Categories of representations of physical systems

Autor(en): **Moore, D.J.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **68 (1995)**

Heft 7-8

PDF erstellt am: **25.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-116759

Helv Phys Acta
Vol. 68 (1995)

0018-0238/95/080658-21$1.50+0.20/0

Categories of Representations of Physical Systems

D. J. Moore

Département de Physique Théorique, 24 quai Ernest-Ansermet, CH-1211 Genève 4

(19.II.1996)

Abstract: I present a review of the mathematical structures used to represent the states and properties of physical systems in the Geneva School approach to the foundations of physics using the language of category theory. After proving the equivalence of the categories of state spaces and property lattices I reformulate the classical decomposition of the property lattice of a physical system as a universal category-theoretical construction and summarise the notions of hemimorphism and adjoint.

1 Introduction

A guiding principle of the Geneva School approach to physics, developed over the last thirty or so years at Geneva, Brussels and Amherst among others, is the conviction that a general framework for the development of specific model theories should be based on reflection upon physically primitive notions. In this work, which is largely expository in nature, I shall formulate a synthesis of the resulting mathematical structures within the language of category theory. Quite apart from the resulting compacity of expression, such an approach enables proofs of the universality, and hence mathematical naturality, of many standard constructions such as the decomposition of a system with respect to its classical variables, an application treated in section 6. A further example, treated briefly in section 7, is the abstract definition of the adjoint of a hemimorphism, introduced by D. J. Foulis [1960] and developed, for example, in [Gudder and Michel 1981; Piron 1995; Pool 1968a,b; Rüttimann 1975]. A final example, which provides the inspiration for this work, is the construction of categories of projective geometries by Cl.-A. Faure and A. Frölicher [1993, 1995b], which has provided an elegant construction of vector spaces from projective geometries and a fundamental theorem proving the representability of general morphisms by semilinear maps [Faure and Frölicher 1994], as well as a general proof of the representability of dualities by sesquilinear forms [Faure and Frölicher 1995a].

I shall not present a general overview of the physical construction of the state space and property lattice of a physical system [Piron 1964, 1976, 1990; Aerts 1982, 1994], although I shall present the basic definitions. Nor shall I comment on the relation of the Geneva School approach to others: for an identification of effects with particular definite experimental projects see Ludwig and Neumann [1981], and for an analysis of a formal scheme motivated by the Geneva School in terms of selection structures see Cattaneo and Nisticò [1993]; for a reformulation of the Geneva School axioms in the language of quasimanuals see Foulis, Piron and Randall [1983] and developments in Randall and Foulis [1983]. Finally, for some discussion of criticisms see Cattaneo and Nisticò [1991] and Foulis and Randall [1984].

The primitive physical notions in the Geneva School approach are 'definite experimental project' and 'particular physical system'. A definite experimental project relative to a physical system is a real experimental procedure where we have defined in advance what would be the positive response should we perform the experiment. These conditions define the response 'yes' — if we perform the experiment and if the conditions are not satisfied then we assign the response 'no'. A given definite experimental project is called certain for a particular realisation of the physical system if it is sure that the positive response would obtain should we perform the experiment.

I shall not enter into a detailed discussion of the notion of certainty, however a few remarks are perhaps in order. First, the certainty of a given definite experimental project is falsifiable since the experimenter always has the right to perform the experiment if the assertion is challenged. Further, 'certain' in no way means 'necessary'; at the very least one would have to assume that no uncontrolled external agent could act upon the system. Finally if a definite experimental project is certain for a particular realisation of the physical system it is so before we perform the corresponding experiment or even if we decide not to perform it.

The collections of definite experimental projects and particular physical systems are provided with natural physical structure which can be encoded mathematically using physically motivated axioms. Let α and β be definite experimental projects. We write $\alpha \prec \beta$ if β is certain in each case that α is certain; in this case we call α stronger than β. The relation $\prec$ can then be demonstrated to be a preorder; the associated equivalence classes are by definition the properties, or potential elements of reality, of the system. The set of properties can then be constructively demonstrated to be a complete lattice.

On the other hand, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be possible realisations of the system. We call $\mathcal{E}_1$ and $\mathcal{E}_2$ orthogonal, written $\mathcal{E}_1 \perp \mathcal{E}_2$, if there exists a definite experimental project α such that α is certain for the particular system $\mathcal{E}_1$ and impossible for the particular system $\mathcal{E}_2$. The orthogonality relation can then be demonstrated to be symmetric and antireflexive.

One can identify each property of the system with the collection of all particular realisations for which it is actual; dually one can identify the state of a particular system with the collection of all of its actual properties. This observation provides the starting point for the development presented here, where I define categories of state spaces and property lattices and extend the physical duality between the two descriptions to a category-theoretical equivalence. For completeness, in the following I have given proofs of the standard results concerning lattices and orthogonality spaces used in the paper. Many of these can be found in Birkhoff [1973] along with historical references; most of the others are reasonably trivial extensions.

2 Category Theory

Category theory provides a compact method of encoding mathematical structures in a uniform way, thereby enabling the use of general theorems on, for example, equivalence and universal constructions. There has been much debate on the relative foundational status of category theory as opposed to set theory; as remarked by J.-P. Marquis [1995] there are four main views that can be held, namely:

(1) categories are structured sets;

(2) sets are unstructured categories;

(3) the two theories are irreducibly complementary in the same way as arithmetic and geometry;

(4) both theories will eventually be superceded as notation systems.

I do not wish to enter into this debate here; for our purposes mathematics will be used to model structures based upon a reflection upon the nature of physical objects and so it will be heuristically convenient to couch my discussion in the language of some underlying set theory: as stated by H. Wang [1974 p.25] "We do feel there is a distinction between the ways in which mathematical and physical propositions are established. One also has the feeling that while objects are basic in physics, relations and structures are more basic than objects in mathematics." In this formulation, a category is a quadruple $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ consisting of:

(C1) a class Ob of objects;

(C2) for each ordered pair (A, B) of objects a set $\mathrm{Hom}(A, B)$ of morphisms;

(C3) for each object A a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$;

(C4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$ a morphism $g \circ f \in \mathrm{Hom}(A, C)$;

which is such that:

(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$;

(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ for all $f \in \mathrm{Hom}(A, B)$;

(M3) the sets $\mathrm{Hom}(A, B)$ are pairwise disjoint.

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B, however it can always be satisfied by replacing $\mathrm{Hom}(A, B)$ by the set $\mathrm{Hom}(A, B) \times (\{A\}, \{B\})$. In the following I shall state the basic definitions and results that will be needed in the following: for more details see, for example, [Adámek, Herrlich and Strecker 1990; Borceux 1994; Mac Lane 1971].

A motivating example in category theory is the category $\underline{\mathrm{Set}}$. Here the objects are sets, the morphisms are maps and we take the usual identity map and map composition. More formally, within ZF set theory we define $\mathrm{Ob} = \{x \mid x = x\}$, and for $A, B \in \mathrm{Ob}$ we define $\mathrm{Hom}(A,B) = \{z \subset (A \times B) \mid (\forall x \in A)(\exists! y \in B)\big((x,y) \in z\big)\}$. If $f \in \mathrm{Hom}(A,B)$ I shall write $f : A \to B : a \mapsto f(a)$, where $f(a)$ is the (by definition) unique element of B such that $\big(a, f(a)\big) \in f$. For $A \in \mathrm{Ob}$ we define $\mathrm{id}_A : A \to A : a \mapsto a$ and for $f \in \mathrm{Hom}(A,B)$, $g \in \mathrm{Hom}(B,C)$ we define $g \circ f : A \to C : a \mapsto g\big(f(a)\big)$.

In the following I shall need two standard constructions used to build new categories from old. First, let $\underline{X}$ be a category and J be a set. We can then form the category $\underline{X}^J$ whose objects A are families $\{A_j | j \in J\}$ of objects in $\underline{X}$ and whose morphisms $f \in \mathrm{Hom}(A,B)$ are families of morphisms $f_j \in \mathrm{Hom}(A_j, B_j)$, with $\big(\mathrm{id}_A\big)_j = \mathrm{id}_{A_j}$ and $(f \circ g)_j = f_j \circ g_j$. Second, let $\underline{X}$ be a category. We can then form the category $\underline{X}^{\mathrm{op}}$, which has the same objects as $\underline{X}$, where the set $\mathrm{Hom}^{\mathrm{op}}(A,B)$ is defined to be $\mathrm{Hom}(B,A)$ and $f * g = g \circ f$.

There are several special types of morphism of particular interest. These generalise important notions for maps such as injectivity, surjectivity and bijectivity. A morphism $f \in \mathrm{Hom}(A,B)$ is called a monomorphism if for all morphisms $g, h \in \mathrm{Hom}(C,A)$ we have that $f \circ g = f \circ h$ implies $g = h$; and a section if there exists a morphism $g \in \mathrm{Hom}(B,A)$ such that $g \circ f = \mathrm{id}_A$. We note that any section is a monomorphism. Indeed, let f be a section with $f^* \circ f = \mathrm{id}$. Then if $f \circ g = f \circ h$ we have that $g = \mathrm{id} \circ g = f^* \circ f \circ g = f^* \circ f \circ h = \mathrm{id} \circ h = h$. The converse does not hold in general, although it does in $\underline{\mathrm{Set}}$, where 'section' and 'monomorphism' both coincide with the notion of injection.

On the other hand, a morphism $f \in \mathrm{Hom}(A,B)$ is called an epimorphism if for all morphisms $g, h \in \mathrm{Hom}(B,C)$ we have that $g \circ f = h \circ f$ implies $g = h$; and a retraction if there exists a morphism $g \in \mathrm{Hom}(B,A)$ such that $f \circ g = \mathrm{id}_B$. Section and retraction are dual notions, as are monomorphism and epimorphism. By this we mean that if $f \in \mathrm{Hom}(A,B)$ is a section in $\underline{X}$, then it is a retraction when considered as an element of $\mathrm{Hom}^{\mathrm{op}}(B,A)$ in $\underline{X}^{\mathrm{op}}$. Hence by duality any retraction is an epimorphism. Indeed, let $f \in \mathrm{Hom}(A,B)$ be a retraction. Then $f \in \mathrm{Hom}^{\mathrm{op}}(B,A)$ is a section and so a monomorphism. But this implies that $f \in \mathrm{Hom}(A,B)$ is an epimorphism. Again, the converse does not hold in general, although it does in $\underline{\mathrm{Set}}$, where 'retraction' and 'epimorphism' both coincide with the notion of surjection.

Finally, a morphism that possesses an inverse (is both a section and a retraction) is called an isomorphism. If there exists an isomorphism $f \in \mathrm{Hom}(A,B)$ then the objects A and B are called isomorphic. The inverse of an isomorphism is unique. Indeed, let $f \in \mathrm{Hom}(A,B)$ be an isomorphism and $f^*, f' \in \mathrm{Hom}(B,A)$ be inverses of f. Then $f^* = \mathrm{id}_A \circ f^* = f' \circ f \circ f^* = f' \circ \mathrm{id}_B = f'$. Note that the classes of sections, monomorphisms, retractions, epimorphisms and isomorphisms are all closed under composition.

