Yangian Y(sl(2)) in Coulomb problem

Helv. Phys. Acta 71 (1998) 586 – 592
0018-0238/98/050586-07 $ 1.50+0.20/0

Helvetica Physica Acta

Yangian $Y(sl(2))$ in Coulomb Problem

By Shengli Zhang

Department of Applied Physics, Xian Jiaotong University, Xi'an 710049
People's Republic of China.

(31.III.98)

abstract In this paper, the Yangian $Y(sl(2))$ is shown excisting in the system that a particle moves in Coulomb field. The generators of $Y(sl(2))$ are constructed in terms of the angular momentum operators and so-called Yangian Runge-Lenz vector. The selection rule and matrix element of $Y(sl(2))$ generators are calculated.

Yangian has drawn much attention in mathematical physics in recent years. It was introduced by Drinfeld [1]. Researchers found there is Yangian symmetry in some practicable physical models [2-6], such as long-range interaction models, one-deimensional Hubberd model, two-dimensional sigma model, and two-dimensional chiral model with or without topological terms. It is more significant that Yangian may play an important role in exploring the multiparton amplitudes in QCD and QED [7].

The Yangian $Y(sl(2))$ will be studied in the Coulomb problem from the point of view of quantum mechanics in this paper. It is well-known that the dynamical symmetry group of the Coulomb problem is $SO(4,2)$. And there is the $SO(4)$ symmetry that is used to determining the energy level in this system [8], which is called energy symmetry. However, the angular momentum and Runge-Lenz vector operators, which is looked as $SO(4)$ generators, become unclosed to the commutative product when the energy is not fixed. In above mentioned sense, those operators cannot form the $so(4)$ algebra. It is neccessary to approach what is the algebra of the operators.

We will research the algebra in the Coulomb problem. It will be seen that the angular momentum and Rung-Lenz vector operators generate the subalgebra of the loop algebra corresponding to $sl(2)$, the limitation of Yangian $Y(sl(2))$ when the quantum deformed parameter α vanish. As the q-hydrogen atom [9], the $Y(sl(2))$ will be realized in terms of the angular momentum $\mathbf{L}$ and Runge-Lenz vector $\mathbf{R}$.

It is well known that the Hamiltonian of the Coulomb problem reads

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{e^2}{|\mathbf{r}|} \tag{1}$$

where μ and e stand for the mass and electric charge of the electron respectively. It has been pointed out [7] that besides the argular momentum $\mathbf{L}$, the Runge-Lenz vector $\mathbf{R}$ is another conserved quantity, which satisfy

$$[\mathbf{L}, H] = [\mathbf{R}, H] = 0 \tag{2}$$

and

$$\mathbf{R} = \frac{1}{2\mu e^2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{\mathbf{r}}{r}. \tag{3}$$

Both $\mathbf{L}$ and $\mathbf{R}$ generate the $SO(4)$ for the fixed energy [8].

Drinfeld [1] defined the Yangian $Y(sl(2))$ as being generated by the generators $\{I_\pm, I_3\}$ and $\{J_\pm, J_3\}$ with the commutation relation

$$\begin{gathered}[I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3 \\ [I_3, J_\pm] = [J_3, I_\pm] = \pm J_\pm, \quad [I_\pm, J_\mp] = \pm 2J_3 \\ [I_3, J_3] = [I_\pm, J_\pm] = 0\end{gathered} \tag{4}$$

and

$$\begin{gathered}[J_3, [J_+, J_-]] = \frac{\alpha^2}{4} I_3(J_- I_+ - I_- J_+) \\ [J_\pm, [J_3, J_\pm]] = \frac{\alpha^2}{4} I_\pm(J_\pm I_3 - I_\pm J_3) \\ 2[J_3, [J_3, J_\pm]] \pm [J_\pm, [J_\pm, J_\mp]] = \frac{\alpha^2}{2}\{I_3(J_\pm I_3 - I_\pm J_3) + I_\pm(I_- J_+ - J_- I_+)\}\end{gathered} \tag{5}$$

where α is the deformation parameter.

