Compatibility in physical theories

Helv. Phys. Acta 70 (1997) 387 - 405
0018-0238/97/030387-19 $ 1.50+0.20/0

Helvetica Physica Acta

Compatibility in Physical Theories

by D. I. Herţia and Al. Ivanov

Institute of Physical Chemistry, 202 Spl. Independenţei, R–77208 Bucharest, Romania

(3.V.1996, revised 8.VII.1996)

Abstract. Only those physical theories are considered which may be constructed from atomic ortholattices. It is proved that an atomic ortholattice becomes a physical theory (i.e. an atomic orthomodular having the covering property lattice) if an appropriate denumerable family $\{F_p \subset L \times L;\ p \geq 0\}$ of symmetric relations, with F_p describing different "degrees of incompatibility", may be defined on it. It is shown that such a family may be explicitly constructed if certain easily understandable physical reasons are taken into account.

1 Introduction

In this work a physical theory is, from the mathematical point of view, an atomic orthomodular lattice which verifies the covering property. The elements of a physical theory will be interpreted as tests ("yes–no" experiments). To be more precise, a test is considered to be a pair consisting of a propositional part (a set of logically equivalent propositions) and an operational component, which is a set of contacts (ideal measuring procedures) which permit us to decide if the propositions of the test are true or not [1].

Our aim is to present an interpretation of the mathematical structure of a physical theory from an unitary point of view. In order to make clear our interpretation, we will first review the current situation in this field. For the sake of convenience we will consider a physical theory $(L, \leq, \perp)$, i.e. an orthomodular lattice defined by the order "$\leq$" and the orthocomplementation "$\perp$". The first remark is that the order and the orthocomplementation on L are physically justified relations [2]. This means that these two relations have a clear physical meaning resulting from empirical facts. The algebraic operations on L, i.e. the meet and join, defined respectively by $a \wedge b$ and $a \vee b$ for all $a,b \in L$, are also considered physically meaningful, but this is not so easy to accept (see, for instance [3]). Nevertheless, it is generally admitted that

any physical theory is at least an atomic ortholattice (concerning atomicity, see [4]). Concerning orthomodularity and the covering law, they are discussed in several works which contain also attempts to interpret them. The orthomodularity is "derived" by using different tools and assumptions in the papers [6-8,1]. Concerning interpretation of the covering we mention the Jauch and Piron's attempt to find such an interpretation by using a reasoning based on the notion of measurement of the first kind [9]. The problem of the covering property appears also in an interesting paper by Pool about quantum mechanics and semimodularity [10].

In our work we try to show that the empirical compatibility is that physical fact which can justify both orthomodularity and the covering law (we say that two tests a and b are empirically compatible if there exists a contact which can measure both a and b in any state). It is therefore important to find relations on L which describe the empirical compatibility. Besides, when there are empirically incompatible tests, we have to consider also an incompatibility relation on L.

Let us assume that the atomic ortholattice L is a virtual physical theory. Then we have to accept that it must exist on L two complementary relations describing compatibility and incompatibility of tests. For our treatment it is necessary to analyze more deeply the incompatibility of tests. If the "definition" of the empirical compatibility is considered, we may say that two tests are incompatible if we can not find a contact able to measure both of them in any state. This fact does not exclude the existence of contacts which measure simultaneously the given tests in some states. A direct consequence of this observation is the idea that **it might exist pairs of incompatible tests which are not equally incompatible.**

All these considerations make reasonable the fundamental hypothesis of our treatment: the incompatibility relation may be split off into a family of mutually disjoint relations, each of them corresponding to a given "degree of incompatibility". More precisely, we will prove that an atomic ortholattice L is a physical theory if there exists $\{F_p;\ p \in \mathbb{N}\}$ a family of relations on L ($\mathbb{N}$ is the set of natural numbers including 0), F_p representing the incompatibility of degree p. In addition, it will be assumed that the set $\mathbb{N} \cup \{\infty\}$ plays the role of an "incompatibility scale", in the sense that, if $(a,b) \in F_p$, $(a',b') \in F_{p'}$ and $p < p'$, then the tests a,b are less incompatible than the tests a',b'.

It is obvious that such a hypothesis makes sense if we are able to construct explicitly a family $\{F_p;\ p \in \mathbb{N}\}$ for some atomic ortholattices. This will be done in the present work. For the moment it is important to note that, if the existence of a degree of incompatibility is accepted, then all relations F_p must be symmetric. Since, taking into account the above mentioned convention concerning "incompatibility scale", F_0 represents obviously the compatibility relation, it is also reflexive.

It will be seen that, once the relation F_0 constructed, the symmetry of the relation F_1 is crucial for defining the "incompatibility scale" and the basic geometrical structure of physical theories. In conclusion we will be in position to affirm that the orthomodularity, the atomicity, the covering property and the dimension function on a physical theory have all a common basis: the fact that in any physical theory must exist a family of relations which describes the empirical compatibility and incompatibility of tests.

Consequently, the structure of the paper is the following. In Paragraph 3 we study certain symmetry properties which are useful in our treatment and the possible physical meaning of some important relations, like modularity. The compatibility relation is defined in Paragraph 4. Paragraph 5 contains several interesting results which point out once again the properties of compatibility in orthomodular lattices which do not satisfy necessarily the covering law. Finally, the construction of the family $\{F_p\}$ is completed in Paragraph 7.

2 Prerequisites

All the results, notions and notations which are used in our work may be found, for instance, in the book [5]. Nevertheless, we decided that it would be easier to read the paper if a list of the most frequently used notions and results is given. In this paragraph we present such a list.

0° Given S a nonempty set and $R \subset S \times S$ a relation, we will denote $(a,b) \in R$ by $(a,b)R$ or aRb and $(a,b) \notin R$ by $(a,b)\overline{R}$ or $a\not{R}b$.

1° Given $(L, \leq)$ a lattice, we define the relation "$\leq_*$" by

$$x \leq_* y \Longleftrightarrow y \leq x. \tag{2.1}$$

Denoting by L, L^* the pairs $(L, \leq)$, $(L, \leq_*)$ respectively, we call L^* the dual of L. The relation "$\leq$" will be occasionally denoted by $\mathcal{O}$.

2° Given $(L, \leq)$ a lattice, we say that $(a,b) \in L \times L$ is a modular (dual modular) pair and write $(a,b)M$ ($(a,b)M^*$) if

$$x \in L,\ x \leq b \Rightarrow (x \vee a) \wedge b = x \vee (a \wedge b) \tag{2.2}$$

$$(x \in L,\ x \geq b \Rightarrow (x \wedge a) \vee b = x \wedge (a \vee b)). \tag{2.2'}$$

L is called modular (dual modular) if $(a,b)M$ ($(a,b)M^*$) for all pairs $(a,b) \in L \times L$. L is called semimodular if $(a,b)M \Rightarrow (b,a)M$.