Much of the utility of category theory lies in the fact that one can relate different categories using the notion of functor. Let $\underline{X}$ and $\underline{Y}$ be two categories. A functor from $\underline{X}$ to $\underline{Y}$ is a family of functions $\mathbf{F}$ which associates to each object A in $\underline{X}$ an object $\mathbf{F}A$ in

$\underline{\mathrm{Y}}$ and to each morphism $f \in \mathrm{Hom}(A,B)$ a morphism $\mathbf{F}f \in \mathrm{Hom}(\mathbf{F}A, \mathbf{F}B)$, and which is such that:

(F1) $\mathbf{F}(g \circ f) = \mathbf{F}g \circ \mathbf{F}f$ for all $f \in \mathrm{Hom}(A,B)$ and $g \in \mathrm{Hom}(B,C)$;

(F2) $\mathbf{F}\,\mathrm{id}_A = \mathrm{id}_{\mathbf{F}A}$ for all $A \in \mathrm{Ob}$.

The map composition of two functors is a functor. Indeed, $(\mathbf{G} \circ \mathbf{F})(g \circ f) = \mathbf{G}(\mathbf{F}g \circ \mathbf{F}f) = (\mathbf{G} \circ \mathbf{F})g \circ (\mathbf{G} \circ \mathbf{F})f$ and $(\mathbf{G} \circ \mathbf{F})\mathrm{id}_A = \mathbf{G}(\mathrm{id}_{\mathbf{F}A}) = \mathrm{id}_{(\mathbf{G}\circ\mathbf{F})A}$. This does not allow us to directly form a category of all categories whose morphisms are functors, since the class of functors from a given category to another need not be a set. One can however form the category $\underline{\mathrm{Cat}}$ whose objects are small categories, that is categories with a set of objects.

For example, let $\underline{\mathrm{X}}$ be a category and J be a set. Then there exists a canonical functor $\mathbf{\Delta}$ from any given category $\underline{\mathrm{X}}$ to the product $\underline{\mathrm{X}}^J$ called the diagonal functor. The object A is mapped to $\{A_j = A\}$ and the morphism f to $\{f_j = f\}$. $\mathbf{\Delta}$ is indeed a functor, since $\mathbf{\Delta}g \circ \mathbf{\Delta}f = \{g_j = g\} \circ \{f_j = f\} = \{h_j = g \circ f\} = \mathbf{\Delta}(g \circ f)$ and $\mathrm{id}_{\mathbf{\Delta}A} = \{h_j = \mathrm{id}_A\} = \mathbf{\Delta}\mathrm{id}_A$.

As with morphisms, there are several special types of functor of particular interest. For example, let $\mathbf{F}$ be a functor from $\underline{\mathrm{X}}$ to $\underline{\mathrm{Y}}$. Then $\mathbf{F}$ is called faithful if the maps $\mathbf{F} : \mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathbf{F}A, \mathbf{F}B)$ are injective, whereas it is called full if they are surjective. A functor that is faithful and injective on objects is called an embedding, whereas a functor that is full, faithful and bijective on objects is called an isomorphism. Again these classes are closed under composition. Finally a functor $\mathbf{F}$ from $\underline{\mathrm{X}}$ to $\underline{\mathrm{Y}}$ is called an equivalence if it is full and faithful, and if for each object B in $\underline{\mathrm{Y}}$ there exists an object A in $\underline{\mathrm{X}}$ such that $\mathbf{F}A$ is isomorphic to B. We say that $\underline{\mathrm{X}}$ is equivalent to $\underline{\mathrm{Y}}$ if there exists an equivalence from $\underline{\mathrm{X}}$ to $\underline{\mathrm{Y}}$, and that $\underline{\mathrm{X}}$ is dual to $\underline{\mathrm{Y}}$ if it is equivalent to $\underline{\mathrm{Y}}^{\mathrm{op}}$. Note that equivalence is indeed an equivalence relation.

Much of the methodological utility of category theory arises from the possibility of encoding many standard constructions in a universal way as the adjoint of certain simple functors. Let $\mathbf{F}$ be a functor from $\underline{\mathrm{X}}$ to $\underline{\mathrm{Y}}$ and $\mathbf{G}$ a functor from $\underline{\mathrm{Y}}$ to $\underline{\mathrm{X}}$. Then $\mathbf{F}$ is called a left adjoint of $\mathbf{G}$ ($\mathbf{G}$ is called a right adjoint of $\mathbf{F}$) if there exists a bijection which associates to each morphism $f \in \mathrm{Hom}(\mathbf{F}A, B)$ a morphism $\phi f \in \mathrm{Hom}(A, \mathbf{G}B)$ such that $\phi(f \circ \mathbf{F}g) = \phi f \circ g$ and $\phi(h \circ f) = \mathbf{G}h \circ \phi f$ for each $g \in \mathrm{Hom}(A', A)$ and $h \in \mathrm{Hom}(B, B')$. Note that any two left (right) adjoints $\mathbf{F}$ and $\mathbf{F}'$ of a given functor are naturally isomorphic in the sense that for each object $A \in \mathrm{Ob}(\underline{\mathrm{X}})$ there exists an isomorphism $\tau_A \in \mathrm{Hom}(\mathbf{F}A, \mathbf{F}'A)$ such that $\mathbf{F}'f \circ \tau_A = \tau_B \circ \mathbf{F}f$ for each $f \in \mathrm{Hom}(A,B)$.

In the following I shall need two such adjoint constructions. First, in a given category the left adjoint of the diagonal functor (if it exists) is called the coproduct and the right adjoint (if it exists) is called the product: in $\underline{\mathrm{Set}}$ the product is the Cartesian product and the coproduct is the disjoint union. Second, let the category $\underline{\mathrm{X}}$ be concrete over some category $\underline{\mathrm{A}}$ in the sense that there exists a faithful functor $\mathbf{U}$ from $\underline{\mathrm{X}}$ to $\underline{\mathrm{A}}$, usually called the forgetful functor. The left adjoint to this functor (if it exists) is then called the free functor. A standard example is the forgetful functor from complete metric spaces to metric spaces, whose left adjoint in the completion functor.

3 The Category State

An object in State is a state space, that is, a pair $(\Sigma, \perp)$, where Σ is a set and $\perp$ is a symmetric antireflexive binary relation which separates the points of Σ:

(SO1) if $\mathcal{E}_i \perp \mathcal{E}_j$ then $\mathcal{E}_j \perp \mathcal{E}_i$;

(SO2) if $\mathcal{E}_i \perp \mathcal{E}_j$ then $\mathcal{E}_i \neq \mathcal{E}_j$;

(SO3) if $\mathcal{E}_i \neq \mathcal{E}_j$ then there exists $\mathcal{E}_k$ such that $\mathcal{E}_i \perp \mathcal{E}_k$ and $\mathcal{E}_j \not\perp \mathcal{E}_k$.

Let $(\Sigma, \perp)$ be a state space and $\mathcal{A} \subseteq \Sigma$. We define $\mathcal{A}^\perp = \{\mathcal{E}' \in \Sigma \,|\, \mathcal{E}' \perp \mathcal{E} \quad \forall \mathcal{E} \in \mathcal{A}\}$. If $\mathcal{A}^{\perp\perp} = \mathcal{A}$ then $\mathcal{A}$ is called biorthogonal. The following results are standard:

Lemma 3.1 *We have the following results:*

(i) $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$ *for each* $\mathcal{A} \subseteq \Sigma$;

(ii) if $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$;

(iii) $\mathcal{A}^{\perp\perp\perp} = \mathcal{A}^\perp$ *for each* $\mathcal{A} \subseteq \Sigma$;

(iv) $\emptyset^\perp = \Sigma$ *and* $\Sigma^\perp = \emptyset$;

(v) $\{\mathcal{E}\}^{\perp\perp} = \{\mathcal{E}\}$ *for each* $\mathcal{E} \in \Sigma$.

Proof: (i) Let $\mathcal{E} \in \mathcal{A}$. Then for each $\mathcal{E}' \in \mathcal{A}^\perp$ we have that $\mathcal{E}' \perp \mathcal{E}$ and so $\mathcal{E} \perp \mathcal{E}'$. Hence $\mathcal{E} \in \mathcal{A}^{\perp\perp}$ and so $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$. (ii) Let $\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{B}^\perp = \{\mathcal{E}' \in \Sigma \,|\, \mathcal{E}' \perp \mathcal{E} \quad \forall \mathcal{E} \in \mathcal{B}\} \subseteq \{\mathcal{E}' \in \Sigma \,|\, \mathcal{E}' \perp \mathcal{E} \quad \forall \mathcal{E} \in \mathcal{A}\} = \mathcal{A}^\perp$. (iii) $\mathcal{A}^\perp \subseteq \mathcal{A}^{\perp\perp\perp}$ by (i). However $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$ by (i) and so $\mathcal{A}^{\perp\perp\perp} \subseteq \mathcal{A}^\perp$ by (ii). (iv) $\emptyset^\perp = \{\mathcal{E}' \in \Sigma \,|\, \mathcal{E}' \perp \mathcal{E} \quad \forall \mathcal{E} \in \emptyset\} = \Sigma$. Next, $\mathcal{E}$ is never orthogonal to itself and so $\Sigma^\perp = \{\mathcal{E}' \in \Sigma \,|\, \mathcal{E}' \perp \mathcal{E} \quad \forall \mathcal{E} \in \Sigma\} = \emptyset$. (v) Let $\mathcal{E}_j \in \{\mathcal{E}_i\}^{\perp\perp}$. Then for all $\mathcal{E} \in \{\mathcal{E}_i\}^\perp$ we have that $\mathcal{E}_j \perp \mathcal{E}$. Hence $\mathcal{E}_j \perp \mathcal{E}$ whenever $\mathcal{E}_i \perp \mathcal{E}$ and so $\mathcal{E}_j = \mathcal{E}_i$. ■

A morphism from $(\Sigma_1, \perp_1)$ to $(\Sigma_2, \perp_2)$ is a partially defined map $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ such that:

(SM1) $\mathcal{K}_1 \cup f^{-1}(\mathcal{F}_2)$ is biorthogonal in Σ_1 for each $\mathcal{F}_2$ which is biorthogonal in Σ_2.

The set $\mathcal{K}_1$ is called the kernel of f. Note that $\mathcal{K}_1$ is necessarily biorthogonal since $\mathcal{K}_1 = \mathcal{K}_1 \cup f^{-1}(\emptyset)$. Clearly the identity maps are morphisms with empty kernel. Hence we need merely show that the composition of two morphisms is again a morphism and that composition is associative.