Equation (4) shows that I_a itself obeys the same commutation relations as L_a. Therefore, we can directly assign

$$I_3 = L_3, \qquad I_\pm = L_\pm = L_1 \pm iL_2. \tag{6}$$

In order to realize $Y(sl(2))$, we must determine the generators J_3 and $J_\pm$. These generators are called Yangian Runge-Lenz vector $\mathbf{R}^{\mathbf{Y}}$ in this paper. It will be shown that the usual Runge-Lenz vector is the limitation of this Yangian Runge-Lenz vector when $\alpha \to 0$. Our intention is to find this deformed Runge-Lenz vector, which play the role of generators $J_3, J_\pm$

in Yangian $Y(sl(2))$, i.e. $\mathbf{J} = \mathbf{R^Y}$. Since the commutation relations of J_3 and $J_\pm$ with I_3 and $I_\pm$ are similar to those of I_3 and $I_\pm$ itself, we have

$$\mathbf{J} = \mathbf{R^Y} = \mathbf{R} + \alpha\mathbf{Q} \tag{7}$$

in which $\mathbf{Q}$ is a vector operator to be found. This expression make the limitation of $\mathbf{R^Y}$ to be $\mathbf{R}$ under $\alpha \to 0$.

It can be verified that Runge-Lenz vector satisfy

$$[L_3, R_\pm] = [R_3, L_\pm] = R_\pm$$

$$[L_3, R_3] = [L_\pm, R_\pm] = 0, \qquad [L_\pm, R_\mp] = \pm 2R_3, \tag{8}$$

with

$$R_\pm = R_1 \pm iR_2.$$

Therefore, the generators $\{Q_3, Q_\pm\}$ may also be determined by the angular momentum and Runge-Lenz vector. Through analysing the equations (5), (7), (8) and the commutation relation of angular momentum, we find that the generators $\{Q_3, Q_\pm\}$ can be chosen as

$$Q_3 = [\mathbf{L}^2, R_3] = \frac{1}{2}(L_+R_- - R_+L_-)$$

$$Q_\pm = [\mathbf{L}^2, R_\pm] = \pm(L_3R_\pm - R_3L_\pm) \tag{9}$$

From (7) and (9), the Yangian Runge-Lenz vector can be defined

$$\mathbf{R^Y} = \mathbf{R} + \alpha[\mathbf{L}^2, \mathbf{R}] \tag{10}$$

Because L_a and R_a are conserved for Hamiltonian (1), i.e. they obey the commutation relation (2), one easily asserts that the generators $\{R_3^Y, R_\pm^Y\}$ are also conserved. They satisfy

$$[H, R_3^Y] = [H, R_\pm^Y] = 0 \tag{11}$$

Therefore, there is Yangian $Y(sl(2))$ symmetry in the Coulomb problem.

In following part, the operators (6) and (10) are verified obeying the commutation relation of $Y(sl(2))$, i.e. equations (4) and (5). Because of the definition (6), in which L_a (a = 1, 2, 3) are usual angular momentum, the operators L_3 and $L_\pm$ must satisfy the first two formulas of (4) obviously. Using (6), (7) and (8), one directly has $[L_3, R_\pm^Y] = \pm R_\pm^Y$. The other formulas of (4) can also verified similarily. Now, let us consider the commutation relations of the generators $\mathbf{R^Y}$. After tedious calculation by making use of (7), (8) and

$$[R_a, R_b] = -\frac{i2}{\mu e^4}H\varepsilon_{abc}L_c$$

we have

$$[R_+^Y, R_-^Y] = \frac{4H\alpha^2}{\mu e^4}L_3(2\mathbf{L}^2 + 1) + 2(\alpha^2 - \frac{2H}{\mu e^4})L_3 \tag{12}$$

Since $[L_3, R_3^Y] = 0$ and

$$[\mathbf{L}^2, R_3^Y] = R_-^Y L_+ - L_- R_+^Y,$$

which can be proved by means of directly calculation, it is derived that

$$[R_3^Y,[R_+^Y,R_-^Y]]=\frac{8\alpha^2 H}{\mu e^4}L_3(R_-^Y L_+ - L_- R_+^Y). \tag{13}$$

