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Some notes on Lidstone's and other approximations to temporary life annuities when the force of mortality is

$$(1 + k) \mu_{x+t}$$

By *Bengt Åkerberg*,

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Suppose that the force of mortality is  $(1 + k) \mu_{x+t}$ , then we have for an endowment assurance

$$\frac{d}{dt} {}_tV_{x:n}^{(k)} = \delta {}_tV_{x:n}^{(k)} + p_{x:n}^{(k)} - (1 + k) \mu_{x+t} (1 - {}_tV_{x:n}^{(k)}).$$

After multiplying both sides with  $\frac{D_{x+t}}{D_x}$  we get

$$d\left(\frac{D_{x+t}}{D_x} {}_tV_{x:n}^{(k)}\right) = p_{x:n}^{(k)} \frac{D_{x+t}}{D_x} dt - \mu_{x+t} \frac{D_{x+t}}{D_x} dt - k \mu_{x+t} \frac{D_{x+t}}{D_x} (1 - {}_tV_{x:n}^{(k)}) dt$$

and after integrating from 0 to  $n$

$$\frac{D_{x+n}}{D_x} = \left(\frac{1}{\bar{a}_{x:n}^{(k)}} - \delta\right) \bar{a}_{x:n} - \left(1 - \delta \bar{a}_{x:n} - \frac{D_{x+n}}{D_x}\right) - k \int_0^n \mu_{x+t} \frac{D_{x+t}}{D_x} (1 - {}_tV_{x:n}^{(k)}) dt.$$

Hence

$$\frac{1}{\bar{a}_{x:n}^{(k)}} = \frac{1}{\bar{a}_{x:n}} + \frac{k}{\bar{a}_{x:n}} \int_0^n \mu_{x+t} \frac{D_{x+t}}{D_x} (1 - {}_tV_{x:n}^{(k)}) dt. \quad (1)$$

This formula turns out to be the key formula, from which a group of approximations can be deduced fairly easily.

If we replace the expression  $1 - {}_tV_{x:n}^{(k)} = \frac{\bar{a}_{x+t:n-t}^{(k)}}{\bar{a}_{x:n}^{(k)}}$  by  $\frac{\bar{a}_{n-t}}{\bar{a}_n}$

we obtain

$$\begin{aligned} \frac{1}{\bar{a}_{xn}^{(k)}} &\sim \frac{1}{\bar{a}_{xn}} + \frac{k}{\bar{a}_{xn}} \cdot \frac{\bar{a}_n - \bar{a}_{xn}}{\bar{a}_n} \\ &\sim \frac{1+k}{\bar{a}_{xn}} - \frac{k}{\bar{a}_n} \end{aligned} \quad (2)$$

and for  $k = 1$ , Lidstone's formula

$$\frac{1}{\bar{a}_{xxn}} \sim \frac{2}{\bar{a}_{xn}} - \frac{1}{\bar{a}_n}. \quad (2a)$$

Supposing  $\frac{\bar{a}_{x+t n-t}^{(k)}}{\bar{a}_{xn}^{(k)}} \sim \frac{\bar{a}_{n-t}}{\bar{a}_{xn}}$  we have, by (1)

$$\begin{aligned} \frac{1}{\bar{a}_{xn}^k} &\sim \frac{1}{\bar{a}_{xn}} + \frac{k}{\bar{a}_{xn}} \cdot \frac{\bar{a}_n - \bar{a}_{xn}}{\bar{a}_{xn}} \\ &\sim \frac{1-k}{\bar{a}_{xn}} + k \frac{\bar{a}_n}{(\bar{a}_{xn})^2} \end{aligned} \quad (3)$$

and for  $k = 1$

$$\bar{a}_{xxn} \sim \frac{(\bar{a}_{xn})^2}{\bar{a}_n}. \quad (3a)$$

And finally if  $\frac{\bar{a}_{x+t n-t}^{(k)}}{\bar{a}_{xn}^{(k)}}$  is replaced by  $\frac{\bar{a}_{n-t}}{\bar{a}_{xn}^{(k)}}$  we obtain

$$\frac{1}{\bar{a}_{xn}^{(k)}} \sim \frac{1}{\bar{a}_{xn}} + \frac{k}{\bar{a}_{xn}} \cdot \frac{\bar{a}_n - \bar{a}_{xn}}{\bar{a}_{xn}^{(k)}}.$$