Lemma 3.2 *Let* $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ *and* $g : \Sigma_2 \setminus \mathcal{K}_2 \to \Sigma_3$ *be morphisms and let* $g \circ f : \Sigma_1 \setminus \mathcal{K} \to \Sigma_3$ *with* $\mathcal{K} = \mathcal{K}_1 \cup f^{-1}(\mathcal{K}_2)$ *be defined by* $(g \circ f)(\mathcal{E}) = g\big(f(\mathcal{E})\big)$. *Then (i)* $g \circ f$ *is a morphism and (ii) composition is associative.*

Proof: (i) If $\mathcal{E} \notin \mathcal{K}$ then $f(\mathcal{E}) \notin \mathcal{K}_2$ so the map is well defined. Let $\mathcal{F}_3 \subset \Sigma_3$ be biorthogonal. Then $\mathcal{K} \cup (g \circ f)^{-1}(\mathcal{F}_3) = \mathcal{K} \cup f^{-1}\big(g^{-1}(\mathcal{F}_3)\big) = \mathcal{K}_1 \cup f^{-1}\big(\mathcal{K}_2 \cup g^{-1}(\mathcal{F}_3)\big)$. However $\mathcal{K}_2 \cup$

$g^{-1}(\mathcal{F}_3)$ is biorthogonal in Σ_2 and so $\mathcal{K} \cup (g \circ f)^{-1}(\mathcal{F}_3)$ is biorthogonal in Σ_1. *(ii)* Let $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$, $g : \Sigma_2 \setminus \mathcal{K}_2 \to \Sigma_3$ and $h : \Sigma_3 \setminus \mathcal{K}_3 \to \Sigma_4$ be morphisms. To prove that $h \circ (g \circ f) = (h \circ g) \circ f$ it suffices to show that the kernels are the same. The kernel of $h \circ (g \circ f)$ is $\big(\mathcal{K}_1 \cup f^{-1}(\mathcal{K}_2)\big) \cup (g \circ f)^{-1}(\mathcal{K}_3) = \mathcal{K}_1 \cup f^{-1}\big(\mathcal{K}_2 \cup g^{-1}(\mathcal{K}_3)\big)$, which is the kernel of $(h \circ g) \circ f$. ∎

Let $\mathcal{K}_1 \subseteq \Sigma_1$ be biorthogonal and $\mathcal{E}_2 \in \Sigma_2$. Then the constant map $c : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2 : \mathcal{E}_1 \mapsto \mathcal{E}_2$ is a morphism. Indeed, let $\mathcal{F}_2 \subseteq \Sigma_2$ be biorthogonal. If $\mathcal{E}_2 \in \mathcal{F}_2$ then $\mathcal{K}_1 \cup c^{-1}(\mathcal{F}_2) = \Sigma_1$ which is biorthogonal. If $\mathcal{E}_2 \notin \mathcal{F}_2$ then $\mathcal{K}_1 \cup c^{-1}(\mathcal{F}_2) = \mathcal{K}_1$ which is biorthogonal by hypothesis. This allows us to prove the following results:

Lemma 3.3 *A morphism $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ is a monomorphism if and only if it is injective with empty kernel.*

Proof: We must prove that for all $g : \Sigma_3 \setminus \mathcal{K}_3 \to \Sigma_1$ and $\tilde{g} : \Sigma_3 \setminus \tilde{\mathcal{K}}_3 \to \Sigma_1$ we have that $f \circ g = f \circ \tilde{g}$ implies $g = \tilde{g}$. We first prove necessity. Let $\mathcal{K}_1 \neq \emptyset$. Let $\mathcal{E}_1 \in \mathcal{K}_1$ and g be the constant morphism onto $\mathcal{E}_1$ with empty kernel. On the other hand let $\tilde{g}$ be the trivial morphism with kernel Σ_3. Then $f \circ g = f \circ \tilde{g}$ but $g \neq \tilde{g}$. Let f not be injective. Then there exist $\mathcal{E}_1 \neq \tilde{\mathcal{E}}_1$ such that $f(\mathcal{E}_1) = f(\tilde{\mathcal{E}}_1)$. Let g be the constant morphism onto $\mathcal{E}_1$ with empty kernel and $\tilde{g}$ be the constant morphism onto $\tilde{\mathcal{E}}_1$ with empty kernel. Then $f \circ g = f \circ \tilde{g}$ but $\tilde{g} \neq g$. Hence the conditions are necessary.

We now prove sufficiency. Let f be injective with empty kernel and $f \circ g = f \circ \tilde{g}$. For the two morphisms to be equal their kernels must be equal so that $\mathcal{K}_3 = \tilde{\mathcal{K}}_3$. Let $\mathcal{E}_3 \notin \mathcal{K}_3$. Then by hypothesis we have $f\big(g(\mathcal{E}_3)\big) = f\big(\tilde{g}(\mathcal{E}_3)\big)$ so that $g(\mathcal{E}_3) = \tilde{g}(\mathcal{E}_3)$ by injectivity. Hence the conditions are sufficient. ∎

Lemma 3.4 *A morphism $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ is an epimorphism if and only if it is surjective.*

Proof: We must prove that for all $g : \Sigma_2 \setminus \mathcal{K}_2 \to \Sigma_3$ and $\tilde{g} : \Sigma_2 \setminus \tilde{\mathcal{K}}_2 \to \Sigma_3$ we have that $g \circ f = \tilde{g} \circ f$ implies $g = \tilde{g}$. We first prove necessity. Let f not be surjective. Then there exists $\mathcal{E}_2 \notin \mathrm{Im} f$. Let $\mathcal{E}_3 \in \Sigma_3$ and g be the constant morphism onto $\mathcal{E}_3$ with empty kernel. On the other hand, let $\tilde{g}$ be the constant morphism onto $\mathcal{E}_3$ with kernel $\{\mathcal{E}_2\}$, which is necessarily biorthogonal. Then $g \circ f = \tilde{g} \circ f$ but $g \neq \tilde{g}$. Hence the condition is necessary.

We now prove sufficiency. Let f be surjective and $g \circ f = \tilde{g} \circ f$. For the two morphisms to be equal their kernels must be equal so that $\mathcal{K}_1 \cup f^{-1}(\mathcal{K}_2) = \mathcal{K}_1 \cup f^{-1}(\tilde{\mathcal{K}}_2)$. However $\mathcal{K}_1$ and $f^{-1}(\mathcal{K}_2)$ are disjoint, as are $\mathcal{K}_1$ and $f^{-1}(\tilde{\mathcal{K}}_2)$, and so $f^{-1}(\mathcal{K}_2) = f^{-1}(\tilde{\mathcal{K}}_2)$. Further, f is surjective so that $\mathcal{K}_2 = \tilde{\mathcal{K}}_2$. Indeed, let $\mathcal{E}_2 \in \mathcal{K}_2$. Then there exists $\mathcal{E}_1 \in \Sigma_1 \setminus \mathcal{K}_1$ such that $\mathcal{E}_2 = f(\mathcal{E}_1)$. Then $\mathcal{E}_1 \in f^{-1}(\mathcal{K}_2) = f^{-1}(\tilde{\mathcal{K}}_2)$ so that $\mathcal{E}_2 \in \tilde{\mathcal{K}}_2$. Let $\mathcal{E}_2 \notin \mathcal{K}_2$. Then there exists $\mathcal{E}_1 \in \Sigma_1 \backslash \mathcal{K}_1$ such that $\mathcal{E}_2 = f(\mathcal{E}_1)$. But then $g(\mathcal{E}_2) = (g \circ f)(\mathcal{E}_1) = (\tilde{g} \circ f)(\mathcal{E}_1) = \tilde{g}(\mathcal{E}_2)$. Hence the condition is sufficient. ■

Lemma 3.5 *A morphism $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ is an isomorphism if and only if it is bijective with empty kernel and $f(\mathcal{F}_1)$ is biorthogonal in Σ_2 for each $\mathcal{F}_1$ which is biorthogonal in Σ_1.*

Proof: We must prove that there exists a morphism $g : \Sigma_2 \setminus \mathcal{K}_2 \to \Sigma_1$ such that $g \circ f = \mathrm{id}_1$ and $f \circ g = \mathrm{id}_2$. Each isomorphism is both a monomorphism and an epimorphism and so it is necessary that f be bijective with empty kernel. Indeed, let $f \circ h = f \circ \tilde{h}$. Then $h = (g \circ f) \circ h = g \circ (f \circ h) = g \circ (f \circ \tilde{h}) = (g \circ f) \circ \tilde{h} = \tilde{h}$ and analogously if $h \circ f = \tilde{h} \circ f$. It remains to prove that $g : \Sigma_2 \to \Sigma_1$ defined by $g(\mathcal{E}_2) = \mathcal{E}_1$ if $\mathcal{E}_2 = f(\mathcal{E}_1)$ is indeed a morphism. Let $\mathcal{F}_1 \subseteq \Sigma_1$ be biorthogonal. Then $g^{-1}(\mathcal{F}_1) = f(\mathcal{F}_1)$ which will be biorthogonal in general if and only if the last condition is satisfied. ■

Note that it is easy to construct morphisms which are mono- and epi- but which are not isomorphisms. Indeed, let $\Sigma_1 = \{\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_k, \mathcal{E}_l\}$ with any two distinct points orthogonal, and $\Sigma_2 = \{\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_j, \tilde{\mathcal{E}}_k, \tilde{\mathcal{E}}_l\}$ with only $\tilde{\mathcal{E}}_i \perp \tilde{\mathcal{E}}_j$ and $\tilde{\mathcal{E}}_k \perp \tilde{\mathcal{E}}_l$. Define $f : \Sigma_1 \to \Sigma_2 : \mathcal{E} \mapsto \tilde{\mathcal{E}}$. Then f is trivially a morphism as every subset of Σ_1 is biorthogonal and is clearly both mono- and epi-. However the inverse map $g : \Sigma_2 \to \Sigma_1 : \tilde{\mathcal{E}} \mapsto \mathcal{E}$ is not a morphism since, for example, $\{\mathcal{E}_i, \mathcal{E}_j\}$ is biorthogonal in Σ_1 but $g^{-1}(\{\mathcal{E}_i, \mathcal{E}_j\}) = \{\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_j\}$ is not biorthogonal in Σ_2.

The paradigm examples of state spaces are the classical entity, where Σ is a manifold with any two distinct points orthogonal, and the quantum entity, where Σ is the set of rays of an underlying Hilbert space, with $[\psi_1] \perp [\psi_2]$ if $\langle \psi_1, \psi_2 \rangle = 0$. In the former case any partial map is a morphism since every subset of the state space is biorthogonal, whereas in the latter case morphisms are represented by semilinear maps.

4 The Category Prop

An object in Prop is a property lattice, that is, a complete atomistic orthocomplemented lattice $(\mathcal{L}, <, ')$:

(PO1) there exists a maximal element $1 \in \mathcal{L}$;

(PO2) the greatest lower bound $\bigwedge \mathcal{A}$ of an arbitrary non-empty family $\mathcal{A}$ exists;

(PO3) $a = \bigwedge\{p' \,|\, p < a',\ p \text{ an atom}\}$ for each $a \in \mathcal{L}$;

(PO4) $a'' = a$ for each $a \in \mathcal{L}$;

(PO5) if $a < b$ then $b' < a'$;

(PO6) $a \wedge a' = 1'$ for each $a \in \mathcal{L}$;

where an atom is an element $p \neq 1'$ such that if $x < p$ then $x = 1'$ or $x = p$. As usual, we shall write 0 for $1'$ and $a \wedge b$ for $\bigwedge\{a, b\}$. The following results are standard:

Lemma 4.1 *We have the following results:*

(*i*) $a < b$ *if and only if* $a \wedge b = a$ *if and only if* $a \vee b = b$;

(*ii*) *The family* $\mathcal{A}$ *has least upper bound* $\bigvee \mathcal{A} = \bigwedge\{b \,|\, a < b \quad \forall a \in \mathcal{A}\}$;

(*iii*) $\bigvee\{a_r'\} = \bigwedge\{a_r\}'$ *and* $\bigwedge\{a_r'\} = \bigvee\{a_r\}'$;

(*iv*) $a = \bigvee\{p \,|\, p < a,\ p$ *an atom*$\}$ *for each* $a \in \mathcal{L}$.

