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Schwarz's Function and Chiral Potts Model

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Abstract

We derive the differential equation of the periods of the hyperelliptic curves arising in chiral Potts model in statistical mechanics, and obtain the Schwarz's (triangle) function representaion of the moduli of this hyperelliptic family of Riemann surfaces. Differential equation of Seiberg-Witten periods of the family is also discussed.

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1 Introduction

The integrable chiral Potts N-state model in statistical mechanics is characterized by the "rapidity" variables lying on the spectral curves, which form an one-parameter family of hyperelliptic curves of genus $N-1$, (see [1] [2] [4] [8] [10] [9] [11], and references therein). These algebraic curves possess a large number of symmetries, a property which has been speculated for the explanation of the role of these curves in solutions of the Yang-Baxter equation. For $N=2$, the curves form the elliptic family, and the statistical model is the eight-vertex model, where the solution was obtained by using the uniformization of Boltzmann weights in terms of Jacobi elliptic functions [7]. For $N>2$, there have been attempts of applying a similar uniformizing method to the problem. In [11], R. J. Baxter found an expression of the hyperelliptic function parametrization of Boltzmann weights via the classical work of Sonya Kowalevski; subsequently a mathematical understanding of Baxter's parametrization was given in [23] through the symmetry principal. However, to the best of the author's knowledge, the use of these quantitative results to calculations of the many interesting physical quantities has not been achieved yet. This paper is a continuation of our qualitative understanding of the chiral Potts hyperelliptic family. Other than for the mathematical reason on the rich geometrical properties revealed by the curves, one hope is that our studies would eventually shed new light on further developments of the physical theory. In this note we study the periods of chiral Potts hyperelliptic curves from the differential equation point of view, and obtain an uniformization of the modulus of the family. We shall present a detailed investigation on roles of Schwarz's (triangle) functions in chiral Potts models, and indicate the similarity of their structures with those appeared in the study of mirror symmetry of elliptic curves in $N=2$

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SUSY Landau-Ginzburg theory [24]. We also notice that chiral Potts curves form a special one-parameter sub-family inside the Seiberg-Witten curves, which appeared in $N = 2$ SUSY Yang-Mill theory on strong-weak coupling duality [25] [26] [5] [12] [14] [17]. In Seiberg-Witten theory, there associates a hyperelliptic Riemann surface for each kind of gauge groups, together with a preferred differential λ_{SW} of the second kind, whose periods describe the spectrum of BPS-saturated states. The degeneracy and monodromy of Seiberg-Witten periods, hence the associated Picard-Fuchs partial differental equations, of the multi-moduli Riemann surfaces play a special role in the theory [3] [20] [21]. While the chiral Potts family is characterized by hyperelliptic curves with a specific type of symmetry structure, their Seiberg-Witten periods will be studied in this note. We find that they are governed by the same type of differential equations as the periods of holomorphic differentials.

The organization of this paper is as follows. In Section 2, we shall discuss some basic properties of Schwarz's (triangle) functions, aiming at a description suited for the latter studies in Section 4. A summary of definitions and basic concepts of Schwarz's functions in [6], which are needed for the discussion of this paper, will be given in Appendix. In Section 3, we shall give a quick review on the "rapidity" curves in chiral Potts models, and give a characterization of these curves by the symmetry structure. In Section 4, we investigate the period matrix of the chiral Potts "rapidity" curves, and obtain an explicit relation of the modulus with Schwarz's functions. In Section 5, we discuss the Seiberg-Witten periods of the chiral Potts family.

2 Schwarz's Function Uniformization

Let us consider the following differential equations of Fuchsian type :

$$(\Theta^2 - \kappa^{-1} z(\Theta+\alpha)(\Theta+\beta))y(z) = 0 \ , \quad \Theta := z\frac{d}{dz} \ , \quad \kappa, \alpha, \beta \in \mathbb{Q} \ , \quad 1 > \alpha \geq \beta > 0 \ , \tag{1}$$

with three regular singular points $z = 0, \kappa, \infty$. By the change of variables $z = \kappa x$, the above equation is equivalent to the hypergeometric equation:

$$x(1-x)\frac{d^2y}{dx^2} + (1-(\alpha+\beta+1)x)\frac{dy}{dx} - \alpha\beta y = 0 \ , \quad \alpha, \beta \in \mathbb{Q} \ , \ 1 > \alpha \geq \beta > 0 \ . \tag{2}$$

Hence the fundamental solutions at $z = 0$ of Eqn. (1) are given by the hypergeomertic series

$$f_1(z) = F(\alpha, \beta; 1; \kappa^{-1} z)$$

together with another solution of the form

$$f_2(z) = \log(z) f_1(z) + \sum_{n=1}^{\infty} d_n z^n .$$

The ratio

$$\mathbf{t}(z) = \frac{f_2(z)}{2\pi \mathrm{i} f_1(z)} \tag{3}$$

is related to Schwarz's function $S(x)(= S(0, \alpha-\beta, 1-\alpha-\beta; x))$ by the following relation:

$$\mathbf{t}(z) = S(\kappa^{-1} z) + \frac{1}{2\pi \mathrm{i}} \log \kappa \ . \tag{4}$$

(For the Schwarz's functions and their basic properties, see Appendix). The function $\mathbf{t}(z)$ is the solution of the non-linear differential equation:

$$\{t, z\} = 2\kappa^{-2} I(0, \alpha - \beta, 1 - \alpha - \beta; \kappa^{-1} z) \ , \tag{5}$$

with the conditions:

$$\lim_{z \to 0} \mathbf{t}(z) = \infty \ , \quad \lim_{\theta \to 2\pi^-} \mathbf{t}(e^{i\theta} z) = \mathbf{t}(z) + 1 \ , \quad \lim_{z \to 0} \frac{e^{2\pi \mathrm{i} \mathbf{t}}}{z} = 1 \ , \tag{6}$$

here $I(0, \alpha - \beta, 1 - \alpha - \beta; x)$ is defined by

$$I(0, \alpha - \beta, 1 - \alpha - \beta; x) = \frac{1 + 2(\alpha + \beta - 2\alpha\beta - 1)x + (1 - (\alpha - \beta)^2)x^2}{4x^2(1 - x)^2} \ ,$$

(see also (26) in Appendix). The multi-valued function $\mathbf{t}(z)$ defines an uniformizing coordinate for the punctured disc near $z = 0$. By introducing the parameter

$$\mathbf{q} = e^{2\pi i \mathbf{t}} \ ,$$

one has a local isomorphism between the z-plane and $\mathbf{q}$-plane near origins with the relation:

$$z = \mathbf{q} + \sum_{n \geq 2} c_n \mathbf{q}^n \ , \ c_n \in \mathbb{C}.$$

The Riemann surface for the Schwarz's function $S(x)$ gives rise a Riemann surface $\mathcal{M}$ over z- and $\mathbf{t}$-planes,

$$\mathbb{P}^1 \xleftarrow{z} \mathcal{M} \xrightarrow{\mathbf{t}} \mathbf{t}(\mathcal{M}) \subset \mathbb{P}^1$$

with

$$\mathrm{Im}(z) = \begin{cases} \mathbb{P}^1 - \{0, \kappa\} & \text{if } \alpha + \beta = 1 \ , \\ \mathbb{P}^1 - \{0\} & \text{otherwise} \ . \end{cases}$$

By the theory of hypergeometric functions, the value $\mathbf{t}(z)$ is always purely imaginary for a real number z lying between 0 and κ. Indeed one has the following result.