3° We write $a \lessdot b$ and say that "b covers a" or "a is covered by b" if $a \leq x \leq b \Rightarrow x = a$ or $x = b$. By $\Omega(L)$ is denoted the set of all atoms of L. We say that L has the covering property if

$$a \in L,\ p \in \Omega(L),\ p \not\leq a \Rightarrow a \lessdot a \vee p. \tag{2.3}$$

We say that L has the exchange property if

$$a\in L,\ p,q\in\Omega(L),\ p\not\leq a,\ p\leq a\vee q \Rightarrow q\leq a\vee p. \tag{2.4}$$

The implication (2.3) ⇒ (2.4) is true. The converse implication is true if L is atomistic (i.e. any $a\in L$ is the join of all atoms under a). In atomistic lattices having the covering property the following implication is also true:

$$a\wedge b\lessdot b \Rightarrow a\lessdot a\vee b,\quad a,b\in L. \tag{2.5}$$

4° A subset B of an atomistic lattice L is said to be a chain if any two elements of B are comparable. A chain B is said to be connected if for any $x\in B$ there exists $x'\in B$ such that $x\lessdot x'$. If B is a chain such that $\wedge B=a$, $\vee B=b$, then B is said to be a chain between a and b. The cardinal of a chain is called the length of that chain. We say that L satisfies the Jordan–Dedekind (JD) property (axiom) if for any two elements $a,b\in L$, all connected chains between a and b have the same length. An element of an atomistic lattice L is called finite if it is the join of a finite family of atoms. If a is a finite element of L and L has the JD property, then all connected chains between 0 and a have the same length, denoted by d(a), which is called the dimension of the element a. The set J(L) of all finite elements of L is an ideal and $J(L)\ni a \rightarrow d(a)\in\mathbb{N}$ is a dimension function on J(L).

5° Let L be a lattice. For any two elements $x,y\in L$, $x<y$, the set $L[x,y]=\{z\in L;x\leq z\leq y\}$ is called the interval/segment between x and y. If $a,b\in L$, (a,b)M, $(b,a)M^*$, then the intervals $L[a\wedge b,b]$, $L[a,a\vee b]$ are isomorphic via the mappings

$$L[a\wedge b,b]\ni x \rightarrow \Theta_1(x)=a\vee x\in L[a,a\vee b], \tag{2.6}$$

$$L[a,a\vee b]\ni y \rightarrow \Theta_2(x)=b\wedge y\in L[a\wedge b,b]. \tag{2.6'}$$

6° The relations C,C^* defined by

$$(a,b)C \Longleftrightarrow b=(a\wedge b)\vee(a^\perp\wedge b) \tag{2.7}$$

$$(a,b)C^* \Longleftrightarrow b=(a\vee b)\wedge(a^\perp\vee b) \tag{2.7'}$$

are called commutativity and dual commutativity on L respectively.

7° Given L an orthomodular lattice and $a\in L$, we can define the corresponding to the element a Sasaki's projections φ_a, $\tilde{\varphi}_a$, as follows:

$$\varphi_a\colon L\rightarrow L,\ \varphi_a(x)=(x\vee a^\perp)\wedge a \tag{2.8}$$

$$\tilde{\varphi}_a\colon L\rightarrow L,\ \tilde{\varphi}_a(x)=(x\wedge a^\perp)\vee a \tag{2.8'}$$

3 Symmetry properties of binary relations

We will study several properties of binary relations on an ortholattice L, which are conventionally called symmetry properties. Our aim is to show that there are some relations on L whose symmetry properties imply the orthomodularity or the semimodularity (covering property) of L. Since the symmetry properties are usually easier to interpret in physical terms, they may be used for justifying orthomodularity or semimodularity.

Before to list the interesting to us symmetry properties, we have to note that only those relations which are invariant under automorphisms of L are considered. This is because, given $P(x_1,...,x_n) \subset L^{xn}$ a relation having a physical meaning and $f: L \to L$ an automorphism, the relation $f(P)=\{(f(x_1),...,f(x_n)); (x_1,...,x_n) \in P\}$ must coincide with P. The most important examples of invariant under automorphisms relations are the order "$\leq$" and the orthocomplementarity "$\perp$", which is defined by $(a,b)\perp \Longleftrightarrow \Longleftrightarrow a \leq b^{\perp}$. It is easy to understand that there exist a lot of relations on L, constructed by using "$\leq$" and "$\perp$", which are also invariant under automorphisms. It will be easy to verify that all particular relations with possible physical implications discussed below have this property.

In what follows we will consider for any relation $R \subset L \times L$ a relation R^*, called the dual of R, and defined by $R^* = \{(x^{\perp},y^{\perp}); (x,y)R\}$.

The symmetry properties listed in the next definition are denoted, for the sake of convenience, by greek letters.

Definition 3.1. Let $R \subset L \times L$ be a binary relation on L. We say that R is:

(i) $\alpha-$symmetric if $(a,b)R \Rightarrow (a^{\perp},b)R$;

(ii) $\delta-$symmetric if $(a,b)R \Rightarrow (b,a)R^*$;

(iii) $\alpha\delta-$symmetric if $R = R^*$;

(iv) $\sigma-$symmetric if $(a,b)R \Rightarrow (b,a)R$.

Obviously, the $\sigma-$symmetry is the usual symmetry. The $\delta-$symmetric and $\alpha\delta-$symmetric relations will be called occasionally dual symmetric respectively selfdual relations. We will use also the letter ε for denoting the obvious implication $(a,b)R \Rightarrow \Rightarrow (a^{\perp},b^{\perp})R^*$. It will be used frequently the relation A defined by

$$(a,b)A \Longleftrightarrow a \wedge b \lessdot b \tag{3.2}$$

It is well known the following result:

Proposition 3.3. The ortholattice L is orthomodular iff C is $\sigma-$symmetric.

It may be proved also

Proposition 3.4. The ortholattice L is orthomodular iff C is $\alpha\delta-$symmetric.

Proof. Suppose C is selfdual and let us take $a,b \in L$, $a<b$. We have $a^{\perp} \vee b \geq a^{\perp} \vee a = 1$, so that $a^{\perp} \vee b = 1$. It results $(a^{\perp} \vee b) \wedge (a \vee b) = b$, so that $(a,b)C^*$ and, by using selfduality of C, we get $(a,b)C$ and finally

$$b = (a \wedge b) \vee (a^{\perp} \wedge b) = a \vee (a^{\perp} \wedge b).$$

There are also known the following two propositions:

Proposition 3.5. If L is an atomic ortholattice, then it satisfies the covering law iff the relation M is σ−symmetric (i.e. L is semimodular).

Proposition 3.6. If L is an orthomodular atomic lattice, then L has the covering property iff the relation A (or A^*) is δ−symmetric.

Propositions (3.3−3.6) might be considered as justifications of the orthomodularity and covering property provided the relations they involve have themselves a clear physical interpretation which supports the symmetry properties required. The existence of such kind of results is important since they suggest a method for justifying "structural properties" of physical theories, like orthomodularity and semimodularity, by proving their equivalence with symmetry properties of adequate relations. Practically, **we have to find physically significant relations, whose symmetry properties imply orthomodularity and semimodularity of physical theories**.

It has been said in Introduction that we intend to construct a family $\{F_p\}$ of binary relations describing "all possible degrees of incompatibility" of tests (elements of L). Naturally, the first step is to establish a minimal set of properties which these relations must have. For the moment we note only, that the empirical definition of incompatibility and the fact that the measurement of a test $a \in L$ is equivalent with the measurement of its negation $a^\perp \in L$ imply that all F_p must be α−symmetric. Henceforth, it becomes obvious that α−property is deeply connected with those relations which are related, in a sense or another, with the possibility of simultaneous measurement of tests. Therefore, it is interesting to see if the relations A and M are α−symmetric. The negative answer is almost evident in the case of A. The modularity is not α−symmetric also, but this statement is not so easy to prove. We give below an example, constructed in the lattice of all closed subspaces of an infinite−dimensional Hilbert space, which proves that, in general, M is not α−symmetric.