Proof: (*i*) Let $a < b$. Then a is a lower bound of $\{a, b\}$ and so $a < a \wedge b < a$. Further b is an upper bound of $\{a, b\}$ and so $b < a \vee b < b$. On the other hand, if $a \wedge b = a$ then $a = a \wedge b < b$ and if $a \vee b = b$ then $a < a \vee b = b$. (*ii*) let $\mathcal{B} = \{b \,|\, a < b \quad \forall a \in \mathcal{A}\}$. Note that $\mathcal{B}$ is nonempty since $1 \in \mathcal{B}$. Each $a \in \mathcal{A}$ is a lower bound of $\mathcal{B}$ and so $a < \bigwedge \mathcal{B}$ since $\bigwedge \mathcal{B}$ is the greatest lower bound of $\mathcal{B}$. Hence $\bigwedge \mathcal{B}$ is an upper bound of $\mathcal{A}$. Let x be such that $a < x$ for each $a \in \mathcal{A}$. Then $x \in \mathcal{B}$ and so $\bigwedge \mathcal{B} < x$ since $\bigwedge \mathcal{B}$ is a lower bound of $\mathcal{B}$. Hence $\bigwedge \mathcal{B}$ is the least upper bound of $\mathcal{A}$. (*iii*) $\bigwedge\{a_r\} < a_{r_0}$ and so $a_{r_0}' < \bigvee\{a_r\}'$ for each a_{r_0}. Hence $\bigvee\{a_r\}'$ is an upper bound of $\{a_r'\}$. Let b be such that $a_{r_0}' < b$ for each a_{r_0}'. Then $b' < a_{r_0}$ and so $b' < \bigwedge\{a_r\}$. Hence $\bigvee\{a_r\}' < b$ and so $\bigvee\{a_r\}'$ is the least upper bound of $\{a_r'\}$. We then have that $\bigwedge\{a_r'\} = \bigwedge\{a_r'\}'' = \bigvee\{a_r''\}' = \bigvee\{a_r\}'$. (*iv*) $a' = \bigwedge\{p' \,|\, p < a''\} = \bigvee\{p \,|\, p < a\}'$ and so $a = \bigvee\{p \,|\, p < a\}$. ■

As we shall see in the following, it is useful to define two dual categories, <u>Prop</u> and <u>Prop</u>*. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be property lattices. A morphism is a map $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that:

(PM1) $\phi(0_1) = 0_2$;

(PM2) $\phi(\bigvee\{a_{1,r}\}) = \bigvee\{\phi(a_{1,r})\}$ for any non-empty family $\{a_{1,r}\}$;

(PM3) ϕ maps atoms of $\mathcal{L}_1$ to either atoms of $\mathcal{L}_2$ or to 0_2.

Dually, a comorphism is a map $\phi^* : \mathcal{L}_2 \to \mathcal{L}_1$ such that

(PM1*) $\phi^*(I_2) = I_1$;

(PM2*) $\phi^*(\bigwedge\{a_{2,r}\}) = \bigwedge\{\phi^*(a_{2,r})\}$ for any non-empty family $\{a_{2,r}\}$;

(PM3*) for each atom $p_1 \in \mathcal{L}_1$ there exists at least one atom $p_2 \in \mathcal{L}_2$ such that $p_1 < \phi^*(p_2)$.

Clearly the identity maps are both morphisms and comorphisms and composition is associative. Hence we need merely show that the composition of two morphisms (comorphisms) is agian a morphism (comorphism).

Lemma 4.2 *The composition of two morphisms (comorphisms) is again a morphism (comorphism).*

Proof: Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ and $\psi : \mathcal{L}_2 \to \mathcal{L}_3$ be morphisms. $(\psi \circ \phi)(0_1) = \psi(0_2) = 0_3$. $(\psi \circ \phi)(\bigvee\{a_{1,r}\}) = \psi\big(\bigvee\{\phi(a_{1,r})\}\big) = \bigvee\{(\psi \circ \phi)(a_{1,r})\}$. Let p_1 be an atom of $\mathcal{L}_1$. Then $\phi(p_1)$ is either an atom of $\mathcal{L}_2$ or 0_2 and so $(\psi \circ \phi)(p_1)$ is either an atom of $\mathcal{L}_3$ or 0_3. Hence $\psi \circ \phi$ is a morphism.

Let $\psi^* : \mathcal{L}_3 \to \mathcal{L}_2$ and $\phi^* : \mathcal{L}_2 \to \mathcal{L}_1$ be comorphisms. $(\phi^* \circ \psi^*)(I_3) = \phi^*(I_2) = I_1$. $(\phi^* \circ \psi^*)(\bigwedge\{a_{3,r}\}) = \phi^*\big(\bigwedge\{\psi^*(a_{3,r})\}\big) = \bigwedge\{(\phi^* \circ \psi^*)(a_{3,r})\}$. Let $p_1 \in \mathcal{L}_1$ be an atom. Then there exists an atom $p_2 \in \mathcal{L}_2$ such that $p_1 < \phi^*(p_2)$. Further, there exists an atom $p_3 \in \mathcal{L}_3$ such that $p_2 < \psi^*(p_3)$. Comorphisms preserve the greatest lower bound and so the order and thus $p_1 < (\phi^* \circ \psi^*)(p_3)$. Hence $\phi^* \circ \psi^*$ is a comorphism. ∎

Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a morphism and $\phi^* : \mathcal{L}_2 \to \mathcal{L}_1$ be a comorphism. We call ϕ and ϕ^* dual in the case that $a_1 < \phi^*(a_2)$ if and only if $\phi(a_1) < a_2$. In this case we shall say that ϕ^* is the dual of ϕ. This is justified by the following result, which establishes the relation between morphisms and comorphisms:

Lemma 4.3 *Each morphism ϕ has a unique dual ϕ^* and each comorphism ϕ^* is the dual of a unique morphism ϕ. We have $\phi^*(a_2) = \bigvee\{x_1 \,|\, \phi(x_1) < a_2\}$ and $\phi(a_1) = \bigwedge\{x_2 \,|\, a_1 < \phi^*(x_2)\}$.*

Proof: We first prove unicity. Let ϕ^* and $\tilde{\phi}^*$ both be duals of ϕ. Then for each $a_2 \in \mathcal{L}_2$ we have that $\phi^*(a_2) < \phi^*(a_2)$ and so $\phi\big(\phi^*(a_2)\big) < a_2$. But then $\phi^*(a_2) < \tilde{\phi}^*(a_2)$. Interchanging the roles of ϕ^* and $\tilde{\phi}^*$ we have that $\tilde{\phi}^* = \phi^*$. A similar argument holds if both ϕ and $\tilde{\phi}$ are dual to ϕ^*.

We now show that the defined maps ϕ^* and ϕ are indeed comorphisms and morphisms respectively. $\phi^*(I_2) = \bigvee\{x_1 \,|\, \phi(x_1) < I_2\} = \bigvee \mathcal{L}_1 = I_1$. ϕ^* preserves the order. Indeed, let $a_2 < b_2$. Then $\phi^*(a_2) = \bigvee\{x_1 \,|\, \phi(x_1) < a_2\} < \bigvee\{x_1 \,|\, \phi(x_1) < b_2\} = \phi^*(b_2)$. Hence $\phi^*\big(\bigwedge\{a_{2,r}\}\big) < \bigwedge\{\phi^*(a_{2,r})\}$. Let $b_1 < \phi^*(a_{2,r_0})$ for all a_{2,r_0}. Then $\phi(b_1) < a_{2,r_0}$ and so $\phi(b_1) < \bigwedge\{a_{2,r}\}$. Hence $b_1 < \phi^*\big(\bigwedge\{a_{2,r}\}\big)$ and so $\phi^*\big(\bigwedge\{a_{2,r}\}\big)$ is the greatest lower bound. Finally, let $p_1 \in \mathcal{L}_1$ be an atom. Then $\phi(p_1)$ is either an atom or 0_2. In either case there exists an atom $p_2 \in \mathcal{L}_2$ such that $\phi(p_1) < p_2$ and so $p_1 < \phi^*(p_2)$.

$\phi(0_1) = \bigwedge\{x_2 \,|\, 0_1 < \phi^*(x_2)\} = \bigwedge \mathcal{L}_2 = 0_2$. ϕ preserves the order. Indeed, let $a_1 < b_1$. Then $\phi(a_1) = \bigwedge\{x_2 \,|\, a_1 < \phi^*(x_2)\} < \bigwedge\{x_2 \,|\, b_1 < \phi^*(x_2)\} = \phi(b_2)$. Hence $\bigvee\{\phi(a_{1,r})\} < \phi\big(\bigvee\{a_{1,r}\}\big)$. Let $\phi(a_{1,r_0}) < b_2$ for all a_{1,r_0}. Then $a_{1,r_0} < \phi^*(b_2)$ and so $\bigvee\{a_{1,r}\} < \phi^*(b_2)$. Hence $\phi\big(\bigvee\{a_{1,r}\}\big) < b_2$ and so $\phi\big(\bigvee\{a_{1,r}\}\big)$ is the least upper bound. Finally, let $p_1 \in \mathcal{L}_1$ be an atom. Then there exists an atom $p_2 \in \mathcal{L}_2$ such that $p_1 < \phi^*(p_2)$. But then $\phi(p_1) < p_2$ and so $\phi(p_1)$ is either an atom or 0_2.

Finally we show that the defined maps ϕ^* and ϕ define duals. If $a_1 < \phi^*(a_2)$ then $\phi(a_1) < a_2$ by definition. To prove the reverse implication we use the fact that $a_1 < \phi^*\big(\phi(a_1)\big)$. Indeed $a_1 = \bigwedge\{x_1 \,|\, a_1 < x_1\} < \bigwedge\{\phi^*(x_2) \,|\, a_1 < \phi^*(a_2)\} = \phi^*\big(\bigwedge\{x_2 \,|\, a_1 < \phi^*(x_2)\}\big) = \phi^*\big(\phi(a_1)\big)$, where we have used the fact that ϕ^* preserves the infimum. Let $\phi(a_1) < a_2$. Then $a_1 < \phi^*\big(\phi(a_1)\big) < \phi^*(a_2)$ since ϕ^* preserves the order. Hence ϕ is dual to ϕ^*. A similar argument shows that ϕ^* is dual to ϕ. ∎

Note that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. Indeed, $(\psi \circ \phi)^*(a_3) = \bigvee\{x_1 \,|\, (\psi \circ \phi)(x_1) < a_3\} = \bigvee\{x_1 \,|\, \phi(x_1) < \psi^*(a_3)\}$, which by definition this is nothing more than $\phi^*\big(\psi^*(a_3)\big)$.

5 The Equivalence of State and Prop

Let $(\mathcal{L}, <, ')$ be a property lattice and define $\Sigma_{\mathcal{L}} = \{p \,|\, p \text{ is an atom}\}$, with $p \perp q$ if and only if $p < q'$.

Lemma 5.1 $\mathbf{S}(\mathcal{L}, <, ') = (\Sigma_{\mathcal{L}}, \perp)$ *is a state space.*

Proof: (SO1) Let $p \perp q$ so that $p < q'$. Then $q = q'' < p'$ and so $q \perp p$. (SO2) Suppose that $a < a'$. Then $a < a \wedge a' = 0$ and so $a = 0$. Since an atom is non-zero by definition we therefore have that $q \not< q'$. (SO3) Suppose that $r < p'$ implies that $r < q'$. Then $p' = \bigvee\{r \,|\, r < p'\} < \bigvee\{r \,|\, r < q'\} = q'$ and so $q = q'' < p'' = p$. Since p and q are atoms we then have that $q = p$. ■

Lemma 5.2 *We have the following results:*

(i) $\mathcal{A}^\perp = \{q \,|\, q < \big(\bigvee \mathcal{A}\big)'\}$ *for each* $\mathcal{A} \subseteq \Sigma_{\mathcal{L}}$;

(ii) $\{p \,|\, p < a\}^\perp = \{q \,|\, q < a'\}$ *for each* $a \in \mathcal{L}$;

(iii) $\mathcal{A} \subseteq \Sigma_{\mathcal{L}}$ *is biorthogonal if and only if* $\mathcal{A} = \{p \,|\, p < \bigvee \mathcal{A}\}$.