Proposition 1. The values of $\mathbf{t}(z)$ varies from $\infty \mathrm{i}$ to $\mathbf{t}(\kappa^-)$ monotonically along the imaginary axis as the real z moves from 0 to κ with

$$\mathbf{t}(\kappa^-) = \frac{1}{2\pi \mathrm{i}}(-2\gamma - \psi(\alpha) - \psi(\beta) - \frac{\Gamma(\alpha)\Gamma(\beta)}{F(\alpha, \beta, 1; 1^-)} + \log \kappa) \ ,$$

here γ is the Euler constant, and $\psi(*)$ is the logarithmic derivative of Γ-function. We have

$$\mathbf{t}(z) - \mathbf{t}(\kappa^-) = \frac{\mathrm{i}}{2\pi} \Gamma(\alpha)\Gamma(\beta)(\frac{F(\alpha, \beta, \alpha + \beta; 1 - \kappa^{-1} z)}{F(\alpha, \beta, 1; \kappa^{-1} z)} - \frac{1}{F(\alpha, \beta, 1; 1^-)}) \ .$$

Note that

$$F(\alpha, \beta, 1; 1^-) = \begin{cases} \infty & \text{if } \alpha + \beta \geq 1 \ , \\ \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} & \text{if } \alpha + \beta < 1 \ . \end{cases}$$

Proof: By (6) and $\frac{\partial \mathbf{t}}{\partial z} \neq 0$, the function $-\mathrm{i}\mathbf{t}$ defines a bijective and decreasing function from $(0, \kappa)$ to $(-\mathrm{i}\mathbf{t}(\kappa^-), \infty)$. Hence one only needs to show the expression of $\mathbf{t}(\kappa^-)$, which by (4), is equivalent to the Schwarz's function $S(x)$ $(= S(0, \alpha - \beta, 1 - \alpha - \beta; x))$ at $x = 1^-$ is given by

$$2\pi \mathrm{i} S(1^-) = -2\gamma - \psi(\alpha) - \psi(\beta) - \frac{\Gamma(\alpha)\Gamma(\beta)}{F(\alpha, \beta, 1; 1^-)} \ .$$

By the relation of hypergeometric functions ([11] p.p. 110),

$$F(\alpha,\beta,\alpha+\beta;x)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!^2\Gamma(\alpha)\Gamma(\beta)}[\tilde{h}_n-\log(1-x)](1-x)^n\ ,$$

one has

$$\Gamma(\alpha)\Gamma(\beta)F(\alpha,\beta,\alpha+\beta;x)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!^2\Gamma(\alpha)\Gamma(\beta)}\tilde{h}_n(1-x)^n-\log(1-x)F(\alpha,\beta,1;1-x)$$

for $|\arg(1-x)|<\pi$, where $\tilde{h}_n=2\psi(1+n)-\psi(\alpha+n)-\psi(\beta+n)$. Therefore for $|\arg(x)|<\pi$, we have

$$\tilde{h}_0F(\alpha,\beta,1;x)-\Gamma(\alpha)\Gamma(\beta)F(\alpha,\beta,\alpha+\beta;1-x)=\log(x)F(\alpha,\beta,1;x)+\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!^2\Gamma(\alpha)\Gamma(\beta)}(\tilde{h}_0-\tilde{h}_n)x^n\ ,$$

which is a solution of the hypergeometry equation (2) for α,β. This implies

$$2\pi\mathrm{i}S(x)=\tilde{h}_0-\Gamma(\alpha)\Gamma(\beta)\frac{F(\alpha,\beta,\alpha+\beta;1-x)}{F(\alpha,\beta,1;x)},\tag{7}$$

hence

$$2\pi\mathrm{i}S(1^-)=\tilde{h}_0-\frac{\Gamma(\alpha)\Gamma(\beta)}{F(\alpha,\beta,1;1^-)}=-2\gamma-\psi(\alpha)-\psi(\beta)-\frac{\Gamma(\alpha)\Gamma(\beta)}{F(\alpha,\beta,1;1^-)}\ .$$

The conclusion of the proposition follows from (4) and (7). □

When $\alpha+\beta=1$, the hypergeometric equation (2) and its corresponding Schwarz's equation are invariant under the linear transformation

$$x\mapsto 1-x\ ,$$

hence Eqns (1), (5), are invariant under the transformation

$$z\mapsto\kappa-z\ .$$

Using $\Gamma(\beta)\Gamma(1-\beta)=\pi\csc(\beta\pi)$ and the identity,

$$\psi(\frac{p}{q})=-\gamma-\log 2q-\frac{\pi}{2}\cot\frac{\pi p}{q}+\sum_{j=1}^{q-1}\cos\frac{2\pi pj}{q}\log\sin\frac{\pi j}{q}\ ,\quad 0<p<q\ ,\quad p,q\in\mathbb{Z}\ ,\tag{8}$$

(see [6]), one obtains the following results by Proposition 1.

Corollary. For $\alpha+\beta=1$, $\beta=\frac{p}{q}$, we have

$$\mathbf{t}(\kappa^-)=\frac{1}{2\pi\mathrm{i}}(\log(4\kappa q^2)-2\sum_{j=1}^{q-1}\cos\frac{2\pi pj}{q}\log\sin\frac{\pi j}{q})\ ,$$

and

$$2\sin(\beta\pi)[\mathbf{t}(z)-\mathbf{t}(\kappa^-)]=\mathrm{i}\frac{F(1-\beta,\beta,1;1-\kappa^{-1}z)}{F(1-\beta,\beta,1;\kappa^{-1}z)}\ ,$$

hence the following duality property of $\mathbf{t}(z)$ holds:

$$2\sin(\beta\pi)[\mathbf{t}(\kappa - z) - \mathbf{t}(\kappa^-)] = \frac{-1}{2\sin(\beta\pi)[\mathbf{t}(z) - \mathbf{t}(\kappa^-)]} .$$

□

By the above corollary, we have

$$\mathbf{t}(\kappa^-) = 0 \quad \text{for} \quad (\alpha, \beta, \kappa^{-1}) = (\frac{1}{2}, \frac{1}{2}, 16),\ (\frac{2}{3}, \frac{1}{3}, 27)\ ,\ (\frac{3}{4}, \frac{1}{4}, 64)\ . \tag{9}$$

For $(\alpha, \beta, \kappa^{-1}) = (\frac{1}{2}, \frac{1}{2}, 16)$, one has the relation

$$\tau := 2\mathbf{t}(z) = \mathrm{i}\frac{F(\frac{1}{2}, \frac{1}{2}, 1; 1 - 16z)}{F(\frac{1}{2}, \frac{1}{2}, 1; 16z)} ,$$

the 16-multiple of the inverse of the above function, $16z(\tau)$, is the classical elliptic modular function $\kappa^2(\tau)$ (see [6] p.p. 99). The remaining two cases in (9) were discussed in [24], where $z(\mathbf{q})$ was explicitly expressed by theta constants. In particular, for $(\alpha, \beta, \kappa^{-1}) = (\frac{3}{4}, \frac{1}{4}, 64)$, it is the case for Ising model, and the Schwarz's function is expressed by

$$\sqrt{2}\mathbf{t}(z) = \mathrm{i}\frac{F(\frac{3}{4}, \frac{1}{4}, 1; 1 - 64z)}{F(\frac{3}{4}, \frac{1}{4}, 1; 64z)} .$$

¿From now on for the rest of this paper, we shall only concern ourselves with another type of generalization of Ising model with the following κ, α, β:

$$\kappa^{-1} = 64\ , \quad \alpha - \beta = \frac{1}{2}\ ,$$

(which implies $\beta < \frac{1}{2}$). The function $\mathbf{t}$ and the Riemann surface $\mathcal{M}$ for the above α, β, κ will be denoted by $\mathbf{t}_\beta, \mathcal{M}_\beta$ respectively. As the function

$$I(0, \alpha - \beta, 1 - \alpha - \beta; x) = I(0, \frac{1}{2}, \frac{1}{2} - 2\beta; x)$$

is invariant by changing β to $\frac{1}{2} - \beta$, we have the identification:

$$\mathbf{t}_\beta = \mathbf{t}_{\frac{1}{2}-\beta}\ , \quad \mathcal{M}_\beta = \mathcal{M}_{\frac{1}{2}-\beta}\ .$$

3 Chiral Pott N-state Curves

First we define the hyperelliptic curves which arise in the chiral Potts model of statistical mechanics systems [11] [23]:

Definition. For $N \geq 2$, a chiral Potts N-state curve (CP N-curve) is a hyperelliptic curve of genus $g := N - 1$ with an order N automorphism which fixes exactly 4 distinct elements. In this paper, the hyperelliptic involution and the order N automorphism of a CP N-curve will always be denoted by σ, θ, respectively.