Hence

$$\bar{a}_{xn}^{(k)} \sim (1+k) \bar{a}_{xn} - k \bar{a}_n \quad (4)$$

and

$$\bar{a}_{xxn} \sim 2 \bar{a}_{xn} - \bar{a}_n. \quad (4a)$$

The formulas (2a), (3a) and (4a) may also be written

$$(2a) \quad \bar{a}_{xn} \sim \frac{2}{\frac{1}{\bar{a}_{xxn}} + \frac{1}{\bar{a}_n}} \quad (\text{the harmonical mean}),$$

$$(3a) \quad \bar{a}_{xn} \sim \sqrt{\bar{a}_{xxn} \bar{a}_n} \quad (\text{the geometrical mean}),$$

$$(4a) \quad \bar{a}_{xn} \sim \frac{\bar{a}_{xxn} + \bar{a}_n}{2} \quad (\text{the arithmetical mean}).$$

Denoting now  $\frac{\bar{a}_{\bar{n}|}}{\bar{a}_{x\bar{n}|}}$  by  $1 + \lambda$  ( $\lambda > 0$ ) it is easily seen that by

$$(2a) \quad \bar{a}_{xx\bar{n}|} \sim \frac{1}{1 + 2\lambda} \bar{a}_{\bar{n}|},$$

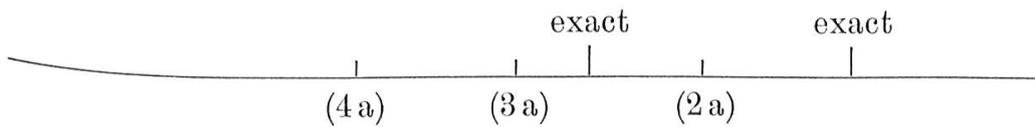
$$(3a) \quad \bar{a}_{xx\bar{n}|} \sim \frac{1}{(1 + \lambda)^2} \bar{a}_{\bar{n}|},$$

$$(4a) \quad \bar{a}_{xx\bar{n}|} \sim \frac{1 - \lambda}{1 + \lambda} \bar{a}_{\bar{n}|}.$$

Since 
$$\frac{1 - \lambda}{1 + \lambda} \bar{a}_{\bar{n}|} < \frac{1}{(1 + \lambda)^2} \bar{a}_{\bar{n}|} < \frac{1}{1 + 2\lambda} \bar{a}_{\bar{n}|}$$

it follows that the values for  $\bar{a}_{xx\bar{n}|}$  by (2a), (3a) and (4a) are decreasing.

We represent those values on a line



and shall prove that the exact value lies either between (3a) and (2a) or, when  $n$  is chosen sufficiently large, to the right of (2a).

It is immediately seen that the following temporary life annuity

$$\bar{a}_{x\bar{n}|} + \bar{a}_{x\bar{n}|} - \bar{a}_{xx\bar{n}|} < \bar{a}_{\bar{n}|}$$

hence the exact value 
$$\bar{a}_{xx\bar{n}|} > 2\bar{a}_{x\bar{n}|} - \bar{a}_{\bar{n}|}.$$

From Schwarz' inequality

$$\int_0^n (f(t))^2 dt \int_0^n (\varphi(t))^2 dt \geq \left[ \int_0^n f(t) \varphi(t) dt \right]^2$$

it follows when

$$f(t) = \frac{l_{x+t}}{l_x} e^{-\frac{\delta t}{2}}$$

and

$$\varphi(t) = e^{-\frac{\delta t}{2}}$$

that the exact value

$$\bar{a}_{xx\bar{n}|} \geq \frac{(\bar{a}_{x\bar{n}|})^2}{\bar{a}_{\bar{n}|}}.$$

Turning now to the study of the formula (2).  
From the formula (1) it is seen that if

$$\frac{\bar{a}_{x+t|n-t}^{(k)}}{\bar{a}_{xn}^{(k)}} > \frac{\bar{a}_{n-t}}{\bar{a}_n}$$

during the whole interval  $(0, n)$  then

$$\frac{1}{\bar{a}_{xn}^{(k)}} > \frac{1+k}{\bar{a}_{xn}} - \frac{k}{\bar{a}_n}. \tag{5}$$

Hence

$$\frac{1}{\bar{a}_{xxn}} > \frac{2}{\bar{a}_{xn}} - \frac{1}{\bar{a}_n}$$

and the exact value  $\bar{a}_{xxn}$  lies between (3a) and (2a).