Example. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}_C(\mathcal{H})$ the lattice of all closed subspaces of $\mathcal{H}$. Obviously, $\mathcal{L}_C(\mathcal{H})$ is orthomodular and atomic. The relation M, considered on $\mathcal{L}_C(\mathcal{H})$, is symmetric, dual symmetric, selfdual, but it does not coincide with $L \times L$ unless $\mathcal{H}$ is a finite dimensional space. When $\mathcal{H}$ is finite dimensional, then M is obviously α−symmetric, so that we have to look for our example in an infinite dimensional Hilbert space. For infinite dimensional case, we have the following characterization of dual modular pairs of subspaces: $(\mathcal{M}, \mathcal{N})M^* \Longleftrightarrow \mathcal{M} + \mathcal{N}$ is a closed subspace $\Longleftrightarrow \mathcal{M} + \mathcal{N} = \mathcal{M} \vee \mathcal{N}$ (Mackey's theorem). Given $\mathcal{M}, \mathcal{N} \in \mathcal{L}_C(\mathcal{H})$, $\mathcal{M} \wedge \mathcal{N} = 0$, we can define "the cosine of the angle between the subspaces $\mathcal{M}$, $\mathcal{N}$" as follows:

$$cos(\mathcal{M}, \mathcal{N}) = \sup\{ |\langle x,y \rangle| ;\ \|x\| = \|y\| = 1,\ x \in \mathcal{M},\ y \in \mathcal{N} \}, \tag{3.7}$$

where $\langle\ ,\cdot\rangle$, $\|\cdot\|$ are the scalar product and the norm in $\mathcal{H}$ respectively. When the dimensions of the subspaces $\mathscr{M}$, $\mathscr{N}$ are finite, we have $cos(\mathscr{M}, \mathscr{N})<1$. This is because the closed unit spheres in both considered subspaces are compact, their Cartesian product is compact also and the function "*cos*" being continuous attains its *sup* for a pair (x,y). If the subspaces have infinite dimension this is not longer true and it becomes possible to have $cos(\mathscr{M}, \mathscr{N})=1$ even if $\mathscr{M} \wedge \mathscr{N}=0$. In such a situation we say that $(\mathscr{M}, \mathscr{N})$ is an asymptotic pair and write $\mathscr{M} \parallel \mathscr{N}$. By using Mackey's theorem we get

$$(\mathscr{M}, \mathscr{N})M^* \Longleftrightarrow \mathscr{M} \not\parallel \mathscr{N}. \tag{3.8}$$

Now we have all necessary elements for constructing our example. Let us consider $\{e_n;\ n\in\mathbb{N}\}$ an orthonormal basis in $\mathscr{H}$ and A, $B\subseteq\mathbb{N}$ two infinite subsets such that $A\cup B=\mathbb{N}$ and $A\cap B=\varnothing$. By changing the notation of the elements of the considered basis, we may replace easily $\{e_n;\ n\in A\}$, $\{e_n;\ n\in B\}$ by $\{f_k;\ k\in\mathbb{N}\}$, $\{g_j;\ j\in\mathbb{N}\}$ respectively. Denoting by [x] the one-dimensional subspace spanned by x, we put $\mathscr{M} = \vee[f_k]$, $\mathscr{M}^{\perp} = \vee[g_j]$. Then we consider two numerical sequences (a_n), (b_n), such that

$$\begin{aligned} &\text{(i)}\quad 0<a_n,\ b_n<1,\ a_n^2+b_n^2=1 \\ &\text{(ii)}\quad a_n\to 1,\ b_n\to 0,\ sup\ b_n<1. \end{aligned} \tag{3.9}$$

We define the subspace $\mathscr{N}=\vee[h_i]$, where $\{h_i=a_ig_i+b_if_i;\ i\in\mathbb{N}\}$ is obviously an orthonormal set. It is easy to see that $\mathscr{M}\wedge\mathscr{N} = \mathscr{M}^{\perp}\wedge\mathscr{N} = 0$. Then $a_n\to 1$ implies $(\mathscr{M}^{\perp}, \mathscr{N})\overline{M}^*$ since $\mathscr{M}^{\perp}\parallel\mathscr{N}$. On the other hand, for arbitrarily fixed $x=\sum_k \langle x,f_k\rangle f_k\in\mathscr{M}$, $y=\sum_i \langle y,h_i\rangle h_i\in\mathscr{N}$, $\|x\|=\|y\|=1$, we find

$$|\langle x,y\rangle| = |\sum_k b_k\langle x,f_k\rangle\langle h_k,y\rangle| \le \Big(\sum_k b_k^2|\langle x,f_k\rangle|^2\Big)^{1/2}\Big(\sum_k|\langle h_k,y\rangle|^2\Big)^{1/2} \le sup\ b_k<1$$

and $cos(\mathscr{M}, \mathscr{N})<1$. It results $\mathscr{M}\not\parallel\mathscr{N}$, which means $(\mathscr{M}, \mathscr{N})M^*$.

Since all F_p must be α-symmetric, we draw the conclusion that relations M and A are not elements of the family (F_p). It is not excluded that M or/and A have physical interpretation, but it is clear that they are not related with compatibility or incompatibility of tests.

The relations R which are α-symmetric have some interesting properties as it results from the following propositions.

Proposition 3.10. Let R be an α-symmetric relation. Then σ- and δ-symmetries are equivalent and each of them imply selfduality ($\alpha\delta$-symmetry).

Proof. It is sufficient to follow the simple chains of implication below.

(i) $\sigma\Rightarrow\delta$: $(a,b)R \overset{\alpha}{\Rightarrow} (a^{\perp},b)R \overset{\sigma}{\Rightarrow} (b,a^{\perp})R \overset{\alpha}{\Rightarrow} (b^{\perp},a^{\perp})R$;

(ii) $\delta\Rightarrow\sigma$: $(a,b)R \overset{\alpha}{\Rightarrow} (a^{\perp},b)R \overset{\delta}{\Rightarrow} (b,a^{\perp})R^* \overset{\alpha}{\Rightarrow} (b^{\perp},a^{\perp})R^* \overset{\varepsilon}{\Rightarrow} (b,a)R$;

(iii) $\sigma\&\alpha\Rightarrow\alpha\delta$: $(a,b)R \overset{\alpha}{\Rightarrow} (a^{\perp},b)R \overset{\sigma}{\Rightarrow} (b,a^{\perp})R \overset{\varepsilon}{\Rightarrow} (b^{\perp},a)R^* \overset{\alpha}{\Rightarrow} (b,a)R^* \overset{\sigma}{\Rightarrow} (a,b)R^*$.

Proposition 3.11. Let R be an α and σ−symmetric relation. Then the following statements are equivalent:

(i) $\mathcal{O} \subset R$;

(ii) $\perp \subset R$.

Proof.

(i) $\Rightarrow$ (ii): $(a,b)\perp \Rightarrow a \leq b^{\perp} \Rightarrow (a,b^{\perp})\mathcal{O} \Rightarrow (a,b^{\perp})R \overset{\alpha}{\Rightarrow} (a^{\perp},b^{\perp})R \overset{\sigma}{\Rightarrow} (b^{\perp},a^{\perp})R \overset{\alpha}{\Rightarrow} (b,a^{\perp})R \overset{\sigma,\alpha}{\Rightarrow} (a,b)R$.

(ii) $\Rightarrow$ (i): $(a,b)\mathcal{O} \Rightarrow a \leq b \Rightarrow a \leq (b^{\perp})^{\perp} \Rightarrow (a,b^{\perp})\perp \Rightarrow (a,b^{\perp})R \overset{\sigma}{\Rightarrow} (b^{\perp},a)R \overset{\alpha}{\Rightarrow} (b,a)R \overset{\sigma}{\Rightarrow} (a,b)R$.

4 Compatibility relation

Let $(L, \leq, \perp)$ be an ortholattice. We will try to construct the relation F_0 (compatibility) on L. We will see that, if such a relation may be constructed on L, then L is orthomodular.

We saw that F_0 must be a reflexive, symmetric, invariant under automorphisms of L and α−symmetric relation. But it is almost obvious that we can not get the form of F_0 by using only this set of properties. On the other hand we want to get the form of F_0 by using as few as possible physical assumptions.