Remark. The hyperellipticity for $N = 2$ simply means an elliptic curve E, which can be identified with an 1-dimensional torus ($E = \mathbb{C}/$lattice). In this situation, the order 2 automorphisms σ and θ are given by

$$\sigma : [\mathrm{z}] \mapsto [-\mathrm{z} + \mathrm{z}_0]\ , \quad \theta : [\mathrm{z}] \mapsto [-\mathrm{z}]\ ,$$

where $[z_0]$ is a 2-torsion element of E. □

For a CP N-curve W, one has the following commutative diagram:

$$\begin{array}{rcccl} & W & \xrightarrow{\Psi} & \mathbb{P}^1 & = W/<\theta> \\ & \downarrow \Pi & & \downarrow \pi & \\ W/<\sigma>= & \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 & = W/<\theta,\sigma> \end{array} \tag{10}$$

where Ψ, ψ, Π, π are the natural projections. For some suitable coordinates of $\mathbb{P}^1$, one can express ψ, π by

$$\psi(t) = t^N \ , \quad \pi(\lambda) = \frac{(1-k'\lambda)(1-k'\lambda^{-1})}{k^2} \ , \quad t, \lambda \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \ ,$$

with $k', k \in \mathbb{C} - \{0, \pm 1\}$ satisfying the relation, $k^2 + k'^2 = 1$. Therefore all the CP N-curves form a one-parameter family depending on $k' \in \mathbb{C}$, and a CP N-curve W is isomorphic to a plane curve of the following form for some k':

$$W_{N,k'} : \ t^N = \frac{(1-k'\lambda)(1-k'\lambda^{-1})}{k^2} \ , \quad (t, \lambda) \in \mathbb{C}^2 \ , \tag{11}$$

with the projections Ψ, Π, and the automorphisms θ, σ described by

$$\Psi(t,\lambda) = \lambda \ , \ \ \Pi(t,\lambda) = t \ , \ \ \theta(t,\lambda) = (\omega t, \lambda) \ , \ \ \sigma(t,\lambda) = (t, \lambda^{-1}) \ ,$$

where $\omega := e^{\frac{2\pi i}{N}}$. Note that $W_{N,k'}$ is isomorphic to $W_{N,\pm k'}, W_{N,\pm\frac{1}{k'}}$, as Riemann surfaces, i.e.

$$W_{N,k'} \simeq W_{N,k'_1} \iff (k' + \frac{1}{k'})^2 = (k'_1 + \frac{1}{k'_1})^2 \ ,$$

hence $(k' + \frac{1}{k'})^2$ is the actual parameter for the isomorphic classes of CP N-curves. The (t, λ)-coordinates of the branch points of Ψ and Π are given by

$$\begin{array}{l} \text{Branch points of } \Psi : \ \mathbf{p} = (\infty, 0), \ \mathbf{p}' = (\infty, \infty), \ \mathbf{q} = (0, k'), \ \mathbf{q}' = (0, k'^{-1}) \ , \\ \text{Branch points of } \Pi : \ \mathbf{b}_j = (\omega^{-j} \sqrt[N]{\frac{1+k'}{1-k'}}, -1), \ \mathbf{b}'_j = (\omega^{-j} \sqrt[N]{\frac{1-k'}{1+k'}}, 1) \ , \ 1 \le j \le N \ , \end{array} \tag{12}$$

where $\sqrt[N]{\frac{1-k'}{1+k'}} := \sqrt[N]{|\frac{1-k'}{1+k'}|} e^{\frac{i}{N} arg \frac{1-k'}{1+k'}}$. By the birational transformation

$$\mathrm{w} = \frac{k'}{k^2}(\lambda - \frac{1}{\lambda}) \ , \quad \lambda = \frac{1}{2k'}\{k^2(\mathrm{w} - t^N) + k'^2 + 1\} \ ,$$

another equivalent expression for a CP N-curve is given by

$$\begin{array}{rcll} W_{N,k'} : \mathrm{w}^2 & = & (t^N - \frac{1-k'}{1+k'})(t^N - \frac{1+k'}{1-k'}) \ , & (t, \mathrm{w}) \in \mathbb{C}^2 \ , \\ & = & t^{2N} - \epsilon t^N + 1 \ , & \epsilon := 2\frac{1+k'^2}{1-k'^2} \ . \end{array} \tag{13}$$

It is convenient to consider the above coordinates (t, w) of $\mathbb{C}^2$ as a local coordinates of the weighted projective plane $\mathbb{P}^2_{(1,1,N)}$ via the identification:

$$[1, t, \mathrm{iw}] = [y_1, y_2, y_3] \in \mathbb{P}^2_{(1,1,N)} \ . \tag{14}$$

The compactification of (13) gives rise the hypersurface in $\mathbb{P}^2_{(1,1,N)}$,

$$Y_{N,\epsilon} : y_1^{2N} + y_2^{2N} + y_3^2 - \epsilon y_1^N y_2^N = 0 \ , \quad [y_1, y_2, y_3] \in \mathbb{P}^2_{(1,1,N)} \ . \tag{15}$$

We have the birational equivalence:

$$W_{N,\pm k'} \simeq Y_{N,\epsilon},$$

under which the branch locus of Π corresponds to $\{y_3 = 0\}$, and the branch locus of Ψ becomes the union of zeros of y_j for $j = 1, 2$, with the corresponding $[y_1, y_2, y_3]$-coordinates given by

$$\mathbf{p} \longleftrightarrow [0, 1, \mathrm{i}], \quad \mathbf{p}' \longleftrightarrow [0, 1, -\mathrm{i}]; \quad \mathbf{q} \longleftrightarrow [1, 0, -\mathrm{i}], \quad \mathbf{q}' \longleftrightarrow [1, 0, \mathrm{i}].$$

In terms of (t, w) or $[y_1, y_2, y_3]$-coordinates, the hyperelliptic involution σ and the order N automorphism θ of $Y_{N,\epsilon}$ are given by

$$\begin{aligned} \sigma &: (t, \mathrm{w}) \longmapsto (t, -\mathrm{w}) \ , \quad \text{equivalently,} \quad [y_1, y_2, y_3] \longmapsto [y_1, y_2, -y_3] \\ \theta &: (t, \mathrm{w}) \longmapsto (\omega t, \mathrm{w}) \ , \quad \text{equivalently,} \quad [y_1, y_2, y_3] \longmapsto [y_1, \omega y_2, y_3]. \end{aligned}$$

It is easy to see that the group of projective linear transformations of $\mathbb{P}^2_{(1,1,N)}$, which preserve Eqn. (15) for a general ϵ, is generated by σ, θ, together with the involution ι,

$$\iota : [y_1, y_2, y_3] \longmapsto [y_2, y_1, y_3] \ , \quad \text{equivalently,} \quad (t, \mathrm{w}) \longmapsto (\frac{1}{t}, \frac{\mathrm{w}}{t^N}) \ .$$

It is easy to see that θ, ι, generate a group isomorphic to the dihedral group D_N. Hence the subgroup $< \sigma, \theta, \iota >$ of the automorphism group $\mathrm{Aut}(Y_{N,\epsilon})$ of Riemann surface $Y_{N,\epsilon}$ is isomorphic $\mathbb{Z}_2 \times D_N$. For $g \geq 2$, σ, θ, ι generate all the symmetries of a CP N-curve $W_{N,k'}$ for a general k' [23]. Indeed one can determine $\mathrm{Aut}(W_{N,k'})$ for every k' as follows.

Lemma 1 . For $g \geq 2$,

$$\mathrm{Aut}(Y_{N,\epsilon})(= \mathrm{Aut}(W_{N,k'})) = \begin{cases} < \sigma, \theta, \iota > \simeq \mathbb{Z}_2 \times D_N & \text{for } \epsilon \neq 0 \ , \ (k'^2 \neq -1) \ , \\ < \sigma, \widetilde{\theta}, \iota > \simeq \mathbb{Z}_2 \times D_{2N} & \text{for } \epsilon = 0 \ , \ (k'^2 = -1) \ , \end{cases}$$

where $\widetilde{\theta}$ is the automorphism on $Y_{N,0}$ defined by

$$(t, \mathrm{w}) \mapsto (\omega^{1/2} t, -\mathrm{w}) \ , \ \text{equivalently} \ , \quad [y_1, y_2, y_3] \longmapsto [y_1, \omega^{1/2} y_2, -y_3] \ .$$

Proof. By identifying $Y_{N,\epsilon}$ with $W_{N,k'}$, we are going to show the result for $\mathrm{Aut}(W_{N,k'})$. The hyperinvolution σ is in the center of $\mathrm{Aut}(W_{N,k'})$. We consider the quotient space $W_{N,k'}/ < \sigma >$ as $\mathbb{P}^1$ via the morphism Π in (10), and regard the quotient group $\overline{\mathrm{Aut}(W_{N,k'})}(:= \mathrm{Aut}(W_{N,k'})/ < \sigma >)$ as an automorphism group of $\mathbb{P}^1$, characterized by the automorphism group of $\mathbb{P}^1$ whose elements permute the branch points of Π (12), $\mathbf{b}_j, \mathbf{b}'_j, 1 \leq j \leq N$. For $|\frac{1-k'}{1+k'}| \neq 1$, it is easy to see that $\overline{\mathrm{Aut}(W_{N,k'})}$ is isomorphic to $< \theta, \iota >$, which implies

$$\mathrm{Aut}(W_{N,k'}) = < \sigma, \theta, \iota > \simeq \mathbb{Z}_2 \times D_N \ .$$