First we shall prove that the inequality (5) holds if  $\mu_{x+t} \bar{a}_{n-t}$  never increases during the whole interval  $(0, n)$ .

Thereafter we shall—under the assumption that  $\mu_{x+t} = \alpha + \beta e^{\nu(x+t)}$ —investigate if possible some simple criterions can be found to decide when  $\mu_{x+t} \bar{a}_{n-t}$  never increases.

The premium to be paid by a constant yearly amount during the whole period of insurance for the annuity  $\bar{a}_n - \bar{a}_{xn}$  we denote by

$$\pi_{xn} = \frac{\bar{a}_n - \bar{a}_{xn}}{\bar{a}_{xn}}$$

From the differential equation

$$\frac{d {}_tV_{xn}}{dt} = (\delta + \mu_{x+t}) {}_tV_{xn} + \pi_{xn} - \mu_{x+t} \bar{a}_{n-t}$$

it follows that  $\pi_{xn} < \mu_x \bar{a}_n$  when  $\mu_{x+t} \bar{a}_{n-t}$  never increases. For otherwise  $\frac{d}{{}_tV_{xn}} > 0$  and  ${}_tV_{xn} > 0$  when  $t > 0$  contrary the fact that  ${}_tV_{xn} = 0$  when  $t = n$ .

$\mu_{x+t} \bar{a}_{n-t}$  never increasing in the interval  $(0, n)$  we thus have

$$\pi_{x+t|n-t} < \mu_{x+t} \bar{a}_{n-t}$$

or

$$\frac{1}{\bar{a}_{x+t|n-t}} - \frac{1}{\bar{a}_{n-t}} < \mu_{x+t}$$

Now 
$$\frac{d}{dt} \left( \frac{\bar{a}_{n-t}}{\bar{a}_{x+t, n-t}} \right) = \frac{\bar{a}_{n-t}}{\bar{a}_{x+t, n-t}} \left[ \frac{1}{\bar{a}_{x+t, n-t}} - \frac{1}{\bar{a}_{n-t}} - \mu_{x+t} \right]$$

so that  $\frac{\bar{a}_{n-t}}{\bar{a}_{x+t, n-t}}$  never increases.

Hence

$$\frac{\bar{a}_n}{\bar{a}_{xn}} > \frac{\bar{a}_{n-t}}{\bar{a}_{x+t, n-t}}$$

or

$$\frac{\bar{a}_{x+t, n-t}}{\bar{a}_{xn}} > \frac{\bar{a}_{n-t}}{\bar{a}_n}.$$

The inequality (5) thus holds for  $k = 0$  and since  $(1+k)\mu_{x+t}\bar{a}_{n-t}$  never increases it holds generally when  $k \neq 0$ .

We shall now enter upon the discussion about some sufficient conditions to decide when  $\mu_{x+t}\bar{a}_{n-t}$  never increases in the whole interval  $(0, n)$  on the assumption that  $\mu_{x+t} = \alpha + \beta e^{\gamma(x+t)}$ .

It is convenient to introduce the notation

$$\varrho_{x+t, n-t} = \mu_{x+t} \bar{a}_{n-t}$$

and from 
$$\frac{d}{dt} \varrho_{x+t, n-t} = \mu_{x+t} [\gamma \bar{a}_{n-t} - e^{-\delta(n-t)}] - \alpha \gamma \bar{a}_{n-t}$$

it is immediately seen that

$$\frac{d}{dt} \varrho_{x+t, n-t} \leq 0$$

in the interval  $0 \leq t \leq n$  and irrespective of  $x$  certainly if

$$\gamma \bar{a}_n - e^{-\delta n} \leq 0.$$

When

$$n \leq \frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma}$$

we thus find that  $\varrho_{x+t, n-t}$  never increases.