Since F_0 is a symmetric relation, we may define the so−called F_0−classes [1]. A subset $K \subseteq L$ is said to be a F_0−class if $a,b \in K \Rightarrow (a,b)F_0$ and K is maximal with this property. Since F_0 must describe the empirical compatibility it is clear that any F_0−class contains only mutually compatible elements. In this case we may argue that, if L is considered to be a "potential" physical theory, then any F_0−class is a sublattice of L which is itself a Boolean algebra,[1]. Taking account of this fact, we can see easily that, if $(a,b)F_0$, then $b=(a \wedge b) \vee (a^{\perp} \wedge b)$, or $(a,b)C$. In other words $F_0 \subseteq C$. The relation C, considered on an ortholattice, has not the minimal set of properties required for F_0 since it is not generally α−symmetric. Nevertheless, it is plausible that C is just that relation which describes compatibility. The next considerations lead to the conclusion that this is indeed so.

First of all let us observe that, since the orthomodularity of L is equivalent with the symmetry of C (Proposition 3.3) and the symmetry of compatibility is a physically justified fact, it seems that the orthomodularity of L and the concrete form of F_0 are intimately related. This assumption is supported also by the following result: **if there exists a relation $R \subset L \times L$ which is reflexive, symmetric, its classes are Boolean algebras and it contains the order, then $R = C$** [1]. The only assumption in this statement which is not easy to accept *a priori* is that that compatibility relation must contain the order relation. This is because, at first sight, the statement $a \leq b$ may be established/verified only if we consider the tests a,b compatible. In fact, a standard reasoning using the common interpretation of the order relation, assures us that the statement $a \leq b$ may be operationally established even if a and b are incompatible (empirically). Indeed, we can imagine the following situation: for $a,b \in L$, $a \leq b$, there exists a finite family of tests $\{x_i;\ 1 \leq i \leq n\}$ such that $a=x_1 \leq x_2 \leq \ldots \leq x_n=b$ and $(x_i,\ x_{i+1})$ is a empirically compatible pair for any i, $1 \leq i \leq n-1$. Translated into the mathematical language of ortholattices, this fact gives the following condition, which is natural to be imposed to any relation R which may play the role of compatibility:

$$a \leq b \Rightarrow \exists\ (x_i),\ 1 \leq i \leq n\ (n \geq 2),\ x_1=a \leq x_2 \leq \ldots \leq x_n=b,\ (x_i,\ x_{i+1})R. \tag{4.1}$$

Taking account of these facts we may formulate and prove a theorem which clarifies—in a sense—the problem of orthomodularity of physical theories and the form of the relation F_0.

Theorem 4.2. Let L be an atomic ortholattice and $R \subset L \times L$ a reflexive, symmetric relation whose classes are Boolean algebras and having the property (4.1). Then L is orthomodular and R = C.

Proof. See [1].

The proof of Theorem 4.2 is strictly based on the atomicity of the ortholattice L. We do not know if an analogue of Theorem 4.2 without atomicity condition can be proved. But, in our opinion, this is not necessary, since atomicity is a property of physical theories with a sufficiently clear physical support [4].

5 Compatible approximation in orthomodular lattices.

In this section L will be an orthomodular lattice. For any nonempty set $S \subset L$ we will define the set

$$C(S)=\{x \in L;\ (x,s)C,\ \forall s \in S\}. \tag{5.1}$$

We will write C(a) instead of C({a}).

Lemma 5.2. For any $S \subset L$, $S \neq \emptyset$, C(S) is an orthomodular sublattice of L. If L is complete, then C(S) is also complete.

Proof of this statement is routine.

For any two arbitrarily fixed elements $a,b \in L$ we define the subsets:

$$\Delta_a(b)=\{x \in L;\ x \leq b,\ (x,a)C\}$$
$$\nabla_a(b)=\{x \in L;\ x \geq b,\ (x,a)C\}. \tag{5.3}$$

$\Delta_a(b)$ ($\nabla_a(b)$) may be called lattice of inferior (superior) compatible approximation of the element b with respect to a. The next proposition confirms that $\Delta_a(b)$, $\nabla_a(b)$ are indeed lattices.

Proposition 5.4. The sets $\Delta_a(b)$, $\nabla_a(b)$ are sublattices of L; $\ell_a(b)=(a\wedge b)\vee(a^\perp\wedge b)$ is the greatest element of $\Delta_a(b)$ and $u_a(b)=(a\vee b)\wedge(a^\perp\vee b)$ is the least element of $\nabla_a(b)$.

Proof. It is easy to see that $\vee\Delta_a(b)\in\Delta_a(b)$, so that $\Delta_a(b)$ is an interval in C(a). By using Lemma 5.2 we get that $\Delta_a(b)$ is a sublattice of L. Since any $x\in\Delta_a(b)$ is compatible with a, we may write $x=(a\wedge x)\vee(a^\perp\wedge x)$, and by using $x\leq b$ we get $x\leq\ell_a(b)$. The corresponding properties for $\nabla_a(b)$ may be obtained by duality.

Remark 5.5. The equality $\Delta_a(b)=L[0,\ell_a(b)]$ is not generally true since the implication $x\leq\ell_a(b)\Rightarrow(x,a)C$ is not true. In order to prove this, let us consider the lattice $\mathscr{L}_C(\mathscr{H})$ and the subspaces $\mathscr{M}$, $\mathscr{N}\in\mathscr{L}_C(\mathscr{H})$ such that and $\mathscr{M}\wedge\mathscr{N}\neq 0$ and $\mathscr{M}^\perp\wedge\mathscr{N}\neq 0$. If we take the vectors $x_1\in\mathscr{M}\wedge\mathscr{N}$, $x_1\neq 0$, $x_2\in\mathscr{M}^\perp\wedge\mathscr{N}$, $x_2\neq 0$, then the vector $x=x_1+x_2$ clearly has the properties $[x]\leq\ell_{\mathscr{M}}(\mathscr{N})$ and $([x],\mathscr{M})\overline{C}$.

We know that in orthomodular lattices compatibility (commutativity) implies modularity, the converse being not true. Therefore, it is interesting to note that there are many elements $y\in L[0,\ell_a(b)]$ such that (a,y)M and (y,a)M.

Proposition 5.6. If $x\in L[a\wedge b,\ell_a(b)]$, $y\in L[u_a(b),a\vee b]$, then (x,a)M, (a,x)M, $(y,a)M^*$, $(a,y)M^*$.

Proof. We know that, if (c,d)M, $c_1\in L[c\wedge d,c]$, $d_1\in L[c\wedge d,d]$, then $(c_1,d_1)M$, so that the proposition is true since $(\ell_a(b),a)M$ and $(a,\ell_a(b))M$.

It is easy to see that $a\wedge b\in\Delta_a(b)$, $a\vee b\in\nabla_a(b)$, so that it is interesting to consider the following lattices:

$$L_a(b)=\{x\in L;\ a\wedge b\leq x\leq b,\ (x,a)C\}$$

$$U_a(b)=\{x\in L;\ b\leq x\leq a\vee b,\ (x,a)C\}. \tag{5.7}$$

Obviously, $L_a(b)\subset\Delta_a(b)$, $U_a(b)\subset\nabla_a(b)$ and they contain all elements of L which are interesting for a compatible approximation of b with respect to a. Intuitively it is clear that, in order to have a satisfactory compatible approximation of b with respect to a, it is necessary to find those elements of $L_a(b)$ ($U_a(b)$) which are "sufficiently close" to b. In other words, the "distance" between b and some elements of $L_a(b)$ ($U_a(b)$) must be sufficiently "small". Of course, the notion of distance or smallness appearing in the last statement is quite vague. In presence of JD−property, or if a dimension function is defined on the lattice, this notion can be made precise. This problem will be discussed in the next paragraph.