Now we consider the cases for $|\frac{1-k'}{1+k'}| = 1$ and $\mathrm{Aut}(W_{N,k'}) \neq < \sigma, \theta, \iota >$. In this situation, one can conclude that the linear tansformation of $\mathbb{P}^1$,

$$\phi(t) = (\sqrt[N]{\frac{1-k'}{1+k'}})^2 t \ ,$$

is an element of $\overline{\mathrm{Aut}(W_{N,k'})}$. Since $\phi(\sqrt[N]{\frac{1-k'}{1+k'}})$ is a branch point of Π, we have

$$(\sqrt[N]{\frac{1-k'}{1+k'}})^3 = \sqrt[N]{\frac{1+k'}{1-k'}}\omega^j \quad \text{for some } j \text{ , hence } (\frac{1-k'}{1+k'})^4 = 1 \text{ .}$$

Therefore

$$\frac{1-k'}{1+k'} = \pm \mathrm{i} \text{ , } \quad k'^2 = -1 \text{ ,}$$

hence it follows the conclusion for $\mathrm{Aut}(W_{N,k'})$. □

Among hyperelliptic curves, the CP N-curves can be characterized by the structure of their symmetry groups. Indeed we have the following result:

Proposition 2 . Let W be a hyperelliptic curve of genus $g = (N-1) \geq 2$. Then

$$\begin{array}{lcl} W \simeq W_{N,k'} \text{ for } k' \neq \pm\mathrm{i} & \Longleftrightarrow & \mathrm{Aut}(W) \simeq \mathbb{Z}_2 \times D_N \text{ ,} \\ W \simeq W_{N,\pm\mathrm{i}} & \Longleftrightarrow & \mathrm{Aut}(W) \simeq \mathbb{Z}_2 \times D_{2N} \text{ .} \end{array}$$

Proof. We have shown the group structure of $\mathrm{Aut}(W_{N,k'})$ in Lemma 1, so only the converse statements need to be considered. Let W be an hyperelliptic curve with the hyperelliptic involution σ. Assume $\mathrm{Aut}(W)$ is isomorphic to $\mathbb{Z}_2 \times D_N$ or $\mathbb{Z}_2 \times D_{2N}$ via an isomorphism f,

$$f : \mathrm{Aut}(W) \longrightarrow \mathbb{Z}_2 \times D \text{ , } \quad D = D_N \text{ or } D_{2N} \text{ .}$$

Then $f(\sigma) = (\rho_1, \rho_2)$ is in the center of $\mathbb{Z}_2 \times D$. Identify $W/ < \sigma >$ with $\mathbb{P}^1$, and regard $\mathrm{Aut}(W)/ < \sigma >$ as an automorphism group of $\mathbb{P}^1$. Claim: the group $\mathrm{Aut}(W)/ < \theta >$ is isomorphic to D. For $\rho_1 = \bar{1}$, (which automatically holds for $D = D_N$ with N odd), by changing the generator of $\mathbb{Z}_2$ of $\mathbb{Z}_2 \times D$ if necessary, one can obtain the result. For $\rho_1 = \bar{0}$, ρ_2 is a non-trivial element in the center of D, hence either $D = D_{2N}$ or $D = D_N$ with N even. The automorphism group $\mathrm{Aut}(W)/ < \sigma >$ of $\mathbb{P}^1$ is isomorphic to $\mathbb{Z}_2 \times (D/ < \rho_2 >)$, where $D/ < \rho_2 > \simeq D_N$ or $D_{N/2}$. Let l be the element of $\mathrm{Aut}(W)/ < \sigma >$ corresponding to the generator of the first factor group of $\mathbb{Z}_2 \times (D/ < \rho_2 >)$. For a suitable coordinate system of $\mathbb{P}^1$, the transformation group $D/ < \rho_2 >$ is generated by

$$t \mapsto e^{2\pi\mathrm{i}/m}t \text{ , } \quad t \mapsto t^{-1} \text{ , } \quad t \in \mathbb{C} \cup \{\infty\} \text{ , } m = \frac{N}{2} \text{ or } N \text{ .}$$

Since l is an involution commuting with the above automorphisms, the fixed point set of the first automorphism is invariant under l, hence one has the following expression of l:

$$l : t \mapsto -t^{-1} \text{ , } \quad \text{or } \; t \mapsto -t \text{ .}$$

If $l(t) = -t^{-1}$, this implies $m = 2$. Then $l \in D/ < \rho_2 >$, which contradicts the description of $\mathrm{Aut}(W)/ < \sigma >$. Therefore $l(t) = -t$ and m is odd. Hence $\mathrm{Aut}(W)/ < \sigma >$ is isomorphic to $D_{2m} = D$. Choose a suitable coordinate system of $\mathbb{P}^1$ such that the automorphism group $\mathrm{Aut}(W)/ < \sigma >$ is generated by the transformations

$$t \longmapsto e^{2\pi\mathrm{i}/m'}t \text{ , } \quad t \longmapsto t^{-1} \text{ , } \quad t \in \mathbb{P}^1 \text{ ,}$$

with $m' = N$ or $2N$ according as $D = D_N$ or D_{2N} respectively. The branch locus of the hyperelliptic cover of W over $\mathbb{P}^1$ consists of $2N$ elements which are permuted under the action of $\mathrm{Aut}(W)/ < \sigma >$.

They are given by $\{\mathbf{b}_j, \mathbf{b}'_j | 1 \le j \le N\}$ in (12) for some k'. Therefore W is isomorphic to $W_{N,k'}$, and the results follow immediately. □

For $k' = \pm \mathrm{i}$, the CP N-curve possesses the maximal symmetries, where the plane curve has the following form:

$$\mathrm{w}^2 = t^{2N} + 1 \ , \quad \text{equivalently} \quad , \ t^N = \frac{\pm \mathrm{i}}{2}(\lambda + \lambda^{-1}) \ .$$

By the substitution,

$$\lambda = \pm \mathrm{i} z, \quad t = \sqrt[N]{\frac{-1}{2}} \frac{u}{\lambda} \ ,$$

one obtains another expression of the same curve:

$$u^N = z^{N-1}(z^2 - 1),$$

which also appeared in the study of minimal surfaces [18].

4 Schwarz's Function Representation of Modulus of Chiral Potts Curves

The space of holomorphic differentials on the Riemann surface $W_{N,k'}$ is a g-dimensional vector space with a basis consisting of

$$t^{j-1} \frac{dt}{\mathrm{w}} \ , \quad 1 \le j \le g \ .$$

Let $(\tau_{jk})_{1 \le j,k \le g}$ be the period matrix of $W_{N,k'}$, which is determined by integrals over two special cycles A, B, together with the symmetries of the Riemann surface, where A is the 1-cycle lying over the segment from $\mathbf{b}'_N (= \sqrt[N]{\frac{1-k'}{1+k'}})$ to $\mathbf{b}_N (= \sqrt[N]{\frac{1+k'}{1-k'}})$ under the projection Π in (10), and B is 1-cycle over a path from $\mathbf{b}'_g (= \omega \sqrt[N]{\frac{1-k'}{1+k'}})$ to $\mathbf{b}_N$ with A, B intersecting at $\mathbf{b}_N$ of intersection number 1. In fact, τ_{jk}'s have the expressions:

$$\tau_{jk} = \frac{1}{N} \sum_{l=1}^{g} (\omega^{jl} - 1)(\omega^{-lk} - 1)\eta_l \ , \qquad \eta_l := \frac{\mathrm{i}\omega^{-l/2} \int_B \frac{t^{l-1}dt}{\mathrm{w}}}{2 \sin \frac{\pi l}{N} \int_A \frac{t^{l-1}dt}{\mathrm{w}}} \ , \tag{16}$$

where $1 \le j, k, l \le g$, (for the details, see [11] [23]). The values of τ_{jk}, η_l, are connected by the following $g \times g$ matrices:

$$(\omega^{-jk})(\tau_{jk})(\omega^{jk}) = N(\delta_{jk}\eta_j) \ . \tag{17}$$

Note that the above periods depend only on ϵ, equivalently the value of k'^2. Furthermore they are functions of $(k' + \frac{1}{k'})^2$ since. under the isomorphism between $W_{N,k'}$ and $W_{N,\frac{\pm 1}{k'}}$,

$$W_{N,k'} \longrightarrow W_{N,\frac{\pm 1}{k'}} \ , \quad (t, \mathrm{w}) \mapsto (\omega^{1/2} t, \mathrm{w}) \ ,$$

the corresponding A, B-cycles are identified, hence $\eta_j(k') = \eta_j(\frac{\pm 1}{k'})$.