When

$$n > \frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma}$$

it is seen that

$$\frac{d}{dt} \varrho_{x+t, n-t} \leq 0$$

wherever

$$\mu_{x+t} \leq \frac{\alpha \gamma \bar{a}_{n-t}}{\gamma \bar{a}_{n-t} - e^{-\delta(n-t)}},$$

which can also be written (using the Makeham expression for  $\mu_{x+t}$ )

$$x + n \leq \frac{1}{\gamma} \log \left[ \frac{\alpha \delta}{\beta} \cdot \frac{e^{(\gamma-\delta)(n-t)}}{\gamma - (\gamma + \delta)e^{-\delta(n-t)}} \right]. \quad (6)$$

Now, in practice, we always have  $\gamma - \delta > 0$ . Thus the right-hand side of (6) considered as a function of  $(n-t)$ , tends to  $+\infty$  when  $(n-t)$  tends to  $\frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma}$  and when  $(n-t)$  tends to  $+\infty$ .

Its derivative is found to be

$$\frac{\gamma - \delta}{\gamma} \cdot \frac{e^{\delta(n-t)} - \frac{\gamma + \delta}{\gamma - \delta}}{e^{\delta(n-t)} - \frac{\gamma + \delta}{\gamma}}$$

and it is thus seen that the function never increases when  $(n-t)$  increases from  $\frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma}$  to a value  $\frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma - \delta}$  and never decreasing when  $(n-t) > \frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma - \delta}$ .

Consequently, the function

$$\frac{1}{\gamma} \log \left[ \frac{\alpha \delta}{\beta} \cdot \frac{e^{(\gamma-\delta)(n-t)}}{\gamma - (\gamma + \delta)e^{-\delta(n-t)}} \right]$$

has a minimum when

$$n - t = \frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma - \delta}$$

and inserting this value, we find the minimum to be

$$\frac{1}{\gamma} \log \left[ \frac{\alpha}{\beta} \left( \frac{\gamma + \delta}{\gamma - \delta} \right)^{\frac{\gamma - \delta}{\delta}} \right].$$

The results reached can be summarized as follows.  $q_{x+t|\overline{n-t}|}$  never increases in the interval  $(0, n)$  when

$$n \leq \frac{1}{\delta} \log \frac{\gamma + \delta}{\gamma}$$

irrespective of  $x$  or when

$$x + n \leq \frac{1}{\gamma} \log \left[ \frac{\alpha}{\beta} \left( \frac{\gamma + \delta}{\gamma - \delta} \right)^{\frac{\gamma - \delta}{\delta}} \right]$$

even if the condition for  $n$  is not satisfied.

For instance, it can be mentioned that with the system of assumptions at present adopted by the Swedish life insurance companies, i. e. a rate of interest of 2,25% and the Swedish mortality table D 37 with a loading of 2,80/100 on the interest and mortality, we have

$$\begin{aligned} n &\leq 9,4, \\ x + n &\leq 64,0. \end{aligned}$$

Finally we shall prove that if  $n$  is chosen sufficiently large, the exact value  $\bar{a}_{x+n|} >$  the value by Lidstone's formula (2a).

From (1) it follows that

$$\frac{1}{\bar{a}_{x+n|}} < \frac{1}{\bar{a}_{xn|}} + \frac{1}{\bar{a}_{xn|}} \int_0^n \frac{D_{x+t}}{D_x} \mu_{x+t} dt$$

or

$$\frac{1}{\bar{a}_{x+n|}} < \frac{1}{\bar{a}_{xn|}} + \frac{1}{\bar{a}_{xn|}} \left[ 1 - \delta \bar{a}_{xn|} - \frac{D_{x+n}}{D_x} \right]$$

and for  $n = +\infty$

$$\frac{1}{\bar{a}_{xx}} < \frac{2}{\bar{a}_x} - \delta.$$

From Lidstone's formula we obtain when  $n = +\infty$

$$\frac{1}{\bar{a}_{xx}} = \frac{2}{\bar{a}_x} - \delta$$

hence the desired result.