Now we will examine some implications of the equivalences:

$$(a,b)C \iff (b=\ell_a(b)=u_a(b)) \iff b\in\Delta_a(b).$$

They permit us to describe two extreme situations concerning compatible approximation of b with respect to a.

i) If (a,b)C, then $\ell_a(b)=b$ and there are no problems concerning the compatible approximation in this case. Analogously, $\ell_b(a)=a$.

ii) The other extreme situation is $\ell_a(b)=a\wedge b$ (dually, $u_a(b)=a\vee b$). We may say that, in some sense, this is the worst compatible approximation of b with respect to a. Of course, when $a\le b$, we have simultaneously (a,b)C and $\ell_a(b)=a\wedge b$.

The case (ii) is interesting also because it may be described in terms of two relations which are, as we will see below, δ−symmetric:

$$(a,b)B \overset{def.}{\iff} \ell_a(b)=a\wedge b$$
$$(a,b)B^* \overset{def.}{\iff} u_a(b)=a\vee b \qquad (5.8)$$

We can show easily that $B\cap C\subset\mathcal{O}^*$, $B^*\cap C\subset\mathcal{O}$. B and B^* are not α−symmetric and, in general, they are not symmetric, but

Theorem 5.9. The relations B, B^* are δ−symmetric is true. This statement can be proved directly by using standard arguments, but we prefer to use the following interesting proposition.

Proposition 5.10. If L is an orthomodular lattice and a,b∈L, then:

(i) $L_a(b)$ and $L_{b^\perp}(a^\perp)$ are isomorphic;

(ii) $U_a(b)$ and $U_{b^\perp}(a^\perp)$ are isomorphic;

(iii) $L_a(b)$ and $U_b(a)$ are dual isomorphic.

Proof. Let $x\in L_a(b)$ be an arbitrarily fixed element. Since (x,a)C, we have $x=(x\wedge a)\vee(x\wedge a^\perp)$. We define the mapping ψ: $L_a(b)\to L$ by the equality

$$\psi(x)=(x\wedge a^\perp)\vee(a^\perp\wedge b^\perp).$$

It is clear that ψ is order preserving. Then, from $x\le b$ it results that $x\wedge a^\perp\le b\wedge a^\perp$, so that $\psi(x)\le\ell_{b^\perp}(a^\perp)$. Clearly, $\psi(x)\ge a^\perp\wedge b^\perp$. Since $x\wedge a^\perp\le x\le b$, we have $((x\wedge a^\perp),b)C$ and, by using Lemma 5.2, we get $(\psi(x), b^\perp)C$. Collecting all these facts we obtain $\psi(x)\in L_{b^\perp}(a^\perp)$. Similarly, the mapping ξ: $L_{b^\perp}(a^\perp)\to L$ defined by $\xi(y)=(y\wedge b)\vee(a\wedge b)$ is order preserving and $\xi(y)\in L_a(b)$. It remains to show that $\xi(\psi(x))=x$, $\psi(\xi(y))=y$ for any $x\in L_a(b)$, $y\in L_{b^\perp}(a^\perp)$. Indeed, we have $\xi(\psi(x))=\xi((x\wedge a^\perp)\vee(a^\perp\wedge b^\perp))=\{[(x\wedge a^\perp)\vee(a^\perp\wedge b^\perp)]\wedge\wedge b\}\vee(a\wedge b)=(x\wedge a^\perp)\vee(a\wedge b)\ge(x\wedge a^\perp)\vee(x\wedge a)=x$, since $(a^\perp\wedge b^\perp,b)M$, $x\wedge a^\perp\le b$ and $x\le b$. On the other hand, $x\in L_a(b)$, $x\ge x\wedge a^\perp$, $x\ge a\wedge b$ and we get $x\ge(x\wedge a^\perp)\vee(a\wedge b)=\xi(\psi(x))$. The equality $\psi(\xi(y))=y$ results in a similar fashion.

(ii) Put $\tilde{\psi}(x)=(x\vee a^\perp)\wedge(b^\perp\vee a^\perp)$ for any $x\in U_a(b)$ and $\tilde{\xi}(y)=(y\vee b)\wedge(a\vee b)$ for $y\in U_{b^\perp}(a^\perp)$

and make a proof which is similar to that in point (i).

(iii) If $y \in L_{b^\perp}(a^\perp)$, we have $a^\perp \wedge b^\perp \leq y \leq a^\perp$ and $(y,b^\perp)C$. Therefore, $a \leq y^\perp \leq a \vee b$ and $(y^\perp,b)C$. In other words, the restriction of the orthocomplementation to $L_{b^\perp}(a^\perp)$ is a dual isomorphism between $L_{b^\perp}(a^\perp)$ and $U_b(a)$. It results that the mapping

$$L_a(b) \ni x \to \tau(x) = \psi(x)^\perp \in U_b(a)$$

is the dual isomorphism we are looking for.

Proof of Theorem 5.9. It is sufficient to consider the equivalence $(a,b)B \iff L_a(b) = \{a \wedge b\}$. It results that $U_b(a)$ is also an one–element set, this being equivalent with $(b,a)B^*$ and so on.

We will give now a series of propositions which reflect different connections between elements of compatible approximation.

Proposition 5.11. $(\ell_a(b),u_a(b))C$.

Proof. $\ell_a(b) \leq b \leq a \vee b$, so that $(\ell_a(b),a \vee b)C$, $(\ell_a(b),b^\perp)C$. Since $(\ell_a(b),a)C$, we get first $(\ell_a(b),a \vee b^\perp)C$ and finally $(\ell_a(b),(a \vee b) \wedge (a \vee b^\perp))C$.

Similarly it can be proved that $(\ell_a(b),\ell_b(a))C$, $(u_b(b),u_b(a))C$.

Proposition 5.12. $\ell_a(b) \wedge u_b(a) = a \wedge b$

$$\ell_a(b) \vee u_b(a) = a \vee b.$$

Proof. By simple computation, we can prove for instance

$$\ell_a(b) \vee u_b(a) = [(a \wedge b) \vee (a^\perp \wedge b)] \vee [(a \vee b) \wedge (a \vee b^\perp)] =$$
$$= (a \wedge b) \vee \{(a^\perp \wedge b) \vee [(a \vee b^\perp) \wedge (a \vee b)]\} = (a \wedge b) \vee (a \vee b) = a \vee b.$$

Proposition 5.13. Let L be an orthomodular lattice and $a,b \in L$. Then the following statements are true:

(i) **$L[u_b(a),a \vee b]$ and $L[a \wedge b,\ell_a(b)]$ are isomorphic;**

(ii) **$L[\ell_a(b),a \vee b]$ and $L[a \wedge b,u_b(a)]$ are isomorphic;**

(iii) **if $\theta_2, \theta_2(x) = x \wedge u_b(a)$ is the isomorphism between the segments in the point (ii) and θ_1, $\theta_1(y) = y \vee \ell_a(b)$ its inverse, then we have also $\theta_2(b) = \varphi_b(a)$, $\theta_1(a) = \tilde{\varphi}_a(b)$, where φ, $\tilde{\varphi}$ are Sasaki's projections.**

Proof. The statements (i) and (ii) result immediately from (2.6), (2.6′), (5.11) and (5.12). For proving (iii) it is sufficient to observe that

$$\theta_2(b) = b \wedge [(a \vee b) \wedge (a \vee b^\perp)] = (a \vee b^\perp) \wedge b = \varphi_b(a).$$

Now we want to give a characterization of the covering law in terms of elements like $\ell_a(b)$. In order to do this, let us remember that any interval L[a,b] with the relative orthocomplementation $L[a,b]\ni x \to x^{\#}=(x^{\perp}\vee a)\wedge b\in L[a,b]$ is an orthomodular lattice. We write $x^{\perp}=b-x$ and, occasionally, $x+y$ when $x\perp y$.