Comparing with the first equation of (5), we see that the generators (6) and (10) indeed satisfy the first commutation relations of $Y(sl(2))$. In equatuion (13), the hamiltonian H can be looked upon as a constant. This is due to its commuting with all generators of $Y(sl(2))$ in our discussion. Similarily, we also determine the other commutation relations of the generators $\mathbf{R^Y}$ as follows,

$$[R_3^Y,R_\pm^Y]=\pm\frac{-2H}{\mu e^4}L_\pm \pm \frac{2\alpha^2 H}{\mu e^4}L_\pm(2\mathbf{L}^2+1)\pm\alpha^2 L_\pm \tag{14}$$

And we obtain

$$[R_\pm^Y,[R_3^Y,R_\pm^Y]]=\frac{8\alpha^2 H}{\mu e^4}L_\pm(R_\pm^Y L_3 - L_\pm R_3^Y). \tag{15}$$

We have applied the equation the commutation relations between the angular momentum and the Runge-Lenz vector, and

$$[\mathbf{L}^2,R_\pm^Y]=2(R_\pm^Y L_3 - L_\pm R_3^Y)$$

in derivating (15). It can been seen that the commutation relation (15) corresponds to the second equation of (5). If we use the commutation relations (12) and (14), the last commutation relation of $Y(sl(2))$ can be determined as follows:

$$2[R_3^Y,[R_3^Y,R_\pm^Y]]\pm[R_\pm^Y,[R_\pm^Y,R_\mp^Y]]$$

$$=\frac{4\alpha^2 H}{\mu e^4}\{L_3(R_\pm^Y L_3 - L_\pm R_3^Y)+L_\pm(L_- R_+^Y - R_-^Y L_+)\} \tag{16}$$

When we consider our realization (6) and (10) for fixed energy, the Hamiltonian in (13), (15) and (16) can be replaced by the energy eigenvalue as the discussion of energy level symmetry in usual hydogen atom. We see that the energy plays the similar role as the deformation parameter in above mentioned case. In the limitation $E_n\rightarrow 0$, i.e. $n\rightarrow\infty$, the Yangian Y(sl(2)) becomes a subalgebra of the loop algebra L(sl(2)). Therefore, the symmetrical algebraic structure in the Coulumb problem is different at the energy critical point at which transition of the state of the system happen from bounded states to free states. The reduced algebra satisfy

$$[L_3,L_\pm]=\pm L_\pm,\quad [L_+,L_-]=2L_3$$

$$[L_3,R_\pm^Y]=[R_3^Y,L_\pm]=\pm R_\pm^Y,\quad [L_\pm,R_\mp^Y]=\pm 2R_3^Y$$
$$[L_3,R_3^Y]=[L_\pm,R_\pm^Y]=0$$

and

$$[R_3^Y,[R_+^Y,R_-^Y]]=0$$

$$[R^Y_\pm, [R^Y_3, R^Y_\pm]] = 0 \tag{17}$$
$$2[R^Y_3, [R^Y_3, R^Y_\pm]] \pm [R^Y_\pm, [R^Y_\pm, R^Y_\mp]] = 0.$$

It should be pointed out that above equations hold true only for the case $E = 0$. The deformation parameter α does not vanish. On the other hand, if the deformation parameter is zero, the above-mentioned Yangian $Y(sl(2))$ also reduces to the same subalgebra of the loop algebra $L(sl(2))$.

Now, let us study the selection rule of the generator of $Y(sl(2))$. It is well known that the spherical harmonics $Y_{lm}(\theta, \phi) = |lm\rangle$ are the eigenstates of L_z and $\mathbf{L}^2$. From the equation (6), we have

$$I_3|lm\rangle = m|lm\rangle$$
$$I_\pm|lm\rangle = [(l \mp m)(l \pm m + 1)]^{1/2}|lm\rangle \tag{18}$$

