $f(x)$ the expressions corresponding with those of the Poisson-, binomial- and negative binomial-distribution, we obtain estimates for λ. So we obtain as estimates for the parameter λ of the Poisson-distribution:

$$_1\hat{\lambda} = \frac{\sum_x \frac{n_x n_{x-1}}{x}}{\sum_x \frac{n_{x-1}^2}{x^2}}, \qquad _3\hat{\lambda} = \frac{\sum_x n_x}{\sum_x \frac{n_{x-1}}{x}},$$

$$_2\hat{\lambda} = \sqrt{\frac{\sum_x \frac{n_x^2 x}{n_{x-1}}}{\sum_x \frac{n_{x-1}}{x}}}, \qquad _4\hat{\lambda} = \frac{\sum_x \frac{n_{x-1}}{x}}{\sum_x \frac{n_{x-1}^2}{x^2 n_{x-1}}}.$$

We are mentioning here this general method of estimating because sometimes in literature one finds a particular example of this method. For instance Sankara Pillai [8] derives in this way $_1\hat{\lambda}$ for the Poisson-distribution. He proves that $_1\hat{\lambda} \xrightarrow{\text{pr.}} \lambda$ and gives numerical results with regard to the relative efficiency of $_1\hat{\lambda}$.

An important actuarial application of such a substitution principle may be found in fitting a curve $ab^x + c$ to a sequence of small frequencies i_x depending on age x. To obtain estimates $\hat{b}$ and $\hat{c}$ we might use the relation $Ei_x = bEi_{x-1} + (1-b)\,c \approx b\,i_{x-1} + (1-b)\,c\,.$

Literature

[1,2,5] *Cramér, H.:* Mathematical Methods of Statistics. Princeton, 1946, pp. 424 to 426.

[3] *Willers:* Practical Analysis. Dover Publications Inc., 1948, pp. 209–213.

[4] *Cramér, H.* and *Wold:* Mortality Variations in Sweden. Skandinavisk Aktuarietidskrift, 18, 1938.
Zwinggi, E.: Versicherungsmathematik. Basel, 1945, Bl. 174, 175.
Saxer, W.: Versicherungsmathematik, Zweiter Teil. Springer Verlag, 1958, Bl. 167–169.

[6] *van Klinken, J.:* Statistical Methods for the Comparison of risks. International Review on Actuarial and Statistical Problems of Social Securtity, Number 2, 1958.

[7] *Cramér, H.:* Mathematical Methods etc., pp. 254, 255.

[8] *van Klinken, J.* en *Prins, H.J.:* Overzicht van Toetsings- en Schattingsmethoden met betrekking tot Poisson-verdelingen. Mathematisch Centrum, Amsterdam, 1954, Bl. 59, 60.

Zusammenfassung

Es wird vorerst das Problem der Parameter-Schätzung auf Basis der χ^2-Minimum-Methode behandelt für den Fall, dass für die Zufallsvariabeln Poisson-Verteilungen gelten. Sodann wird eine spezielle Methode der kleinsten Quadrate betrachtet, welche die modifizierten χ^2-Minimum-Methoden als Sonderfall enthält und welche sich für die Parameter-Schätzung einiger wichtiger diskreter Verteilungen eignet.

Résumé

L'auteur traite tout d'abord le problème de l'évaluation des paramètres au moyen de la méthode «χ^2-minimum» pour le cas où les répartitions de Poisson sont applicables aux variables aléatoires. Il considère ensuite une «méthode des moindres carrés» spéciale qui englobe comme cas particulier des méthodes «χ^2-minimum» modifiées et qui se prête à l'évaluation des paramètres de quelques répartitions discrètes importantes.

Riassunto

L'autore allude al problema della valutazione dei parametri secondo il metodo «χ^2 minimo» per i casi in cui sono applicabili le distribuzioni di Poisson. In seguito considera un metodo speziale dei quadrati minimi speciale che comprende come caso particulare i metodi «χ^2 minimi» modificati e un metodo adattabile all'evatuazione dei parametri di qualche distribuzione discreta importante.