Proposition 3. The functions η_j's have the following properties: .
(i) The equality, $\eta_j = \eta_{N-j}$, holds for $1 \le j \le g$.
(ii) For the value ϵ near 2, we have $\lim_{\epsilon \to 2} \eta_j = \infty$. For $\epsilon = 2 + re^{2\pi\xi}$ along the path from $\xi = 0$ to $\xi = 2\pi^-$ for $|r| \ll 1$, the change of values of η_j is given by

$$\eta_j \mapsto \eta_j + 1 \ , \quad 1 \le j \le g \ .$$

(iii) For real $\epsilon \to \infty$, (equivalently real $k' \to 1^-$), we have

$$\eta_j \longrightarrow \begin{cases} \frac{i\omega^{j/2}}{2\sin\frac{\pi j}{N}} & \text{if } j \le \frac{N}{2} , \\ \frac{-i\omega^{-j/2}}{2\sin\frac{\pi j}{N}} & \text{if } j \ge \frac{N}{2} . \end{cases}$$

Proof. On the Riemann surface $W_{N,k'}$, the following relations of holomorphic differentials hold:

$$\iota^*(\frac{t^{j-1}dt}{w}) = \frac{-t^{N-j-1}dt}{w} , \quad \theta^*(\frac{t^{j-1}dt}{w}) = \omega^j \frac{t^{j-1}dt}{w} . \tag{18}$$

By the equivalence of homologous 1-cycles,

$$\iota(B) \sim \theta^{-1}(B) , \quad \iota(A) \sim -A ,$$

the equality in (i) follows immediately. Let S be the algebraic surface composed of CP N-curves:

$$S = \{([y_1, y_2, y_3], [\epsilon_1, \epsilon_0]) \in \mathbb{P}^2_{(1,1,N)} \times \mathbb{P}^1 \mid \epsilon_1(y_1^{2N} + y_2^{2N} + y_3^2) - \epsilon_0 y_1^N y_2^N = 0 \} ,$$

and denote p the projection of second factor,

$$p : S \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \infty \quad , \text{ where } [0,1] \leftrightarrow \infty \quad , \quad [1,\epsilon] \in \mathbb{P}^1 - \{[0,1]\} \leftrightarrow \epsilon \in \mathbb{C}.$$

Then S is a non-singular surface over the ϵ-line. For ϵ close to 2, the curves degenerate via the equation:

$$(y_1^N - y_2^N + iy_3)(y_1^N - y_2^N - iy_3) = (\epsilon - 2)y_1^N y_2^N .$$

At $\epsilon = 2$, the degenerate fiber $p^{-1}(2)$ is a union of two rational curves intersecting normally at the following N elements:

$$[y_1, y_2, y_3] = [1, \omega^j, 0] , \quad 1 \le j \le N .$$

Around each of the above elements, there exists a local coordinates (t_+, t_-) such that the local description of the projection p is given by

$$\epsilon - 2 = t_+ t_- .$$

(A similar conclusion holds also for the degeneration near $\epsilon = -2$). By the theory of degeneration of Riemann surfaces, the vanishing cycle on a general curve $p^{-1}(\epsilon)$ near 2 is generated by the cycle A. As ϵ belongs to a small punctured disc near 2 which is of the form $2 + re^{2\pi\xi}$ with $0 \le \xi < 2\pi$, the change of cycles of a general fiber is described by the Picard-Lefschetz transformation L with the following description:

$$L(A) \sim A , \quad L(B) \sim B - A + \theta(A) .$$

By (18), the linear map of the cohomology group induced by L^{-1} gives rise to the change of periods:

$$\frac{\int_B \frac{t^{j-1}dt}{w}}{\int_A \frac{t^{j-1}dt}{w}} \mapsto \frac{\int_B \frac{t^{j-1}dt}{w}}{\int_A \frac{t^{j-1}dt}{w}} + 1 - \omega^j ,$$

hence we obtain (ii). For $0 < k' < 1$, the cycle B in $W_{N,k'}$ is homologous to $-A + B'$ where B' is the 1-cycle lying over the path

$$e^{-i\xi} \sqrt[N]{\frac{1-k'}{1+k'}}\omega , \quad 0 \le \xi \le \frac{2\pi}{N} ,$$

with $A \cdot B' = 1$. As k' tends to 1^-, one has

$$\sqrt{1-k'^2}w = \sqrt{(1-k'^2)t^{2N} - 2(1+k'^2)t^N + (1-k'^2)} \longrightarrow \sqrt{-4t^N} ,$$

hence

$$\frac{\int_B \frac{t^{j-1}dt}{\mathrm{w}}}{\int_A \frac{t^{j-1}dt}{\mathrm{w}}} + 1 = \frac{\int_{B'} \frac{t^{j-1}dt}{\mathrm{w}}}{\int_A \frac{t^{j-1}dt}{\mathrm{w}}} \longrightarrow \frac{\mathrm{i}\int_{2\pi/N}^0 e^{\mathrm{i}(j-N/2)\xi}d\xi}{\int_0^\infty t^{j-1-N/2}dt} \lim_{k'\to 1^-} (\frac{1-k'}{1+k'})^{j/N-1/2} = 0 \quad \text{for} \quad j \geq \frac{N}{2} ,$$

which implies

$$\lim_{k'\to 1^-} \eta_j = \frac{-\mathrm{i}\omega^{-j/2}}{2\sin\frac{\pi j}{N}} \qquad \text{for} \quad j \geq \frac{N}{2} .$$

By (i), one obtains the conclusion of (iii) for $j \leq \frac{N}{2}$. □

Remark. (I). One can also obtain (i) of the above proposition by (17) and the symmetric property of matrices (τ_{jk}) and (ω^{jk}) with the relation

$$(\omega^{jk}) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} (\omega^{-jk}) .$$

(II). For k' being real or purely imaginary, the curve $W_{N,k'}$ possesses a canonical real structure with the conjugation given by $c : (t, \mathrm{w}) \mapsto (\bar{t}, \overline{\mathrm{w}})$. The value of η_j for these curves has the following property:

$$\overline{\eta_j} = \begin{cases} -\eta_j - 1 & \text{for small } k' \in \mathbb{R} - \{0\} , \\ -\eta_j & \text{for } k' \in \mathrm{i}\mathbb{R} - \{0\} . \end{cases}$$

In fact, by the relations of cycles in $W_{N,k'}$,

$$c(A) \sim -A , \quad c(B) \sim \begin{cases} -\theta^{-1}(B) + A - \theta^{-1}(A) & \text{for small } k' \in \mathbb{R} - \{0\} , \\ -\theta^{-1}(B) & \text{for } k' \in \mathrm{i}\mathbb{R} - \{0\} . \end{cases}$$

one has

$$\overline{(\int_B \frac{t^{j-1}dt}{\mathrm{w}})/(\int_A \frac{t^{j-1}dt}{\mathrm{w}})} = (\int_{c(B)} \frac{t^{j-1}dt}{\mathrm{w}})/(\int_{c(A)} \frac{t^{j-1}dt}{\mathrm{w}}) ,$$

which equals to either

$$(\int_{B+A} \theta^{-1}\frac{t^{j-1}dt}{\mathrm{w}})/(\int_A \frac{t^{j-1}dt}{\mathrm{w}}) - 1 = \omega^{-j}(\int_B \frac{t^{j-1}dt}{\mathrm{w}})/(\int_A \frac{t^{j-1}dt}{\mathrm{w}}) + \omega^{-j} - 1$$

for small $k' \in \mathbb{R} - \{0\}$; or

$$(\int_B \theta^{-1}\frac{t^{j-1}dt}{\mathrm{w}})/(\int_A \frac{t^{j-1}dt}{\mathrm{w}}) = \omega^{-j}(\int_B \frac{t^{j-1}dt}{\mathrm{w}})/(\int_A \frac{t^{j-1}dt}{\mathrm{w}})$$

for $k' \in \sqrt{-1}\mathbb{R} - \{0\}$. Then the conclusion follows immediately. □