Proof: (i) Let $p \in \mathcal{A}$ and $q < \big(\bigvee \mathcal{A}\big)'$. Then $p < \bigvee \mathcal{A} < q'$ and so $q \in \mathcal{A}^\perp$. On the other hand, let $q \in \mathcal{A}^\perp$ so that $q < p'$ for each $p \in \mathcal{A}$. Then $q < \bigwedge\{p' \,|\, p \in \mathcal{A}\} = \big(\bigvee \mathcal{A}\big)'$. (ii) Let $p < a$ and $q < a'$. Then $q < a' < p'$ and so $q \in \{p \,|\, p < a\}^\perp$. On the other hand, let $q \in \{p \,|\, p < a\}^\perp$. Then $p < q'$ for each $p < a$ and so $q < \bigwedge\{p' \,|\, p < a\} = \big(\bigvee\{p \,|\, p < a\}\big)' = a'$. (iii) By (i) we have that $\mathcal{A}^\perp = \{q \,|\, q < \big(\bigvee \mathcal{A}\big)'\}$. By (ii) we then have that $\mathcal{A}^{\perp\perp} = \{\{q \,|\, q < \big(\bigvee \mathcal{A}\big)'\}^\perp = \{p \,|\, p < \bigvee \mathcal{A}\}$. ■

Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a morphism and define $f_\phi : \Sigma_{\mathcal{L}_1} \setminus \mathcal{K}_1 \to \Sigma_{\mathcal{L}_1} : p_1 \mapsto \phi(p_1)$, where $\mathcal{K}_1 = \{p_1 \in \Sigma_1 \,|\, \phi(p_1) = 0_2\}$. Note that f_ϕ is well defined by (PM3).

Lemma 5.3 $\mathbf{S}\phi = f_\phi$ *is a morphism.*

Proof: Let $\mathcal{A}_2$ be biorthogonal so that $\mathcal{A}_2 = \{p_2 \,|\, p_2 < \bigvee \mathcal{A}_2\}$. Then $f(p_1) \in \mathcal{A}_2$ if and only if $\phi(p_1) \neq 0_2$ and $\phi(p_1) < \bigvee \mathcal{A}_2$, which is the case if and only if $\phi(p_1) \neq 0_2$ and

$p_1 < \phi^*(\bigvee \mathcal{A}_2)$. On the other hand, $p_1 \in \mathcal{K}_1$ if and only if $\phi(p_1) = 0_2$. In this case also $\phi(p_1) < \bigvee \mathcal{A}_2$ and so $p_1 < \phi^*(\bigvee \mathcal{A}_2)$. Hence $\mathcal{K}_1 \cup f^{-1}(\mathcal{A}_2) = \{p_1 \,|\, p_1 < \phi^*(\bigvee \mathcal{A}_2)\}$, which is biorthogonal. ∎

Lemma 5.4 **S** *is a functor.*

Proof: It is trivial that $\mathbf{S}(\mathrm{id}_{\mathcal{L}}) = \mathrm{id}_{\Sigma}$. We must then prove that the correspondence preserves composition. Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ and $\psi : \mathcal{L}_2 \to \mathcal{L}_3$ be property lattice morphisms. Let the state space morphisms corresponding to ϕ, ψ and $\psi\circ\phi$ be f, g and h respectively. To show that $h = g \circ f$ it suffices to show that the kernels are the same. Let the kernels of f, g and h be $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}$ respectively. Now $\mathcal{K}_2$ is biorthogonal since $\mathcal{K}_2 = \{p_2 \,|\, p_2 < \phi^*(0_3)\}$. Then $\mathcal{K}_1 \cup f^{-1}(\mathcal{K}_2) = \{p_1 \,|\, p_1 < \phi^*(\bigvee \mathcal{K}_2)\} = \{p_1 \,|\, p_1 < (\phi^* \circ \psi^*)(0_3)\} = \{p_1 \,|\, p_1 < (\psi \circ \phi)^*(0_3)\} = \mathcal{K}$. ∎

Let $(\Sigma, \perp)$ be a state space and define $\mathcal{L}_\Sigma = \{\mathcal{A} \subset \Sigma \,|\, \mathcal{A}^{\perp\perp} = \mathcal{A}\}$.

Lemma 5.5 $\mathbf{P}(\Sigma, \perp) = (\mathcal{L}_\Sigma, \subseteq, {}^\perp)$ *is a property lattice with* $\bigwedge\{\mathcal{A}_r\} = \bigcap\{\mathcal{A}_r\}$.

Proof: (PO1) We have that $\Sigma^{\perp\perp} = \emptyset^\perp = \Sigma$ and so $\mathcal{L}_\Sigma$ has a maximal element. (PO2) The intersection of a family of biorthogonal subsets is itself biorthogonal since $\bigcap\{\mathcal{A}_r\} = \bigcap\{\mathcal{A}_r^{\perp\perp}\} = \bigcap\{\{\mathcal{E} \in \Sigma \,|\, \mathcal{E} \perp \mathcal{E}' \;\; \forall \mathcal{E}' \in \mathcal{A}_r^\perp\}\} = \{\mathcal{E} \in \Sigma \,|\, \mathcal{E} \perp \mathcal{E}' \;\; \forall \mathcal{E}' \in \bigcup\{\mathcal{A}_r^\perp\}\} = \big(\bigcup\{\mathcal{A}_r^\perp\}\big)^\perp$, and for any subset $\mathcal{A}$ we have that $\mathcal{A}^{\perp\perp\perp} = \mathcal{A}^\perp$. The intersection is then the greatest lower bound since $\bigcap\{\mathcal{A}_r\} \subseteq \mathcal{A}_{r_0}$ and if $\mathcal{B} \subseteq \mathcal{A}_{r_0}$ for all $\mathcal{A}_{r_0}$ then $\mathcal{B} \subseteq \bigcap\{\mathcal{A}_r\}$.

(PO3) The atoms of $\mathcal{L}_\Sigma$ are exactly the singletons $\{\mathcal{E}\}$ since each singleton is biorthogonal and if $\mathcal{E} \in \mathcal{A}$ then $\{\mathcal{E}\} \subseteq \mathcal{A}$. Note that the least upper bound of a family $\{\mathcal{A}_r\}$ is given by $\big(\bigcup\{\mathcal{A}_r\}\big)^{\perp\perp}$. Indeed $\mathcal{A}_{r_0} \subseteq \bigcup\{\mathcal{A}_r\} \subseteq \big(\bigcup\{\mathcal{A}_r\}\big)^{\perp\perp}$. Let $\mathcal{B}$ be biorthogonal and such that $\mathcal{A}_{r_0} \subseteq \mathcal{B}$ for all $\mathcal{A}_{r_0}$. Then $\bigcup\{\mathcal{A}_r\} \subseteq \mathcal{B}$ so that $\mathcal{B}^\perp \subseteq \big(\bigcup\{\mathcal{A}_r\}\big)^\perp$ and $\big(\bigcup\{\mathcal{A}_r\}\big)^{\perp\perp} \subseteq \mathcal{B}^{\perp\perp} = \mathcal{B}$. Finally, let $\mathcal{A}$ be biorthogonal. Then $\mathcal{A} = \mathcal{A}^{\perp\perp} = \big(\bigcup\{\{\mathcal{E}\} \,|\, \mathcal{E} \in \mathcal{A}\}\big)^{\perp\perp} = \bigvee\{\{\mathcal{E}\} \,|\, \{\mathcal{E}\} \subseteq \mathcal{A}\}$.

(PO4) The map $\mathcal{A} \mapsto \mathcal{A}^\perp$ is well defined since $\mathcal{A}^\perp$ is biorthogonal. By definition, if $\mathcal{A} \in \mathcal{L}_\Sigma$ then $\mathcal{A}^{\perp\perp} = \mathcal{A}$. (PO5) For any subsets $\mathcal{A}$ and $\mathcal{B}$ of Σ we have that $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$. (PO6) If $\mathcal{E} \in \mathcal{A}^\perp$ then $\mathcal{E} \notin \mathcal{E}$ since $\mathcal{E} \not\perp \mathcal{E}_k$, and so $\mathcal{A} \cap \mathcal{A}^\perp = \emptyset$. ∎

Let $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ be a morphism and define $\phi_f : \mathcal{L}_{\Sigma_1} \to \mathcal{L}_{\Sigma_2} : \mathcal{A}_1 \mapsto f(\mathcal{A}_1 \setminus \mathcal{K}_1)^{\perp\perp}$.

Lemma 5.6 $\mathbf{P}f = \phi_f$ *is a morphism with* $\phi_f^*(\mathcal{A}_2) = \mathcal{K}_1 \cup f^{-1}(\mathcal{A}_2)$.

Proof: We must show that (i) ϕ_f^* is a comorphism, and (ii) ϕ_f and ϕ_f^* are dual. (i) Note that ϕ_f^* is well defined by (SM1). (PM1*) $\phi^*(\Sigma_2) = \mathcal{K}_1 \cup f^{-1}(\Sigma_2) = \Sigma_1$. (PM2*)

$\phi^*(\bigcap\{\mathcal{A}_{2,r}\}) = \mathcal{K}_1 \cup f^{-1}(\bigcap\{\mathcal{A}_{2,r}\}) = \mathcal{K}_1 \cup (\bigcap\{f^{-1}(\mathcal{A}_{2,r})\}) = \bigcap\{(\mathcal{K}_1 \cup f^{-1}(\mathcal{A}_{2,r}))\} = \bigcap\{\phi^*(\mathcal{A}_{2,r})\}$. (PM3*) Let $\mathcal{E}_1 \in \Sigma_1$. If $\mathcal{E}_1 \notin \mathcal{K}_1$ then $\{\mathcal{E}_1\} \subseteq f^{-1}(\{f(\mathcal{E}_1)\}) \subseteq \mathcal{K}_1 \cup f^{-1}(\{f(\mathcal{E}_1)\}) = \phi^*(\{f(\mathcal{E}_1)\})$. If $\mathcal{E}_1 \in \mathcal{K}_1$ then for any $\mathcal{E}_2 \in \Sigma_2$ we have $\{\mathcal{E}_1\} \subseteq \mathcal{K}_1 \subseteq \mathcal{K}_1 \cup f^{-1}(\{\mathcal{E}_2\}) = \phi^*(\{\mathcal{E}_2\})$. *(ii)* $\phi(\mathcal{A}_1) = \bigcap\{\mathcal{B}_2 \,|\, \mathcal{A}_1 \subseteq \phi^*(\mathcal{B}_2)\} = \bigcap\{\mathcal{B}_2 \,|\, \mathcal{A}_1 \subseteq \mathcal{K}_1 \cup f^{-1}(\mathcal{B}_2)\}$. Let $\mathcal{A}_1 \subseteq \mathcal{K}_1 \cup f^{-1}(\mathcal{B}_2)$. Then $\mathcal{A}_1 \setminus \mathcal{K}_1 \subseteq f^{-1}(\mathcal{B}_2)$ since $\mathcal{K}_1$ and $f^{-1}(\mathcal{B}_2)$ are disjoint, so that $f(\mathcal{A}_1 \setminus \mathcal{K}_1) \subseteq \mathcal{B}_2$. Hence $f(\mathcal{A}_1 \setminus \mathcal{K}_1)^{\perp\perp} \subseteq \mathcal{B}_2$ and so $f(\mathcal{A}_1 \setminus \mathcal{K}_1)^{\perp\perp} \subseteq \phi(\mathcal{A}_1)$. Finally, $f(\mathcal{A}_1 \setminus \mathcal{K}_1) \subseteq f(\mathcal{A}_1 \setminus \mathcal{K}_1)^{\perp\perp}$ and so $\mathcal{A}_1 \setminus \mathcal{K}_1 \subseteq f^{-1}(f(\mathcal{A}_1 \setminus \mathcal{K}_1)) \subseteq f^{-1}(f(\mathcal{A}_1 \setminus \mathcal{K}_1)^{\perp\perp})$. Hence $\phi(\mathcal{A}_1) \subseteq f(\mathcal{A}_1 \setminus \mathcal{K}_1)^{\perp\perp}$, completing the proof. ∎