Proposition 5.14. Let L be an orthomodular lattice and $x<y$ two elements of L. Then the intervals $L[0,y-x]$ and $L[x,y]$ are orthoisomorphic.

Proof. Take $z = y-x$ and put $a=x$, $b=z$. We have $a\wedge b=0$, $a\vee b=y$, $(a,b)\perp$, therefore $(a,b)C$. It results that $(a,b)M$, $(b,a)M^*$, so that we can apply (2.5). Consider now the mapping $L[0,z]\ni t \to \theta_1(t)=t\vee x\in L[x,y]$ and let us prove that $\theta_1(t^{\#})=[\theta_1(t)]^{\#}$, where $\#$ and $\#$ stand for relative orthocomplementations in the segments above. But $t^{\#}=z-t$ and we may write:

$$\theta_1(t^{\#})=x\vee(z-t)=x\vee(z\wedge t^{\perp})=x\vee[(y\wedge x^{\perp})\wedge t^{\perp}]=t^{\perp}\wedge[x\vee(y\wedge x^{\perp})]=t^{\perp}\wedge y=(t\vee x)^{\#}.$$

Proposition 5.15. If L is an orthomodular lattice, then $a\lessdot b$ iff $b-a\in\Omega(L)$.

Proof. If $a\lessdot b$, then obviously $b-a\in\Omega(L)$. Conversely, if $b-a=p\in\Omega(L)$, then it is sufficient to apply Proposition 5.14 to the intervals $L[0,p]$ and $L[a,b]$.

Theorem 5.16. If L is an orthomodular atomic lattice, then the following statements are equivalent:

- **(i) L has the covering property;**
- **(ii) L^* has the covering property;**
- **(iii) $a\in L$, $p\in\Omega(L)$, $(a,p)\overline{C} \Rightarrow a\gtrdot \ell_p(a)$;**
- **(iv) $a\in L$, $p\in\Omega(L)$, $p\nleq a^{\perp} \Rightarrow \varphi_a(p)=(p\vee a)\wedge a\in\Omega(L)$.**

Proof. (i) $\Rightarrow$ (ii) since L, L^* are isomorphic. For proving (ii) $\Rightarrow$ (iii) we remark that (iii) is the dual covering law in the particular case $p\nleq a^{\perp}$. Indeed, since $p\wedge a=0$, we have $\ell_p(a)=p^{\perp}\wedge a$. Then $p\nleq a^{\perp} \iff p^{\perp}\leq_* a$, where "$\leq_*$" denotes the order in L^* and $p^{\perp}\in\Omega(L^*)$. The covering law in L^* for the pair $(p^{\perp},a)$ is $a\lessdot_* p^{\perp}\vee_* a$, hence $a\gtrdot p^{\perp}\wedge a=\ell_p(a)$.

The statement (iv) is obviously satisfied when $(p,a)C$. If $(p,a)\overline{C}$, we use Proposition 5.13 for obtaining the isomorphism between intervals $L[\ell_p(a),a\vee p]$ and $L[0=a\wedge p,u_a(p)]$. Hence, from $a\gtrdot \ell_p(a)$ we get $\theta_2(a)\gtrdot\theta_2(\ell_p(a))=0$. Therefore $\theta_2(a)\in\Omega(L)$ and $\theta_2(a)=\varphi_a(p)$. The implication (iv) $\Rightarrow$ (i) is a known result.

The results obtained in this paragraph have an obvious technical character. Nevertheless, we consider that they illustrate quite convincingly the close connection we expect to exist between compatibility and the geometrical structure of physical theories.

We want to end this paragraph with some considerations concerning relations B and A. In this short comment we prefer to denote $(a,b)B$ by $a\wedge b\lessdot b$. From the definition of the relation B we know that between $a\wedge b$ and b there are no elements compatible with a. On the other hand, as it has been proved in Theorem 5.9, B is a δ−symmetric relation, i.e.

$$a \wedge b \lessdot b \Longleftrightarrow a \lessdot a \vee b. \tag{5.17}$$

If we replace the sign "$\lessdot$" by "$<$", then we get

$$a \wedge b < b \Longleftrightarrow a < a \vee b, \tag{2.5}$$

which is, in orthomodular lattices, equivalent with the covering law and expresses the δ−symmetry of the relation A. It is clear that (2.5) is a much stronger condition than (5.17). A question arises, if we might use the formal similarity between (5.17) and (2.5) for postulating that (2.5) is true in any orthomodular lattice which may be a physical theory. Unfortunately, the relations B and A are not α−symmetric and have only an indirect connection with the compatibility/incompatibility relation, so that they cannot be used in our tentative construction of the family $\{F_p\}$.

In the next paragraph we will find a solution for constructing F_1 and proving in what conditions the covering law appears as a necessary property of physical theories.

6 Quasicompatibility relation

Let L be an orthomodular atomic lattice. We want to construct the relation F_1, which we prefer – for some reasons which will become clear below – to denote by Q. In fact we will propose a form of the relation Q suggested by the form of C and certain other considerations.

We know that $(a,b)C \Longleftrightarrow b=(a \wedge b) \vee (a^{\perp} \wedge b)$. It follows that $(a,b)\overline{C} \Longleftrightarrow b > (a \wedge b) \vee (a^{\perp} \wedge b) = \ell_a(b)$. Therefore, it is quite natural to think that the element $b-\ell_a(b)$ is strictly related to the measure of incompatibility of the pair (a,b). It is also natural to suppose that the "smallest degree of incompatibility" corresponds to the situation $b-\ell_a(b) \in \Omega(L)$ since the lattice L does not admit nonzero elements smaller than atoms. Taking account of these facts we will define the relation Q as follows:

$$(a,b)Q \Longleftrightarrow b \gtrdot \ell_a(b) \Longleftrightarrow b-\ell_a(b) \in \Omega(L). \tag{6.1}$$

The dual of Q is obviously the relation

$$(a,b)\, Q^* \Longleftrightarrow b \lessdot u_a(b) \Longleftrightarrow u_a(b)-b \in \Omega(L). \tag{6.1'}$$

We can verify easily that Q is invariant under automorphisms of L and is an α−symmetric relation. Consequently, it is plausible that the choice of Q for describing the "incompatibility of degree one" is an appropriate option. Q will be named often "quasicompatibility relation".

Once Q admitted as describing the smallest degree of incompatibility, we have to assume also that it is a symmetric relation. We will see soon that this assumption is crucial for our purpose: it is equivalent to the fact that L has the covering property. This very important result will be proved later. Now we intend to prove some interesting properties of the relations Q and Q^*.

Proposition 6.2. Let L be an orthomodular atomic lattice. Then the following statements are true:

(i) $(a,b)Q \iff \varphi_b(a) \gtrdot a\wedge b$; in this case $b-\ell_a(b)=\varphi_b(a)-(a\wedge b)$;

(ii) $(a,b)\,Q^* \iff \tilde{\varphi}_b(a) \lessdot a\vee b$; in this case $u_a(b)-b=(a\vee b)-\tilde{\varphi}_b(a)$;

(iii) if $(a,b)Q$ and $p=b-\ell_a(b)\in\Omega(L)$, then $a\vee b=u_a(b)$.