Now we are going to derive the differential equation for η_j. Let us consider a variation of (15) by introducing more parameters in the equation:

$$s_1 y_1^{2N} + s_2 y_2^{2N} + y_3^2 - s_0 y_1^N y_2^N = 0 , \quad [y_1, y_2, y_3] \in \mathbb{P}^2_{(1,1,N)} , \quad s_0, s_1, s_3 \in \mathbb{C} .$$

The periods of the above curves can be obtained by Dwork-Griffiths-Katz reduction method [13] [16] [27] of residuum expression:

$$\hat{\omega}_j(s_0,s_1,s_2) = \int_\gamma \int_{\Gamma_i} y_1^{N-1-j} y_2^{j-1} \frac{y_1 dy_2 \wedge dy_3 - y_2 dy_1 \wedge dy_3 + N y_3 dy_1 \wedge dy_2}{s_1 y_1^{2N} + s_2 y_2^{2N} + y_3^2 - s_0 y_1^N y_2^N} , \quad 1 \le j \le g , \quad (19)$$

where γ is a small circle in $\mathbb{P}^2_{(1,1,N)}$ normal to the curve, Γ_i are 1-circles on the curve. In terms of (t,w)-coordinate in (13), the periods of $W_{N,k'}$ are given by

$$\hat{\omega}_j(\epsilon,1,1) = \frac{-1}{2} \int_{\Gamma_i} t^{j-1} \frac{dt}{\mathrm{w}} , \quad \epsilon = 2\frac{1+k'^2}{1-k'^2} .$$

One has the following symmetry properties for the periods $\hat{\omega}_j$'s:

$$\begin{aligned} \hat{\omega}_j(\lambda^{2N}s_0, \lambda^{2N}s_1, \lambda^{2N}s_2) &= \lambda^{-N}\hat{\omega}_j(s_0,s_1,s_2) , \\ \hat{\omega}_j(s_0, \lambda^{2N}s_1, \lambda^{-2N}s_2) &= \lambda^{-(N-2j)}\hat{\omega}_j(s_0,s_1,s_2) , \quad \text{for } \lambda \in \mathbb{C}^* , \end{aligned} \quad (20)$$

whose infinitesimal forms give rise the differential equations:

$$\begin{aligned} (s_0\tfrac{\partial}{\partial s_0} + s_1\tfrac{\partial}{\partial s_1} + s_2\tfrac{\partial}{\partial s_2} + \tfrac{1}{2})\ \hat{\omega}_j &= 0 , \\ (s_1\tfrac{\partial}{\partial s_1} - s_2\tfrac{\partial}{\partial s_2} + N - 2j)\ \hat{\omega}_j &= 0 . \end{aligned}$$

With the trivial relation $y_1^{2N}y_2^{2N} = (y_1^N y_2^N)^2$, one obtains a further equation:

$$(\frac{\partial^2}{\partial s_1 \partial s_2} - \frac{\partial^2}{\partial s_0^2})\ \hat{\omega}_j = 0 . \quad (21)$$

Introduce the parameter

$$\zeta := \frac{s_1 s_2}{s_0^2} = \frac{1}{\epsilon^2} ,$$

and define the function

$$\omega_j(\zeta) := \epsilon^{(N-j)/N}\hat{\omega}_j(\epsilon,1,1) .$$

By (20), one has the relation

$$\hat{\omega}_j(s_0,s_1,s_2) = \frac{s_2^{(N-2j)/2N}}{s_0^{(N-j)/N}}\omega_j(\zeta) .$$

Eqn. (21) can be transformed into the form:

$$\{\zeta\frac{\partial}{\partial\zeta}(4\zeta-1)\frac{\partial}{\partial\zeta} + \frac{3N-2j}{2N}(4\zeta-1)\frac{\partial}{\partial\zeta} + \frac{(2N-j)(N-j)}{N^2}\}\omega_j(\zeta) = 0 , \quad (22)$$

which has three regular singular points at $\zeta = 0, \infty, \frac{1}{4}$. By the change of coordinates, $z = \frac{1}{64} - \frac{\zeta}{16}$, and the relation

$$\zeta\frac{\partial}{\partial\zeta} = (z - \frac{1}{64})\frac{\partial}{\partial z} , \quad (4\zeta-1)\frac{\partial}{\partial\zeta} = 4z\frac{\partial}{\partial z} ,$$

Eqn. (22) takes a form of (1):

$$(\Theta^2 - 64z(\Theta+\alpha_j)(\Theta+\beta_j))\omega_j = 0 \ , \Theta = z\frac{\partial}{\partial z} , \ \beta_j = \frac{N-j}{2N} , \ \alpha_j = \beta_j + \frac{1}{2} , \quad 1 \le j \le g . \quad (23)$$

We shall denote the Schwarz's function (5) for the above system with the value β_j simply by $\mathbf{t}_j(z)$:

$$\mathbf{t}_j := \mathbf{t}_{\beta_j} \ , \quad 1 \le j \le g \ .$$

By $\beta_j + \beta_{N-j} = \frac{1}{2}$, we have

$$\mathbf{t}_j = \mathbf{t}_{N-j} \ ,$$

which is the solution of the equation

$$\{t, z\} = 2^{13} I(0, \frac{1}{2}, \frac{2j-N}{2N}; 64z) = \frac{1 - 64(1 - \frac{(N-j)j}{N^2}))z + 3(32z)^2}{2z^2(1-64z)^2} \ , \tag{24}$$

with the condition (6). Note that the parameter z is related to the parameter k' of CP N-curves $W_{N,k'}$ by the expression:

$$z = \frac{1}{64} - \frac{1}{16\epsilon^2} = \frac{1}{16}(k' + \frac{1}{k'})^{-2} \ .$$

The value η_j in (16) for the period of $W_{N,k'}$ can be considered as a (multi-valued) function of z,

$$\eta_j = \eta_j(z) \ .$$

We are going to determine the relation between the function $\eta_j(z)$ and Schwarz's function $\mathbf{t}_j(z)$.

Theorem 1. The following relations hold:

(i) $\eta_j(z) = \mathbf{t}_j(z) - \frac{1}{2} + \frac{\mathrm{i}}{\pi}(\log \frac{N}{2} - \sum_{k=1}^{N-1} \cos \frac{2\pi jk}{N} \log \sin \frac{\pi k}{N}) \ ,$ for $1 \le j \le g$,

(ii) $\eta_j(z) = \eta_{N-j}(z) = \frac{\mathrm{i}}{2\pi}\Gamma(\beta_j + \frac{1}{2})\Gamma(\beta_j)\frac{F(\beta_j+\frac{1}{2},\beta_j,2\beta_j+\frac{1}{2};1-64z)}{F(\beta_j+\frac{1}{2},\beta_j,1;64z)} - \frac{1}{1-\omega^{-j}} \ ,$ for $1 \le j \le \frac{N}{2}$,

where $\beta_j = \frac{N-j}{2N}$.