Lemma 5.7 **P** *is a functor.*

Proof: It is trivial that $\mathbf{P}(\mathrm{id}_\Sigma) = \mathrm{id}_\mathcal{L}$. We must then prove that the correspondence preserves composition. Let $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ and $g : \Sigma_2 \setminus \mathcal{K}_2 \to \Sigma_3$ be state space morphisms. Let the property lattice morphisms corresponding to f, g and $g \circ f$ be ϕ, ψ and χ respectively. Then $\chi^*(\mathcal{A}_3) = (\mathcal{K}_1 \cup f^{-1}(\mathcal{K}_2)) \cup (g \circ f)^{-1}(\mathcal{A}_3) = \mathcal{K}_1 \cup f^{-1}(\mathcal{K}_2 \cup g^{-1}(\mathcal{A}_3)) = \mathcal{K}_1 \cup f^{-1}(\psi^*(\mathcal{A}_3)) = (\phi^* \circ \psi^*)(\mathcal{A}_3) = (\psi \circ \phi)^*(\mathcal{A}_3)$. Hence $\chi^* = (\psi \circ \phi)^*$. ∎

Finally we have the following result, which establishes the relationship between <u>State</u> and <u>Prop</u>:

Theorem 5.8 <u>*State*</u> *and* <u>*Prop*</u> *are equivalent.*

Proof: We must show that *(i)* **S** is full and faithful (bijective on each Hom-set), and *(ii)* $(\mathbf{SP})(\Sigma, \perp)$ is isomorphic to $(\Sigma, \perp)$. *(i)* Let $\phi, \tilde{\phi} : \mathcal{L}_1 \to \mathcal{L}_2$ be such that $\mathbf{S}(\phi) = \mathbf{S}(\tilde{\phi}) = f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$. Then $\mathcal{K}_1 = \{p_1 \,|\, \phi(p_1) = 0_2\} = \{p_1 \,|\, \tilde{\phi}(p_1) = 0_2\}$ and if $p_1 \notin \mathcal{K}_1$ we have $\phi(p_1) = f(p_1) = \tilde{\phi}(p_1)$. Hence **S** is injective on each Hom-set. On the other hand, let $f : \Sigma_1 \setminus \mathcal{K}_1 \to \Sigma_2$ be a morphism and define $\phi : \mathcal{L}_1 \to \mathcal{L}_1 : a_1 \mapsto \bigvee\{f(p_1) \,|\, p_1 \notin \mathcal{K}_1,\ p_1 < a_1\}$. Then ϕ is a morphism. Indeed $\phi(0_1) = 0_2$ and $\phi(\bigvee\{a_{1,r}) = \bigvee\{f(p_1) \,|\, p_1 \notin \mathcal{K}_1,\ p_1 < \bigvee\{a_{1,r}\}\} = \bigvee\{(\bigvee\{f(p_1) \,|\, p_1 \notin \mathcal{K}_1,\ p_1 < a_{1,r}\})\} = \bigvee\{\phi(a_{1,r})\}$. Finally, if $p_1 \in \mathcal{L}$ is an atom then either $p_1 \in \mathcal{K}_1$ and $\phi(p_1) = 0_2$ or $p_1 \notin \mathcal{K}_1$ and $\phi(p_1) = f(p_1)$, which is an atom. Trivially we have that $\mathbf{S}(\phi) = f$ so that **S** is surjective on each Hom-set. *(ii)* Let $(\Sigma, \perp)$ be a state space and $\mathcal{L} = \mathbf{P}(\Sigma)$ so that $\mathbf{S}(\mathcal{L}) = \{\{\mathcal{E}\} \,|\, \mathcal{E} \in \Sigma\}$. The map $f : \{\mathcal{E}\} \mapsto \mathcal{E}$ is then an isomorphism. It is clearly bijective with empty kernel. Further $\mathcal{E}_i \perp \mathcal{E}_j$ if and only if $\mathcal{E}_i \in \{\mathcal{E}_j\}^\perp$ or equivalently $\{\mathcal{E}_i\} \subseteq \{\mathcal{E}_j\}^\perp$; that is $\{\mathcal{E}_i\} \perp \{\mathcal{E}_j\}$. Hence the inverse image of a biorthogonal subset of $\mathbf{S}(\mathcal{L})$ is biorthogonal and the image of a biorthogonal subset of Σ is biorthogonal, completing the proof. ∎

6 Classical Variables

A property $a \in \mathcal{L}$ is called classical if for each atom $p \in \mathcal{L}$ either $p < a$ or $p < a'$. Clearly 0 and 1 are both classical. Recall that a lattice is called distributive if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in \mathcal{L}$. On the other hand, a state space $(\Sigma, \perp)$ is called classical if any two distinct states are orthogonal. The following result is standard:

Lemma 6.1 *Let $(\Sigma, \perp)$ be a state space with corresponding property lattice $\mathcal{L}$. Then the following are equivalent:*

(i) $(\Sigma, \perp)$ is classical,

(ii) $\mathcal{L}$ is distributive,

(iii) each property $a \in \mathcal{L}$ is classical.

Proof: (i) $\Longrightarrow$ (ii): Let $(\Sigma, \perp)$ be a classical state space and $\mathcal{L}$ be the corresponding property lattice. All subsets of Σ are biorthogonal so that $\bigvee\{\mathcal{A}_r\} = \bigcup\{\mathcal{A}_r\}$ and so the greatest lower bound and least upper bound distribute.

(ii) $\Longrightarrow$ (iii): Let $\mathcal{L}$ be a distributive property lattice, $a \in \mathcal{L}$ and p be an atom of $\mathcal{L}$. Then $p = p \wedge 1 = p \wedge (a \vee a') = (p \wedge a) \vee (p \wedge a')$. However $p \wedge a < p$ so that either $p \wedge a = p$ and so $p < a$ or $p \wedge a = 0$. But in this case $p = 0 \vee (p \wedge a') = p \wedge a'$ and so $p < a'$.

(iii) $\Longrightarrow$ (i): Let $\mathcal{L}$ be a property lattice for which each property is classical and let p and q be atoms of $\mathcal{L}$. Then, since q is classical by hypothesis, either $p < q$ and so $p = q$ or $p < q'$ and so $p \perp q$. ■

Let us define the category PSet, whose objects are sets and whose morphisms are partially defined maps.

Theorem 6.2 *The free state space over PSet is classical.*

Proof: We must show that (i) State is concrete over PSet, and (ii) each free state space is classical. (i) The map $(\Sigma, \perp) \to \Sigma$, $f \mapsto f$ is clearly a functor, since the composition law in the two categories is the same, and is trivially faithful. (ii) Let Σ be a set and set $\mathcal{E}_1 \perp \mathcal{E}_2$ if $\mathcal{E}_1 \neq \mathcal{E}_2$ so that $(\Sigma, \perp)$ is classical. Now every subset of Σ is biorthogonal and so any partially defined map from $(\Sigma, \perp)$ to a state space is a morphism. Define $\mathbf{F} : \Sigma \mapsto (\Sigma, \perp)$, $f \mapsto f$. We associate to each morphism $f : \mathbf{F}\Sigma \to (\Sigma', \perp')$ the map $\phi f : \Sigma \to \Sigma' : \mathcal{E} \mapsto f(\mathcal{E})$. It is then trivial that ϕ is a bijection, and that $\mathbf{F}$ is a left adjoint of $\mathbf{U}$ since the composition laws are the same in the two categories. ■

The rest of this section will be devoted to proving that any property lattice can be decomposed into its so-called irreducible components, a property lattice being called irreducible if its only classical properties are 0 and 1. First we define the centre $\mathcal{Z}$ of $\mathcal{L}$ to be the set of classical properties. The next two results are standard:

Lemma 6.3 *The centre is a distributive property sublattice of $\mathcal{L}$.*

Proof: We first show that $\mathcal{Z}$ is a complete subalgebra of $\mathcal{L}$; that is that it is closed under the orthocomplementation and greatest lower bound. If a is classical then a' is trivially classical. Let $\{a_r\}$ be a family of classical properties and set $a = \bigwedge\{a_r\}$. Let $p \not< a$. Then since either $p < a_r$ or $p < a'_r$ there exists a_{r_0} such that $p < a'_{r_0}$. But $a' = \bigvee\{a'_r\}$ and so $p < a'$. Hence a is classical and $\mathcal{Z}$ is a complete subalgebra of $\mathcal{L}$.

We next show that the atoms of $\mathcal{Z}$ are exactly the elements $c_p = \bigwedge\{a \in \mathcal{Z} \,|\, p < a\}$ for p an atom of $\mathcal{L}$. The sets are nonempty since $1 \in \mathcal{Z}$. Let $x \in \mathcal{Z}$ with $x < c_p$. Then, since x is classical, either $p < x$ in which case $c_p < x$ and so $x = c_p$ or $p < x'$. In this case $c_p < x'$ so that $x < c_p \wedge c'_p = 0$. Hence each c_p is an atom of $\mathcal{Z}$. Now let x be an atom of $\mathcal{Z}$. Then x is non-zero and so there exists an atom p of $\mathcal{L}$ such that $p < x$ by atomisticity in $\mathcal{L}$. But in this case $c_p < x$ so that $x = c_p$. The atoms of $\mathcal{Z}$ are then exactly the c_p.

We now show that $\mathcal{Z}$ is atomistic. It is trivial that $\bigvee\{c_p \,|\, c_p < x\} < x$. Further, for x classical, $p < x$ if and only if $c_p < x$. But then by atomisticity in $\mathcal{L}$ we have that $x = \bigvee\{p \,|\, p < x\} < \bigvee\{c_p \,|\, p < x\} = \bigvee\{c_p \,|\, c_p < x\}$ and so $\mathcal{Z}$ is atomistic. Finally it is trivial that $\mathcal{Z}$ is distributive by theorem 6.1. Indeed let c_p and c_q be atoms of $\mathcal{Z}$. Then either $p < c_q$ so that $c_p < c_q$ and hence $c_p = c_q$ or $p < c'_q$ so that $c_p < c'_q$ and the corresponding states are orthogonal. ■

Next, let $c_p \in \mathcal{Z}$ be an atom and define $\mathcal{L}_{c_p} = [0, c_p] = \{a \in \mathcal{L} \,|\, a < c_p\}$. We have the following result:

Lemma 6.4 *$\mathcal{L}_{c_p}$ is an irreducible property lattice with order $<$ and orthocomplementation $a^r = a' \wedge c_p$.*

Proof: The existence of a greatest lower bound in $\mathcal{L}_{c_p}$ is trivial and the map $a \mapsto a^r$ is well defined on $\mathcal{L}_{c_p}$ since $a^r < c_p$ for all a. $a \wedge a^r = a \wedge (a' \wedge c_p) = (a \wedge a') \wedge c_p = 0$. Let $a < b$ so that $b' < a'$ and $a' \wedge b' = b'$. Then $a^r \wedge b^r = (a' \wedge c_p) \wedge (b' \wedge c_p) = (a' \wedge b') \wedge c_p = b' \wedge c_p = b^r$ and $b^r < a^r$. Let $a < c_p$ so that $c'_p < a'$. Then by atomisticity $a' = \bigvee\{q \,|\, q < a'\} = \big(\bigvee\{r < c_p \,|\, r < a'\}\big) \vee \big(\bigvee\{s < c'_p \,|\, s < a'\}\big) = (a' \wedge c_p) \vee c'_p$. Hence $a = a'' = (a' \wedge c_p)' \wedge c_p = (a^r)^r$ and so $a \mapsto a^r$ is an orthocomplementation on $\mathcal{L}_{c_p}$. Finally $\mathcal{L}_{c_p}$ is trivially atomistic since $\mathcal{L}$ is, and is trivially irreducible since c_p is a minimal non-zero classical property of $\mathcal{L}$. ■

We are now in a position to prove that any property lattice can be decomposed into its irreducible components:

Theorem 6.5 Σ *is the coproduct of the* Σ_{c_p}.