Proof. (i): By Proposition 5.13 we know that the intervals $L[\ell_a(b),a\vee b]$ and $L[a\wedge b,u_b(a)]$ are isomorphic via $x \to \theta_2(x)=x\wedge u_b(a)$. Hence, if $(a,b)Q$, then $b\gtrdot \ell_a(b)$ and $\theta_2(b)\gtrdot \theta_2(\ell_a(b))$. It remains to observe that $\theta_2(b)=\varphi_b(a)$ and $\theta_2(\ell_a(b))=a\wedge b$. The elements $p_1=b-\ell_a(b)$ and $p_2=\varphi_b(a)-(a\wedge b)$ are atoms. We have $p_2\perp(a\wedge b)$, $p_2\le\varphi_b(a)=(a\vee b^\perp)\wedge b$, so that $p_2\le b$, $p_2\le a\vee b^\perp$ or $p_2\perp(a^\perp\wedge b)$. It follows $p_2\le p_1$, i.e. $p_1=p_2$.

The proof of (ii) may be obtained by duality.

(iii): We have $p\not\le a$, $p\not\le a^\perp$ since otherwise $(p,a)C$ and, taking into account $(\ell_a(b),a)C$, we would obtain $(b,a)C$, which contradicts $(a,b)Q$. Furthermore, $p\perp\ell_a(b)$ implies $p\le a\vee b^\perp$ and since $p\le a\vee b$, we get $p\le(a\vee b)\wedge(a\vee b^\perp)$. Then, $a\le u_b(a)$ implies $a\vee p\le u_b(a)$. It remains to prove that $u_b(a)-(a\vee p)=0$ or, equivalently $(a\vee b)\wedge(a\vee b^\perp)\wedge(a\vee p)^\perp=0$, (+).

By using the equality $b=p+\ell_a(b)=(a\wedge b)\vee(a^\perp\wedge b)\vee p$, we get $(a\vee p)^\perp\wedge(a\vee b)= (a\vee p)^\perp\wedge[a\vee(a\wedge b)\vee(a^\perp\wedge b)\vee p]=(a\vee p)^\perp\wedge[(a\vee p)\vee(a^\perp\wedge b)]$. We have $a^\perp\wedge b\le a^\perp$, $a^\perp\wedge b\le p^\perp$, so that $a^\perp\wedge b\le a^\perp\wedge p^\perp=(a\vee p)^\perp$. Since L is orthomodular, we have $((a\vee p),(a\vee p)^\perp)M$ and, by using the modular identity, we get

$$(a\vee p)^\perp\wedge[(a\vee p)\vee(a^\perp\wedge b)]=a^\perp\wedge b,$$

which proves (+).

Proposition 6.3. If L is an orthomodular atomic lattice having the covering property, then $(a,b)Q \Rightarrow (a,b)Q^*$.

Proof. It is sufficient to observe (see Proposition 6.2 (iii)) that $u_b(a)=a\vee p\gtrdot a$, since $p\not\le a$.

Now we can prove the main result of this section which – according to the accepted by us physical interpretation of Q/F_1 – is a justification of the assumption that a physical theory must satisfy the covering law.

Theorem 6.4. If L is an orthomodular lattice, then it satisfies the covering property if and only if $Q\subset L\times L$ is symmetric.

Proof. If L has the covering property, then by combining Proposition 6.3 with

Proposition 3.10 we obtain that Q is symmetric. Conversely, suppose that Q is symmetric and consider $p \in \Omega(L)$, $a \in L$, $p \nleq a$. If $p \leq a^{\perp}$, then we get immediately $a \lessdot a \vee p$. If $p \nleq a^{\perp}$, then $\ell_a(p)=(a \wedge p) \vee (a^{\perp} \wedge p)=0 \lessdot p$, or (a,p)Q. Since Q is symmetric, we have also (p,a)Q or $a \gtrdot \ell_p(a)=(p \wedge a) \vee (p^{\perp} \wedge a)=p^{\perp} \wedge a$. According to Proposition 5.16 (iii) this statement is equivalent with the covering law.

Remark 6.5. It is interesting to translate Theorem 6.4 in the Hilbert−space language.

Theorem 6.4′. Let $\mathcal{M}$, $\mathcal{N} \in \mathcal{L}_C(\mathcal{H})$. Suppose that $\mathcal{M}$ has an orthonormal basis such that all its vectors, except one, are elements of $\mathcal{N}$ or $\mathcal{N}^{\perp}$. Then $\mathcal{N}$ has the same property: it admits an orthonormal basis such that all its vectors, except one, are elements of $\mathcal{M}$ or $\mathcal{M}^{\perp}$.

It would be interesting to see how looks a "purely Hilbertian" proof of this result. This case serves as a good illustration of the fact that there are geometrical problems which may be easier solved by using lattice−theoretical methods instead of Hilbertian techniques.

Finally we want to remark a striking similarity between the properties of the relation C in orthomodular lattices and the corresponding ones of the relation Q in atomic orthomodular lattices having the covering property. The properties of the relation C which we have in view are listed below:

(C1) C is α−symmetric;
(C2) C is symmetric (δ−symmetric) and its symmetry is equivalent with orthomodularity of L;
(C3) C is selfdual and its selfduality is equivalent with orthomodularity;
(C4) $C \subset M, M^*$, these inclusions being also equivalent with orthomodularity;
(C5) $(a,b)C \iff \varphi_b(a)=a \wedge b$; $(a,b)C^* \iff \tilde{\varphi}_b(a)=a \vee b$.

The corresponding to (Ck) property of Q will be denoted by (Qk).

(Q1) Q is α−symmetric;
(Q2) Q is symmetric (δ−symmetric) and its symmetry is equivalent with the covering property;
(Q3) ?
(Q4) $Q \subset M, M^*$;
(Q5) $(a,b)Q \iff \varphi_b(a) \gtrdot a \wedge b$; $(a,b)Q^* \iff \tilde{\varphi}_b(a) \lessdot a \vee b$.

We do not know if the selfduality of Q implies the semimodularity of L since we could not find a proof or a counterexample. For proving (Q4) we may use the following known result: if L is an atomic orthomodular satisfying the covering law lattice and $a,b \in L$ such that $b=b_1 \vee b_2$, $b_1 \perp b_2$, $(b_1,a)C$, then $(a,b_2)M$ and $(a,b_2)M^* \Rightarrow (a,b)M$, and $(a,b)M^*$. Now, if we put $b_1=\ell_a(b)$, $b_2=b-\ell_a(b)$ and assume (a,b)Q, then $b_2 \in \Omega(L)$. Since any atom is a modular element of L, we have $(a,b_2)M,M^*$ and $(a,b_1)C$. By using the above mentioned result, we get easily $Q \subset M,M^*$. It must be noted that $Q \subset M$

does not imply the symmetry of Q.

7 Finite compatibility

Collecting the main results of the preceding paragraphs we may formulate the following statement: **if L is an atomic ortholattice such that a compatibility and a quasicompatibility may be defined on it, then L is an orthomodular atomic lattice having the covering property**. In this section we will prove that, given L such a lattice, a family $\{F_p;\ p \geq 2\}$ of symmetric relations can be constructed which describes completely incompatibilities of degrees larger than 1 (see Introduction).

Since L satisfies the covering law, a dimension function exists on it, defined by $J(L) \ni a \to d(a)$ (see 2.4°).

The relations F_n are defined as follows:

$$(a,b)F_n \iff b - \ell_a(b) \in J(L) \text{ and } d(b - \ell_a(b)) = n, \qquad (7.1)$$

or

$$(a,b)F_n \iff u_b(a) - b \in J(L) \text{ and } d(u_b(a) - b) = n \qquad (7.1')$$

Since the relation Q is symmetric, the JD−axiom is satisfied and we can say that $(a,b)F_n$ if and only if there exists a connected chain $\ell_a(b) = x_0 \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_n = b$, so that it becomes clear that the relations $F_0 = C$ and $F_1 = Q$ are also defined by (7.1). It is also obvious that all F_n are α−symmetric.

We will prove now that the orthomodularity together with the semimodularity of L ensure the symmetry of the relations F_n and the inclusions $F_n \subset M, M^*$.