Furthermore, using (4) and (7), we can derive

$$\langle l'm'|R^Y_\pm|lm\rangle = \delta_{m'm\pm1}\langle l'm \pm 1|R^Y_\pm|lm\rangle \tag{19}$$

and

$$\langle l'm'|R^Y_3|lm\rangle = \delta_{m'm}\langle l'm|R^Y_3|lm\rangle \tag{20}$$

respectively. The above two formulas just indicate the selection rules of the generators $\mathbf{R}^Y$. From the commutation relation $[I_+, J_+] = [I_- J_-] = 0$, we can determine the recurrence relation of the matrix elements of generators $\mathbf{R^Y}$ as:

$$\langle l'm + 1|R^Y_+|lm\rangle = \sqrt{\frac{(l' - m)(l' + m + 1)}{(l + m)(l - m + 1)}}\langle l'm|R^Y_+|lm - 1\rangle$$

$$\langle l'm - 1|R^Y_-|lm\rangle = \sqrt{\frac{(l' + m)(l' - m + 1)}{(l - m)(l + m + 1)}}\langle l'm|R^Y_-|lm + 1\rangle$$

Making use of $[L_+, R^Y_-] = [R^Y_+, L_-] = 2J_3$, we also obtain

$$2\langle l'm|R^Y_3|lm\rangle = \sqrt{(l' + m)(l' - m + 1)}\langle l'm - 1|R^Y_-|lm\rangle$$
$$-\sqrt{(l - m)(l + m + 1)}\langle l'm|R^Y_-|lm + 1\rangle$$
$$= \sqrt{(l + m)(l - m + 1)}\langle l'm|R^Y_+|lm - 1\rangle - \sqrt{(l' - m)(l' + m + 1)}\langle l'm + 1|R^Y_+|lm\rangle$$

Equation (18) shows that the generators $\mathbf{I}$ represent the transition in same quantum number l. And equations (19) and (20) indicate that the generators $\mathbf{R^Y}$ may play the role of transition between states with different angular quantum number l, which selection rule is similar to the generators $\mathbf{I}$ for the magnetic quantum number m. It must be pointed out the above conclusion is universal for $Y(sl(2))$, not confined to the realization of (6) and (10).

It is well known that the Hilbert space of the Coulumb problem is given by the wave function

$$|nlm\rangle = R_{nl}(r)Y_{lm}(\theta, \phi)$$

where

$$R_{nl}(r) = -\{(\frac{2}{na_0})^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}\}^{1/2} e^{-\frac{r}{na_0}} (\frac{2r}{na_0})^l L_{n+l}^{2l+1}(\frac{2r}{na_0})$$

and L_{n+l}^{2l+1} stands for the associated Laguerre polynomial. Using our realization (6) and (10), we have derived the commutator brackets (12) and(14). From the commutation relations (12), (14) and selection rules (19) and (20), one has

$$\langle n'l'm|R_+^Y|n''l''m-1\rangle\langle n''l''m-1|R_-^Y|nlm\rangle - \langle n'l'm|R_-^Y|n''l''m+1\rangle\langle n''l''m+1|R_+^Y|nlm\rangle$$

$$= \delta_{nn'}\delta_{ll'} m\{\frac{4\alpha^2 E_n}{\mu e^4}[2l(l+1)+1] + 2(\alpha^2 - \frac{2E_n}{\mu e^4}\} \quad (21)$$

and

$$\langle n'l'm\pm 1|R_3^Y|n''l''m\pm 1\rangle\langle n''l''m\pm 1|R_\pm^Y|nlm\rangle - \langle n'l'm\pm 1|R_\pm^Y|n''l''m\rangle\langle n''l''m|R_3^Y|nlm\rangle$$

$$= \pm\delta_{nn'}\delta_{ll'}\sqrt{(l\mp m)(l\pm m+1)}\{-\frac{2\alpha^2 E_n}{\mu e^4} + \frac{2E_n}{\mu e^4}[2l(l+1)+1] + \alpha^2\} \quad (22)$$

Since $R_+^{Y+} = R_-^Y$, the following relation

$$\langle n'l'm-1|R_-^Y|nlm\rangle = \langle nlm|R_+^Y|n'l'm-1\rangle,$$

holds true. Therefore, we can obtain

$$\langle n'l'm+1|R_+^Y|nlm\rangle = \delta_{nn'}\delta_{ll'}\{\frac{2\alpha^2 E_n}{\mu e^4}[2l(l+1)+1] + [\alpha^2 - \frac{2E_n}{\mu e^4}]\}^{1/2}\sqrt{(l-m)(l+m+1)} \quad (23)$$

and

$$\langle n'l'm-1|R_-^Y|nlm\rangle = \delta_{nn'}\delta_{ll'}\{\frac{2\alpha^2 E_n}{\mu e^4}[2l(l+1)+1] + [\alpha^2 - \frac{2E_n}{\mu e^4}]\}^{1/2}\sqrt{(l+m)(l-m+1)} \quad (24)$$