Proof. The periods of $W_{N,k'}$, $\int_B \frac{t^{j-1}dt}{\mathrm{w}}, \int_A \frac{t^{j-1}dt}{\mathrm{w}}$, as functions of z satisfy Eqn. (23), hence the function $\eta_j(z)$ is a solution the corresponding Schwarzian equation (24). By Propostion 3 (ii) , $\eta_j(z)$ satisfies the conditions:

$$\lim_{z\to 0} \eta_j(z) = \infty \ , \quad \lim_{\xi \to 2\pi^-} \eta_j(e^{i\xi} z) = \eta_j(z) + 1.$$

By the characterization of the Schwarz's function $\mathbf{t}_j(z)$, we have

$$\eta_j(z) = \mathbf{t}_j(z) - \mathbf{t}_j(\frac{1}{64}^-) - c_j \ , \quad z \in \mathbb{C} - \{0, \frac{1}{64}\} \ ,$$

for some constant c_j. As z is a real number tending to $\frac{1}{64}^-$, one obtains the expression of c_j by Propostion 3 (iii):

$$c_j = \begin{cases} \frac{1}{1-\omega^{-j}} & j \le \frac{N}{2} \ , \\ \frac{1}{1-\omega^{j}} & j \ge \frac{N}{2} \ . \end{cases}$$

Note that $c_j = c_{N-j}$. In order to obtain the expression of $\mathbf{t}_j(z) - \eta_j(z)$, one needs only to consider the case $j \le \frac{N}{2}$, where the inequality, $\frac{1}{2} + 2\beta_j \ge 1$, holds. By (8) and Proposition 1, one obtains (ii) and the relation

$$\mathbf{t}_j(\frac{1}{64}^-) + c_j = \frac{1}{2} + \frac{\mathrm{i}}{\pi}(\sum_{k=1}^{N-1} \cos \frac{2\pi jk}{N} \log \sin \frac{\pi k}{N} - \log \frac{N}{2}) \ .$$

Then the conclusion (i) of the theorem follows immediately. □

Remark. (I) When $N = 2$, $W_{2,k'}$ is an elliptic curve and we have

$$\eta_1 = \frac{1}{2}\tau \ , \quad \tau = \frac{\int_B \frac{dt}{\mathrm{w}}}{\int_A \frac{dt}{\mathrm{w}}} \ .$$

By Theorem 1, the relation of τ and the Schwarz's function $\mathbf{t}(z)(:= \mathbf{t}_1(z))$ is given by

$$\tau + 1 = 2\mathbf{t},$$

which was obtained in [24] by another geometrical argument.

(II) For $N = 3$, there is only one function η_j in our consideration, denoted by $\rho := \eta_1 = \eta_2$. By Theorem 1 (i), the functions $\rho(z)$, $\mathbf{t}_1(z)$, are related by

$$\rho(z) = \mathbf{t}_1(z) - (\frac{1}{2} + \frac{1}{2\pi \mathrm{i}} \log \frac{27}{16}) \ .$$

The period matrix of $W_{N,k'}$ takes the form:

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rho$$

which was appeared in [11]. Hence the theta function of $W_{N,k'}$ has the following expression :

$$\vartheta(\mathrm{s}; \tau) = \sum_{m_1, m_2} e^{2\pi i(m_1 \mathrm{s}_1 + m_2 \mathrm{s}_2)} e^{2\pi i(m_1^2 + m_1 m_2 + m_2^2)\rho}$$

which can be decompsed as a sum of products of Jacobi elliptic theta functions associated to the modulus ρ and 3ρ:

$$\vartheta(\mathrm{s}_1, s_2; \tau) = \vartheta_2(\mathrm{s}_1 + s_2; 3\rho)\vartheta_2(\mathrm{s}_1 - s_2; \rho) + \vartheta_3(\mathrm{s}_1 + s_2; 3\rho)\vartheta_3(\mathrm{s}_1 - s_2; \rho) \ .$$

□

By Theorem 1, $\eta_j(z)$ is characterized as a solution of Eqn. (24) with the conditions:

$$\lim_{z \to 0} \eta_j(z) = \infty \ , \quad \lim_{\xi \to 2\pi^-} \eta_j(e^{i\xi} z) = \eta_j(z) + 1 \ , \quad \lim_{z \to 0} \frac{e^{2\pi i \eta_j(z)}}{z} = \frac{-4}{N^2} e^{2\sum_{k=1}^{N-1} \cos \frac{2\pi jk}{N} \log \sin \frac{\pi k}{N}} \ .$$

For $j = [\frac{N}{2}]$, Eqn. (24) becomes

$$\{t, z\} = 2^{13} I(0, \frac{1}{2}, 0; 64z) \ , \qquad \text{or} \qquad 2^{13} I(0, \frac{1}{2}, \frac{-1}{2N}; 64z) \ .$$

By the theory of Schwarz's functions, the Riemann surface $\mathcal{M}_{[\frac{N}{2}]}(z))$ is the uniformation of the z-plane. The function u $(:= \eta_{[\frac{N}{2}]}(z))$ provides a uniformizing coordinate of the variable z. We obtain a single-valued function $\eta_j(u)$ for each j, which is compatible with the z-functions.

5 Seiberg-Witten Differential of Chiral Potts Curves

The spectral curve of $N = 2$ SUSY $SU(n)$ Yang-Mills theory is of the form [17] [22],

$$\mu + \frac{\Lambda^{2N}}{\mu} + P(x) = 0 \ , \quad (\mu, x) \in \mathbb{C}^2 \ , \quad \text{where} \quad P(x) := x^N + u_2 x^{n-2} + u_3 x^{N-3} + \ldots + u_N \ ,$$

which, by the change of variables, $y = \mu - \frac{\Lambda^{2N}}{\mu}$, is equivalent to the plane curve:

$$y^2 = P(x)^2 - 4\Lambda^{2N} \ , \quad (y, x) \in \mathbb{C}^2 \ .$$

The Seiberg-Witten differential is the following abelian differential of second kind:

$$\lambda_{SW} = x\frac{d\mu}{\mu} = -x\frac{dP(x)}{y} \ ,$$

with the electric and magnetic masses a_i, a_i^D, given by their periods along some special closed paths on the Riemann surfaces:

$$a_i = \int_{\gamma_i} \lambda_{SW} \ , \quad a_i^D = \int_{\gamma_i^D} \lambda_{SW} \ .$$

The divisor of λ_{SW} is given by

$$\mathrm{div}(\lambda_{SW}) = \sum_{j=1}^{2N} \mathbf{z}_i - 2(\mathbf{p} + \mathbf{p}') \ , \quad \sum_{j=1}^{2N} \mathbf{z}_i = (x\frac{dP(x)}{dx} = 0), \quad \mathbf{p} + \mathbf{p}' = (x = \infty) \ .$$

By the relation

$$\frac{\partial y}{\partial u_i} = \frac{P(x)x^{N-i}}{y} \ , \quad \frac{\partial y^{-1}}{\partial u_i} = \frac{-P(x)x^{N-i}}{y^3} \ ,$$

the derivative of λ_{SW} with respective to $u_i, 2 \leq i \leq N$, is a holomorphic differential mudulus an exact form:

$$\begin{aligned} \frac{\partial \lambda_{SW}}{\partial u_i} &= -(N-i)x^{N-i}y^{-1}dx + x^{N-i+1}y^{-3}P(x)dP(x) \\ &= -(N-i)x^{N-i}y^{-1}dx - x^{N-i+1}dy^{-1} \\ &= x^{N-i}y^{-1}dx - d(x^{N-i+1}y^{-1}) \end{aligned}$$

Among the Seiberg-Witten family of hyperelliptic curves, there lies the following one-parameter family of CP N-curves,

$$\mu + \frac{1}{\mu} + x^N + u_N = 0 \ ,$$

whose relation with (11) is given by the following identification of variables:

$$\Lambda = 1 \ , \ \mu = \lambda \ , \ x^N = \frac{k^2}{k'}t^N \ , \quad u_2 = \cdots = u_{N-1} = 0 \quad , \quad u_N = \frac{1 + k'^2}{-k'} \ .$$

The expression of Seiberg-Witten differential of the above family is:

$$\lambda_{SW} = -N\frac{x^N dx}{y} = \sqrt[N]{\frac{k^2}{k'}}\frac{t d\lambda}{\lambda} \ .$$

With the coordinate (t, w) in (13), one has the relation:

$$\sqrt[N]{\frac{k'}{k^2}}\lambda_{SW} = -N\frac{t^N dt}{\mathrm{w}} \ .$$

Along the parameter of CP N-curve family, $\sqrt[N]{\frac{k'}{k^2}}a_i$, $\sqrt[N]{\frac{k'}{k^2}}a_i^D$, are determined by the periods of $\frac{t^N dt}{\mathrm{w}}$ over the cycles A and B. By replacing the $(j-1)$ by N in (19), one obtains the equation of $\int_A \frac{t^N dt}{\mathrm{w}}, \int_B \frac{t^N dt}{\mathrm{w}}$ along CP N-curves:

$$(\Theta^2 - 64z(\Theta + \frac{N-1}{2N})(\Theta - \frac{1}{2N}))\omega = 0 \quad , \Theta = z\frac{\partial}{\partial z} \ , \quad z = \frac{1}{64} - \frac{1}{16\epsilon^2} \ .$$

Appendix: Schwarz's Function

The hypergeometric equation (2) has three regular singular points,

$$x = 0, 1, \infty \ ,$$

with the solution expressed by Riemann P-function :