Theorem 7.2. If L is an atomic orthomodular lattice having the covering property, then the relations F_n are symmetric.

Proof. It is sufficient to prove the dual symmetry of the relations F_n: $(a,b)F_n \Rightarrow$ $\Rightarrow (b,a)F_n^{*}$. Let us take $a,b \in L$, $(a,b)F_n$ and $(p_1,\ldots,p_n)$ an orthogonal decomposition in atoms of the element $b - \ell_a(b)$. We have obviously $(p_i,p_j)\perp$ for $i \neq j$, $(p_i, \ell_a(b))\perp$, $p_i \leq b$ for all i, j, $1 \leq i,j \leq n$. Therefore,

$$b = (a \wedge b) \vee (a^{\perp} \wedge b) \vee p_1 \vee p_2 \vee \ldots \vee p_n.$$

Since $a \not\mathcal{C} p_i$ (otherwise p_i would be compatible with a), we can not expect that $(a,p_1,\ldots,p_n)$ is an orthogonal decomposition of $u_b(a)$. Nevertheless, the following statements are true:

$$a \lessdot a \vee p_1 \lessdot a \vee p_1 \vee p_2 \lessdot \ldots \lessdot a \vee p_1 \vee \ldots \vee p_n, \qquad (A)$$

$$u_b(a) = a \vee p_1 \vee p_2 \vee \ldots \vee p_n. \qquad (B)$$

If (A) and (B) are true, then it results that between a and $u_b(a)$ there exists a connected chain of length n+1, which represents in fact the statement $d(u_b(a) - a) = n$, or $(b,a)F_n^{*}$.

Let us prove (A). We have $p_1 \nleq a$, since otherwise $(p_1,a)C$, $(p_1 \vee \ell_a(b),a)C$, $p_1 \vee \ell_a(b) \leq b$, $p_1 \vee \ell_a(b) \gneqq \ell_a(b)$, contradiction. By applying the covering law we get $a \lessdot p_1 \vee a$. If we might show that $p_2 \nleq p_1 \vee a$, then we would have similarly $p_1 \vee a \lessdot p_1 \vee p_2 \vee a$ and so on. Suppose that $p_2 \leq p_1 \vee a$. Since L is finite modular (all its finite elements are modular) and $(a,p_1)M^*$, there exists $r \in \Omega(L)$, $r \leq a$, $p_2 \leq r \vee p_1$. Obviously, $p_2 \neq r$ ($(p_2,a)\overline{C}$ and $(p_1,p_2)\perp$). By using the exchange law we get $r \leq p_1 \vee p_2$. Therefore, $r \perp \ell_a(b)$, $r \leq b$, $(r,a)C$ and we obtain a contradiction ($b \geq r \vee \ell_a(b) > \ell_a(b)$ and $(r \vee \ell_a(b),a)C$), which proves (A). In order to prove (B), let us observe that $p_i \leq b \leq a \vee b$, $p_i \perp (a^\perp \wedge b)$, i.e. $p_i \leq a \vee b^\perp$. It results that $p_i \leq (a \vee b) \wedge (a \vee b^\perp) = u_b(a)$. Hence, $a \vee p_1 \vee ... \vee p_n \leq (a \vee b) \wedge (a \vee b^\perp) = u_b(a)$. It remains to show that $u_b(a) - (a \vee p_1 \vee ... \vee p_n) = 0$ or, equivalently, $(a \vee b) \wedge (a \vee p_1 \vee ... \vee p_n)^\perp = a^\perp \wedge b = (a \vee b^\perp)^\perp$. By considering the equality $b = \ell_a(b) \vee p_1 \vee ... \vee p_n$, we get $(a \vee b) \wedge (a \vee p_1 \vee ... \vee p_n)^\perp = [(a^\perp \wedge b) \vee (a \vee p_1 \vee ... \vee p_n)] \wedge (a \vee p_1 \vee ... \vee p_n)^\perp = a^\perp \wedge b$, and the theorem is completely proved.

Proposition 7.3. If L is an orthomodular atomic lattice satisfying the covering law, then $F_n \subset M, M^*$.

Proof. Let us take $a,b \in L$, $(a,b)F_n$. The element $b_2 = b - \ell_a(b)$ is finite and, since L is finite modular, b_2 is modular. It results $(a,b_2)M,M^*$ and $(a,b)M,M^*$.

The family $\{F_p;\ p \geq 0,1,2,...,n,...\}$ being constructed, we may define the relation $F_\infty = (L \times L) \setminus \cup F_n$, which is clearly symmetric, invariant under automorphisms of L and α-symmetric. It remains to observe that $\{F_p;\ p \in \mathbb{N} \cup \{\infty\}\}$ is a family of relations which describes almost completely the incompatibility of tests of the theory L.

8 Conclusion

A physical theory is a triple (L, F_0, F_1), where:

(i) L is an atomic ortholattice;
(ii) $F_0 \subset L \times L$ is a relation describing compatibility of tests;
(iii) $F_1 \subset L \times L$ is a relation describing quasicompatibility – or incompatibility of degree 1 – of tests.

Both F_0 and F_1 have physical interpretations resulting from a careful analysis of empirical compatibility and of its mathematical representation as a relation on L. Such a theory is automatically orthomodular and semimodular. The set $\{F_0, F_1\}$ may be enlarged up to a set $\{F_p;\ F_p \subset L \times L,\ p \in \mathbb{N} \cup \{\infty\}\}$ describing all possible degrees of incompatibility of tests.

References

[1] Al. Ivanov, *Helv. Phys. Acta*, **64**, 97(1991).
[2] C. Piron, *Foundations of Quantum Physics* (W.A.Benjamin, Inc., 1976); Dirk Aerts, *The one and the many*, Ph.D.Thesis, (Vrije Universitait Brussel, 1980 - 1981).

[3] Al. Ivanov, *Helv. Phys. Acta*, **67**, 117(1994).
[4] Al. Ivanov, *Helv. Phys. Acta*, **65**, 641(1992).
[5] F. Maeda, S. Maeda, *Theory of Symmetric Lattices* (Springer Verlag, Berlin Heidelberg New York, 1970).
[6] M. J. Maczinski, *Int. J. Theor. Phys.*, **11**, 149(1974).
[7] A. R. Marlow in *Mathematical Foundations of Quantum Theory*, (A. R. Marlow ed., Academic Press, New York, 1978), pp. 59-70.
[8] D. Foulis, C. Piron and C. Randall, *Found. Phys.*, **13**, 813(1983).
[9] J. M. Jauch and C. Piron, *Helv. Phys. Acta*, **42**, 842(1969).
[10] J. C. T. Pool, *Comm. Math. Phys.*, **9**, 212(1968).

Appendix

In his Ph. D. Thesis Aerts defines "lattices of properties of an entity consisting of two separated entities" and proves that they are not orthomodular and do not satisfy the covering law, [2]. We will show here that this fact does not contradict our conclusion concerning properties which a physical theory must have.

Indeed, in the Aerts' reasoning a particular class of systems - those constituted from separated entities - is analyzed and the lattices of properties for such systems are constricted account taking of certain specific physical assumptions. Naturally, the lattices obtained in such a way have not necessarily the properties we found that a physical theory must have. The main result of our paper is the justification of the fact that **a physical theory, which is at least an ortholattice, is orthomodular and semimodular if a measure of incompatibility is defined on it**. So, it is clear that the Aerts' and our results do not contradict each other. We can say even more: if an ortholattice L describes some physical systems and if it is accepted that any physical theory considers the compatibility in the sense of this work, then it must be also accepted the existence of a physical theory T and of an injective mapping φ: L → T which preserves the order and the orthocomplementation (see, for instance, [3]).