Furthermore, we have

$$\langle nlm|R_3^Y|n'l'm\rangle = \delta_{nn'}\delta_{ll'} m\{\frac{2\alpha^2 E_n}{\mu e^4}[2l(l+1)+1] + [\alpha^2 - \frac{2E_n}{\mu e^4}]\}^{1/2} \quad (25)$$

This means that the $Y(sl(2))$ generators realized by (6) and (10) also represent the transition within the states with same principal quantum number n and angular quantum number l. It must be pointed out above conclusion is consistent with the general discussion in the last paragraph.

When the quantum parmeter α vanishes, the generator (10) reduced to the usual Runge-Lenz vector, i.e. $\mathbf{R}^Y \rightarrow \mathbf{R}$. The commutation relation (13), (15) and (16) become

$$[R_3, [R_+, R_-]] = 0$$

$$[R_\pm, [R_3, R_\pm]] = 0$$

$$2[R_3, [R_3, R_\pm]] \pm [R_\pm, [R_\pm, R_\mp]] = 0. \quad (26)$$

Equations (25), (8) and the commutation relation of angular momentum **L** show that our realization of Yangian $Y(sl(2))$ indeed reduce to the sub-algebra of loop algebra $L(sl(2))$. Correspondingly, the selection rule and matrix element given as above formulas are become those of usual Runge-Lenz vector in quantum mechanics [8]. It must be pointed out that equations (26) holds true for all energy eigenstate. This is different from the equations (17), which holds only for vanish energy. In (17), the generators are also different from this expression.

In this paper, we deform the Runge-Lenz vector to so-called Yangian Runge-Lenz vector in Coulumb problem. It is shown that the usual angular momentum and Yangian Runge-Lenz vector construct the Yangian $Y(sl(2))$. They are also the conservative quantities of the Coulumb problem. When the quantum deformation parameter α vanishes, the Yangian Runge-Lenz reduces to the usual Runge-Lenz vector. For the state with fixed energy, the energy plays the role as a deformation parameter in our realization of the $Y(sl(2))$ generators. It has been seen that the algebraic structures of the bounded state, free state and the critical point transiting from a bounded state to a free state, are different, which are Yangian $Y(sl(1,1))$, $Y(sl(2))$ and the loop algebra $L(sl(2))$ (more pricisely, the sub-algebra of $L(sl(2))$) [4] respectively. The selection rule and matrix elements of the $Y(sl(2))$ generators for the eigenstates of the Coulumb problem have also been discussed.

References

[1] V. G. Drinfeld, Sov. Math. Dokl. **32** (1985) 254; Proc. ICM-86 (Berkeley, 1987) p. 798.

[2] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard and Pasquier, Phys. Rev. Lett., **69** (1992) 2021; Y. Kato, Y. Kuramoto, Phys. Rev. Lett., **74** (1995) 1222; K. Hikami, J. Phys. **A28** (1995) L131.

[3] D. B. Uglov and V. E. Korepin, Phys. Lett., **A190** (1994) 238; F. Gohmann and V. Inozemtsev, Phys. Lett., **A214** (1996) 161.

[4] D. Bernard, Int. J. Mod. Phys., **B7** (1993) 3517.

[5] D. Bernard, Commun. Math. Phys. **137** (1991) 191; K. Schoutens, Phys. Lett., **B331** (1994) 335.

[6] S. L. Zhang, Int. J. Theor. Phys., bf 36 (1997) 2997.

[7] W.A. Bardeen, Prog. Theor. Phys. **Supp. No.123** (1996) 1; L. N. Lipatov, Talk in 1997 Nankai Symposium, (Tianjin 1997).

[8] L. I. Schiff, Quantum Mechanics, (McGraw-Hill, 1968).

[9] M. Kibler and T. Negadi, J. Phys., **A24** (1991) 2583; S. L. Zhang, Int. J. Theor. Phys., **34** (1995) 2217.