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ;x \\ 0 & 1-\alpha-\beta & \beta & \end{array}\right\} = ay_1(z) + by_2(z) \ , \quad a, b \in \mathbb{C},$$

where y_i's are the fundamental solutions at $x = 0$ with the hypergeomertic series $y_1(x)$,

$$y_1(x) = F(\alpha, \beta; 1; x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!^2} x^n \ ,$$

and $y_2(x)$ the another solution uniquely determined by form

$$y_2(x) = \log(x)F(\alpha, \beta; 1; x) + \sum_{n=1}^{\infty} a_n x^n \ .$$

The ratio of the above functions defines the Schwarz's (triangle) function:

$$S(x) \ (= S(0, \alpha-\beta, 1-\alpha-\beta; x)) := \frac{y_2(x)}{2\pi i y_1(x)} \ . \tag{25}$$

In general, the local system for Eqn. (2) is described by the analytic continuation of $y_1(x)$, $y_2(x)$, or that of any other fundamental solutions:

$$ay_1(x) + by_2(x) \quad , \quad cy_1(x) + dy_2(x) \ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \ .$$

The ratio

$$s(x) = \frac{ay_1(x) + by_2(x)}{ay_1(x) + by_2(x)} \ ,$$

is invariant under the substitution

$$y(x) \mapsto g(x)y(x)$$

for an arbitrary given function $g(x)$, which transfers Eqn. (2) into another second order linear differential equation. By choosing

$$g(x) = x^{\frac{-1}{2}}(1-x)^{\frac{-(\alpha+\beta)}{2}} \ ,$$

the equation is put into the form,

$$\frac{d^2y}{dx^2} + I(0, \alpha - \beta, 1 - \alpha - \beta; x)y = 0 \ ,$$

where

$$I(\rho, \mu, \nu; x) := \frac{1 - \rho^2}{4x^2} + \frac{1 - \nu^2}{4(1 - x)^2} + \frac{1 + \mu^2 - \rho^2 - \nu^2}{4x(1 - x)} \ . \tag{26}$$

Eliminating y in the system of equations:

$$\begin{cases} (\frac{d^2}{dx^2} + I(0, \alpha - \beta, 1 - \alpha - \beta; x))y = 0 \ , \\ (\frac{d^2}{dx^2} + I(0, \alpha - \beta, 1 - \alpha - \beta; x))(sy) = 0 \ , \end{cases}$$

one obtains the non-linear Schwarzian differential equation for $s(x)$,

$$\{s, x\} = 2I(0, \alpha - \beta, 1 - \alpha - \beta; x) \ , \tag{27}$$

here the Schwarzian derivative is defined by $\{s, x\} = \frac{\partial_x^3 s}{\partial_x s} - \frac{3}{2}(\frac{\partial_x^2 s}{\partial_x s})^2$. All solutions of the above equation are equivalent under the $SL_2(\mathbb{C})$-action :

$$s \mapsto \frac{as + b}{cs + d} \ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \ .$$

Each solution gives rise to a local uniformization of a punctured disc near $x = 0$. It determines the element

$$s(0) := \lim_{x \to 0} s(x) \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \ ,$$

and a parabolic transformation fixing $s(0)$, which is given by the local monodromy around $x = 0$. The Schwarz's function $S(x)$ is characterized as a solution of Eqn. (27) with the conditions:

$$\lim_{x \to 0} S(x) = \infty \ , \quad \lim_{\xi \to 2\pi^-} S(e^{\xi i}x) = S(x) + 1 \ , \quad \lim_{x \to 0} \frac{e^{2\pi i S}}{x} = 1 \ .$$

It forms an uniformizing coordinate of the punctured disc at $x = 0$, and one has the power series expansion:

$$x = e^{2\pi i S} + \sum_{n \geq 2} c_n e^{n 2\pi i S} \ , \quad , c_n \in \mathbb{C},$$

which defines an local isomorphism between the x-plane and $e^{2\pi i S}$-plane near origins. The analytical continuation of $S(x)$ gives rise to a Riemann surface $\mathcal{M}^\circ$ which spreads over the S-plane and infinitely covers over x-plane outside $\{0, \frac{1}{\lambda}, \infty\}$. By $\frac{dS}{dx} \neq 0$, the projection of $\mathcal{M}^\circ$ to the S-plane defines a local isomorphism with an open domain in $\mathbb{P}^1$ as its image. One has the following relations between Riemann surfaces:

$$\mathbb{P}^1 - \{0, 1, \infty\} \xleftarrow{x} \mathcal{M}^\circ \xrightarrow{S} S(\mathcal{M}^\circ) \subset \mathbb{P}^1 \ . \tag{28}$$

For $\alpha - \beta > 0$, one can extend $\mathcal{M}^\circ$ to a Riemann surface over $x = \infty$ as follows. Since the fundamental solutions of Eqn. (1) near $x = \infty$ can take the form:

$$x^{-\alpha} p_\alpha(\frac{1}{x}) \ , \quad x^{-\beta} p_\beta(\frac{1}{x}) \ , \quad |x| \gg 0 \ ,$$

for p_α and p_β power series in $\frac{1}{x}$ with the constant term 1, on a connected region of $\mathcal{M}^\circ$ near $x=\infty$, one has

$$S(x) = \frac{ax^{-\alpha}p_\alpha(\frac{1}{x}) + bx^{-\beta}p_\beta(\frac{1}{x})}{cx^{-\alpha}p_\alpha(\frac{1}{x}) + dx^{-\beta}p_\beta(\frac{1}{x})}\ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})\ , \quad \text{for } |x| \gg 0\ .$$

Therefore

$$\lim_{x\to\infty} S(x) = \frac{b}{d} \in \mathbb{P}^1\ .$$

Write

$$\alpha - \beta = \frac{l}{k}$$

with k and l two relatively prime positive integers. By the expression of $S(x)$ near $x=\infty$, there exists a local coordinate system w near $x=\infty$, and $\tilde{s}$ near $s=\frac{b}{d}$, such that over a small punctured disc near $x=\infty$, a connected region of $\mathcal{M}^\circ$ has the local description:

$$w^l = \tilde{s}^k\ , \quad (w,\tilde{s}) \neq (0,0)\ .$$

Let u be the local coordinate of the desingularization of the above equation. The description of (28) on a small connected region over a disc near $x=\infty$ is now equivalent to the diagram:

$$\{0<|w|<\delta'\} \longleftarrow \{0<|u|<\epsilon\} \longrightarrow \{0<|\tilde{s}|<\delta\}\ , \quad u^k = w \leftarrow u \rightarrow \tilde{s} = u^l\ ,$$

which can be extended to the following one:

$$\{0<|w|<\delta'\} \longleftarrow \{|u|<\epsilon\} \longrightarrow \{|\tilde{s}|<\delta\}\ , \quad u^k = w \leftarrow u \rightarrow \tilde{s} = u^l\ .$$

The above local construction provides the data for the extended Riemann surface of $\mathcal{M}^\circ$ over $x=\infty$ with the multiplicity k. By the assumption on α,β, we have $|1-(\alpha+\beta)|<1$. For $1-(\alpha+\beta)\neq 0$, write

$$|1-(\alpha+\beta)| = \frac{l'}{k'},\ \ \gcd(l',k') = 1\ .$$

With the same procedure as before, $\mathcal{M}^\circ$ can again be extended to one over $x=1$. The resulting extended Riemann surface, denoted by $\mathcal{M}$, will be called the Riemann surface associated to the Schwarz's function $S(x)$. One obtains the extension of the diagram (28) for $\alpha-\beta>0$:

$$\mathbb{P}^1 \xleftarrow{x} \mathcal{M} \xrightarrow{S} S(\mathcal{M}) \subset \mathbb{P}^1\ ,$$

with

$$\mathrm{Im}(x) = \begin{cases} \mathbb{P}^1 - \{0,1\} & \text{if } \alpha+\beta = 1\ , \\ \mathbb{P}^1 - \{0\} & \text{otherwise}\ . \end{cases}$$

Note that if $\alpha-\beta$ is equal to the reciprocal value of an integer, the Riemann surface $\mathcal{M}$ is local isomorphic to $S(\mathcal{M})$ via the map S near an element in $x^{-1}(\infty)$. A similar phenomenon holds for elements of $x^{-1}(1)$ for the same description of the value of $1-(\alpha+\beta)$. In this situation, the function $\tau = S(x)$ has a meromorphic single-valued inverse function $x = S^{-1}(\tau)$ on a certain simply connected region of τ-plane, which is called an automorphic function.

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