Proof: A coproduct of a family $\{A_j \mid j \in J\}$ of objects in a given category $\underline{X}$, if it exists, can be proved to be an object A of $\underline{X}$ and a family of morphisms $g_j : A_j \to A$ such that for any object B of $\underline{X}$ and any family of morphisms $h_j : A_j \to B$ there exists a unique morphism $f : A \to B$ such that $h_j = f \circ g_j$.

Now $p < q'$ in $\mathcal{L}$ if and only if $c_p \neq c_q$ or $p < q^r$ in $\mathcal{L}_{c_p}$. Indeed, if $c_p \neq c_q$ then $p < c_q' < q'$, and if $p < q^r$ in c_p then $p < q^r = q' \wedge c_p < q'$. On the other hand, let $p < q'$. Then either $q < c_p'$ in which case $p < c_p < q'$, or $q < c_p$ in which case $p = p \wedge q' = (p \wedge c_p) \wedge q' = p \wedge (q' \wedge c_p) = p \wedge q^r$ so that $p < q^r$. Hence Σ is the disjoint union of the family $\{(\Sigma_j, \perp_j) \mid j \in J\}$, with $\mathcal{E} \in \Sigma_j$ and $\tilde{\mathcal{E}} \in \Sigma_k$ orthogonal if and only if $j \neq k$ or $j = k$ and $\mathcal{E} \perp \tilde{\mathcal{E}}$.

We now show that $(\Sigma, \perp)$ is the coproduct of $\{(\Sigma_j, \perp_j) \mid j \in J\}$. We use the fact that $\mathcal{A}$ is biorthogonal in Σ if and only if each $\mathcal{A}_j = \mathcal{A} \cap \Sigma_j$ is biorthogonal in Σ_j. Indeed, if $\mathcal{E}_j \in \mathcal{A}_j$ then $\mathcal{E}_j \in \mathcal{A}^\perp$ if and only if $\mathcal{E}_j \in \mathcal{A}_j^{\perp_j}$ since for any $\mathcal{E} \notin \mathcal{A}_j$ we have that $\mathcal{E}_j \perp \mathcal{E}$. Hence $\mathcal{A}^\perp = \bigcup_j \mathcal{A}_j^{\perp_j}$. Now $\mathcal{A}_j^{\perp_j} \subseteq \Sigma_j$ by definition and so $\mathcal{A}^{\perp\perp} = \bigcup_j \mathcal{A}^{\perp_j \perp_j}$. We then need merely note that $\mathcal{A} = \bigcup_j \mathcal{A}_j$. Let us define the maps $g_j : \Sigma_j \to \Sigma : \mathcal{E}_j \mapsto \mathcal{E}_j$. Let $\mathcal{A}$ be biorthogonal in Σ. Then $g_j^{-1}(\mathcal{A}) = \mathcal{A} \cap \Sigma_j$ is birthogonal in Σ_j by the above. Hence the g_j are morphisms with empty kernel. Finally, let $(\Sigma', \perp')$ be a state space and $h_j : \Sigma_j \setminus \mathcal{K}_j \mapsto \Sigma'$ be morphisms. If there exists a morphism $f : \Sigma \setminus \mathcal{K} \to \Sigma'$ such that $h_j = f \circ g_j$ then necessarily we must have $\mathcal{K}_j = g_j^{-1}(\mathcal{K}) = \mathcal{K} \cap \Sigma_j$. Hence $\mathcal{K} = \bigcup_j \mathcal{K}_j$. Further, if $\mathcal{E}_j \notin \mathcal{K}_j$ then we must have $h_j(\mathcal{E}_j) = f(\mathcal{E}_j)$. We must show that h_j is indeed a morphism. Let $\mathcal{A}$ be biorthogonal in Σ'. Then since the h_j are morphisms we have that $\mathcal{K}_j \cup h_j^{-1}(\mathcal{A})$ is biorthogonal in Σ_j. However $\mathcal{K} \cup f^{-1}(\mathcal{A}) = \bigcup_j \left(\mathcal{K}_j \cup h_j^{-1}(\mathcal{A})\right)$ which is biorthogonal by the above. Hence f is a morphism and $(\Sigma, \perp)$ is the coproduct. ■

We note that the atoms of the centre $\mathcal{Z}$ are often called superselection rules in a usage derived from that of G. C. Wick, A. S. Wightman and E. P. Wigner [1952]. The existence of such classical variables is important in discussions of, for example, elementary particles as defined via imprimitivity systems [Giovannini and Piron 1979], the two body system [Piron 1965], unstable systems [Piron 1969], the quantum electromagnetic field [D'Emma 1980] and chirality [Amann 1988; Pfeifer 1983].

7 Hemimorphisms

In the lattice context it is useful to also consider a larger set of maps, called hemimorphisms. These are maps $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ which satisfy (PM1) and (PM2):

(PM1) $\phi(0_1) = 0_2$,

(PM2) $\phi(\bigvee_\alpha a_{1,\alpha}) = \bigvee_\alpha \phi(a_{1,\alpha})$ for any non-empty family $\{a_{1,\alpha}\}$.

Clearly the identity maps are hemimorphisms and the composition of two hemimorphisms is again a hemimorphism. A hemimorphism preserves the order since it preserves the supremum. It is easy to construct hemimorphisms which are not morphisms. Indeed let $\mathcal{L}_1$ and $\mathcal{L}_2$ be property lattices with $a_1 \in \mathcal{L}_1$ and $a_2 \in \mathcal{L}_2$. We define the constant map onto a_2 with kernel a_1 by $c(x_1) = 0_2$ if $x_1 < a_1$ and $c(x_1) = a_2$ otherwise. Then c is a hemimorphism, but is not a morphism unless a_2 is either an atom or 0_2.

I now introduce the important notion of adjoint [Foulis 1960]. Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ and $\psi : \mathcal{L}_2 \to \mathcal{L}_1$ be hemimorphisms. Then ϕ and ψ are called adjoint if $\psi\big(\phi(a_1)'\big) < a_1'$ and $\phi\big(\psi(a_2)'\big) < a_2'$ for all $a_1 \in \mathcal{L}_1$ and $a_2 \in \mathcal{L}_2$. The following result is due to C. Piron [1995]:

Theorem 7.1 *Any hemimorphism $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ has a unique adjoint $\phi^\dagger : \mathcal{L}_2 \to \mathcal{L}_1$ given by $\phi^\dagger(a_2) = \bigwedge\{x_1' \,|\, \phi(x_1) < a_2'\}$. Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ and $\psi : \mathcal{L}_2 \to \mathcal{L}_3$ be hemimorphisms. Then $\phi^{\dagger\dagger} = \phi$ and $(\psi \circ \phi)^\dagger = \phi^\dagger \circ \psi^\dagger$.*

Proof: We first show unicity. Let $\phi^\dagger$ and ϕ^* both be adjoint to ϕ. Then $a_2 < \big(\phi(\phi^\dagger a_2)'\big)'$ and so $\phi^*(a_2) < \phi^*\big((\phi(\phi^\dagger a_2)')'\big) < \phi^\dagger(a_2)$. Inversing the argument we have $\phi^* = \phi^\dagger$.

We now show existence. $\phi^\dagger\big(\phi(a_1)'\big) = \bigwedge\{x_1' \,|\, \phi(x_1) < \phi(a_1)\} < a_1'$ by considering $x_1 = a_1$. $\phi\big(\phi^\dagger(a_2)'\big) = \phi\big(\bigwedge\{x_1' \,|\, \phi(x_1) < a_2'\}'\big) = \phi\big(\bigvee\{x_1 \,|\, \phi(x_1) < a_2'\}\big) = \bigvee\{\phi(x_1) \,|\, \phi(x_1) < a_2'\} < a_2'$.

It therefore remains to show that $\phi^\dagger$ is a hemimorphism. $\phi^\dagger(0_2) = \bigwedge_\alpha\{x_1' \,|\, \phi(x_1) < 1_2\} = \bigwedge \mathcal{L}_1 = 0_1$. $\phi^\dagger$ preserves the order. Indeed let $a_2 < b_2$. Then $\phi^\dagger(a_2) = \bigwedge\{x_1 \,|\, a_2 < \phi(x_1')'\} < \bigwedge\{x_1 \,|\, b_2 < \phi(x_1')'\} = \phi^\dagger(b_2)$. Hence $\bigvee_\alpha \phi^\dagger(a_{2,\alpha}) < \phi^\dagger(\bigvee_\alpha a_{2,\alpha})$. Let $b_1 < \phi^\dagger(a_{2,\alpha})'$ for all $a_{2,\alpha}$. Then $\phi(b_1) < \phi\big(\phi^\dagger(a_{2,\alpha})'\big) < a_{2,\alpha}'$ so that $a_{2,\alpha} < \phi(b_1)'$ for all $a_{2,\alpha}$. But then $\bigvee_\alpha a_{2,\alpha} < \phi(b_1)'$ so that $\phi^\dagger(\bigvee_\alpha a_{2,\alpha}) < \phi^\dagger(\phi(b_1)') < b_1'$. Setting $b_1 = \big(\bigvee_\alpha \phi^\dagger(a_{2,\alpha})\big)'$ completes the proof.

$\phi^{\dagger\dagger} = \phi$ trivially since the conditions on an adjoint pair are symmetric. $\phi^\dagger\psi^\dagger\big((\psi\phi)(a_1)'\big) < \phi^\dagger\big(\phi(a_1)'\big) < a_1'$ and $\psi\phi\big((\phi^\dagger\psi^\dagger)(a_3)'\big) < \psi\big(\psi^\dagger(a_3)'\big) < a_3'$, the desired result then following from unicity. ■

I shall need the following simple lemma in the following:

Lemma 7.2 *Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a hemimorphism. If $a_2 < \phi(1_1)$ and $\phi^\dagger(a_2) = 0_1$ then $a_2 = 0_2$.*

Proof: Let $\phi^\dagger(a_2) = 0_1$. Then $\phi^\dagger(a_2)' = 1_1$ so that $\phi(1_1) = \phi\big(\phi^\dagger(a_2)'\big) < a_2'$ and $a_2 < \phi(1_1)'$. However $a_2 < \phi(1_1)$ so that $a_2 < \phi(1_1)' \wedge \phi(1_1) = 0_2$. ■

A morphism $\phi : \mathcal{L}_1 \mapsto \mathcal{L}_2$ is called a homomorphism if the hemimorphism $\phi^\dagger$ is also a morphism. It is easy to construct morphisms which are not homomorphisms. Indeed, let $\mathcal{L}_1 = \{0_1, p_1, p_1', 1_1\}$ and $\mathcal{L}_2 = \{0_2, p_2, p_2', q_2, q_2', 1_2\}$. Define $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ by $\phi(0_1) = 0_2$, $\phi(p_1) = p_2$, $\phi(p_1') = p_2'$ and $\phi(1_1) = 1_2$. Then ϕ is clearly a morphism, however $\phi^\dagger(q_2) = \bigwedge\{x_1 \mid q_2 < \phi(x_1')'\} = 1_1$ and so ϕ is not a homomorphism.

The notion of adjoint enables the definition of standard operator theoretic concepts in the lattice context. For example, a hemimorphism $u : \mathcal{L}_1 \to \mathcal{L}_2$ is called an isometry if $u^\dagger \circ u = \mathrm{id}_1$ and unitary if in addition $u \circ u^\dagger = \mathrm{id}_2$.

Theorem 7.3 *Let $u : \mathcal{L}_1 \to \mathcal{L}_2$ be a hemimorphism. Then the following are equivalent:*

(i) u is an isometry,

(ii) $a_1 < b_1'$ iff $u(a_1) < u(b_1)'$,

(iii) $u^\dagger(1_2) = 1_1$ and $u(a_1') < u(a_1)'$ for all $a_1 \in \mathcal{L}_1$.

Proof: *(i)* $\Longrightarrow$ *(ii)*: Let $a_1 < b_1'$. Then $a_1 < u^\dagger u(b_1)'$ and so $u(a_1) < u\big(u^\dagger u(b_1)'\big) < u(b_1)'$. Let $u(a_1) < u(b_1)'$. Then $a_1 = u^\dagger u(a_1) < u^\dagger\big(u(b_1)'\big) < b_1'$.

(ii) $\Longrightarrow$ *(iii)*: $a_1 = a_1''$ so that $u(a_1) < u(a_1')'$ and $u(a_1') < u(a_1)'$. Let $u^\dagger(1_2) = a_1$. Then $u^\dagger(1_2)' = a_1'$ and so $u(a_1') = u\big(u^\dagger(1_2)'\big) < 1_2' = 0_2$. Thus $1_2 = 0_2' = u(a_1')'$ and so $u(1_1) < 1_2 = u(a_1')'$. In this case $1_1 < a_1'' = a_1$ and so $u^\dagger(1_2) = 1_1$.

(iii) $\Longrightarrow$ *(i)*: $u^\dagger u(a_1) = u^\dagger u(a_1'') < u^\dagger\big(u(a_1')'\big) < a_1'' = a_1$. Let $p_1 \in \mathcal{L}_1$ be an atom. Then $u^\dagger u(p_1) < p_1$ and so $u^\dagger u(p_1) = 0_1$ or p_1. Let $u^\dagger u(p_1) = 0_1$. Then $u(p_1) = 0_2$ since $u(p_1) < u(1_1)$. However $p_1 < 1_1 = u^\dagger(1_2)$ so that $p_1 = 0_1$, which is impossible. Hence for each atom $p_1 \in \mathcal{L}_1$ we have that $u^\dagger u(p_1) = p_1$. Let $a_1 \in \mathcal{L}_1$. By atomisticity $a_1 = \bigvee\{p_1 \mid p_1 < a_1\}$. Hence $u^\dagger u(a_1) = u^\dagger u\big(\bigvee\{p_1 \mid p_1 < a_1\}\big) = u^\dagger\big(\bigvee\{u(p_1) \mid p_1 < a_1\}\big) = \bigvee\{u^\dagger u(p_1) \mid p_1 < a_1\} = \bigvee\{p_1 \mid p_1 < a_1\} = a_1$. ∎

The two conditions in *(iii)* are independent. Indeed $\phi : a_1 \mapsto 0_2$ is a hemimorphism which satisfies $\phi(a_1') < \phi(a_1)'$ but is not an isometry. Note that an isometric hemimorphism need not be a morphism. Indeed, let $\mathcal{L}_1 = \{0_1, 1_1\}$ and $\mathcal{L}_2$ be any property lattice. Define $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ by $\phi(0_1) = 0_2$ and $\phi(1_1) = 1_2$. Then ϕ is an isometry since $\phi^\dagger\phi(1_1) = \phi^\dagger(1_2) = \bigwedge\{x_1 \mid 1_2 < \phi(x_1')'\} = 1_1$. However ϕ is not a morphism if $\mathcal{L}_2$ has more than two elements.

Lemma 7.4 *Let $u : \mathcal{L}_1 \to \mathcal{L}_2$ be a homomorphism such that $u(a_1) < u(b_1)'$ implies $a_1 < b_1'$. Then u is an isometry.*

Proof: $u\big(u^\dagger u(p_1)'\big) < u(p_1)'$ and so $u^\dagger u(p_1)' < p_1'$. Thus $p_1 < u^\dagger u(p_1)$. However u is a homomorphism and so $u^\dagger u(p_1)$ is either an atom or 0_1. Hence $u^\dagger u(p_1) = p_1$ and so $u^\dagger u(a_1) = a_1$ by atomisticity. ∎

Theorem 7.5 *Let $u : \mathcal{L}_1 \to \mathcal{L}_2$ be a hemimorphism. Then u is unitary if and only if it is bijective and $u(a_1') = u(a_1)'$ for each $a_1 \in \mathcal{L}$.*

Proof: Let u be unitary. Let $a_2 \in \mathcal{L}_2$ and set $a_1 = u^\dagger(a_2)$. Then $u(a_1) = uu^\dagger(a_2) = a_2$ and so u is surjective. Let $u(a_1) = u(b_1)$. Then $a_1 = u^\dagger u(a_1) = u^\dagger u(b_1) = b_1$ and so u is injective. Since u is an isometry we have that $u(a_1') < u(a_1)'$. Further, since $u^\dagger$ is an isometry we have that $u^\dagger(a_2') < u^\dagger(a_2)'$. Setting $a_2 = u(a_1)$ we have that $u(a_1)' = uu^\dagger\big(u(a)'\big) < u\big(u^\dagger u(a_1)'\big) = u(a_1')$ and so $u(a_1') = u(a_1)'$.

Let u be bijective with $u(a_1') = u(a_1)'$ for each $a_1 \in \mathcal{L}_1$. Note that $u(1_1) = u(0_1') = u(0_1)' = 0_2' = 1_2$ Then $u^\dagger(1_2) = \bigwedge\{x_1 \mid 1_2 < u(x_1')'\} = \bigwedge\{x_1 \mid 1_2 < u(x_1)\} = 1_1$ since u is injective. Hence u is an isometry. Let $a_2 \in \mathcal{L}_2$. Then since u is surjective there exists $a_1 \in \mathcal{L}_1$ such that $a_2 = u(a_1)$. Hence $u^\dagger(a_2') = u^\dagger u(a_1)' = u^\dagger u(a_1') = a_1' = \big(u^\dagger u(a_1)\big)' = u^\dagger(a_2)'$. Hence $u^\dagger$ is an isometry and so u is unitary. ∎

REFERENCES

J. Adámek, H. Herrlich and G. E. Strecker 1990; "Abstract and Concrete Categories" John Wiley & Sons (New York).

D. Aerts 1982; "Description of many separated physical entities without the paradoxes encountered in quantum mechanics" *Found. Phys.* **12** 1131-1170.

D. Aerts 1994; "Quantum structures, separated physical entities and probability" *Found. Phys.* **24** 1227-1259.

A. Amann 1988; "Chirality as a classical observable in algebraic quantum mechanics" in A. Amann, L. Cederbaum and W. Gans (eds.) *Fractals, Quasicrystals, Chaos, Knots and Algebraic Quantum Mechanics* Kluwer Academic (Dordrecht).

G. Birkhoff 1973; "Lattice Theory" Third Edition, American Mathematical Society (Providence). First Edition 1940.

F. Borceux 1994; "Handbook of Categorical Algebra" Cambridge University Press (Cambridge).

G. Cattaneo and G. Nisticò 1991; "Axiomatic foundations of quantum physics: critiques and misunderstandings. Piron's question-proposition system" *Int. J. Theor. Phys.* **30** 1293-1336.

G. Cattaneo and G. Nisticò 1993; "A model of Piron's preparation-question structures in Ludwig's selection structures" *Int. J. Theor. Phys.* **32** 407-431.

G. C. D'Emma 1980; "On quantization of the electromagnetic field" *Helv. Phys. Acta* **53** 535-551.

Cl.-A. Faure and A. Frölicher 1993; "Morphisms of projective geometries and of corresponding lattices" *Geom. Ded.* **47** 25-40.

Cl.-A. Faure and A. Frölicher 1994; "Morphisms of projective geometries and semilinear maps" *Geom. Ded.* **53** 937-969.

Cl.-A. Faure and A. Frölicher 1995a; "Dualities for infinite-dimensional projective geometries" *Geom. Ded.* **56** 225-236.

Cl.-A. Faure and A. Frölicher 1995b; "Categorical aspects in projective geometry" Preprint.

D. J. Foulis 1960; "Baer *-semigroups" *Proc. AMS* **11** 648-654.

D. J. Foulis, C. Piron and C. H. Randall 1983; "Realism, operationalism, and quantum mechanics" *Found. Phys.* **13** 813-841.

D. J. Foulis and C. H. Randall 1984; "A note on misunderstandings of Piron's axioms for quantum mechanics" *Found. Phys.* **14** 65-88.

N. Giovannini and C. Piron 1979; "On the group-theoretical foundations of classical and quantum physics: kinematics and state spaces" *Helv. Phys. Acta* **52** 518-540.

S. P. Gudder and J. R. Michel 1981; "Representations of Baer *-semigroups" *Proc. AMS* **81** 157-163.

G. Ludwig and H. Neumann 1981; "Connections between different approaches to the foundations of quantum mechanics" in H. Neumann (ed.) *Interpretations and Foundations of Quantum Theory* B. I. Wissenschaftsverlag (Mannheim).

S. Mac Lane 1971; "Categories for the Working Mathematician" Springer-Verlag (New York).

J.-P. Marquis 1995; "Category theory and the foundations of mathematics: philosophical excavations" *Synthese* **103** 421-447.

P. Pfeifer 1983; "Molecular structure derived from first-principles quantum mechanics: two examples" in J. Hinze (ed.) *Energy Storage and Redistribution in Molecules* Plenum (New York).

C. Piron 1964; "Axiomatique quantique" *Helv. Phys. Acta* **37** 439-468.

C. Piron 1965; "Sur la quantification du système de deux particules" *Helv. Phys. Acta* **38** 104-108.

C. Piron 1969; "Les règles de supersélection continues" *Helv. Phys. Acta* **42** 330-338.

C. Piron 1976; "Foundations of Quantum Physics" W. A. Benjamin (Reading, Massachusetts).

C. Piron 1990; "Mécanique quantique bases et applications" Presses polytechniques et universitaires romandes (Lausanne).

C. Piron 1995; "Morphisms, hemimorphisms and Baer *-semigroups" *Int. J. Theor. Phys*

34 1681-1687.

J. C. T. Pool 1968a; "Baer*-semigroups and the logic of quantum mechanics" *Comm. Math. Phys.* **9** 118-141.

J. C. T. Pool 1968b; "Semimodularity and the logic of quantum mechanics" *Comm. Math. Phys.* **9** 212-228.

C. H. Randall and D. J. Foulis 1983; "Properties and operational propositions in quantum mechanics" *Found. Phys.* **13** 843-857.

G. T. Rüttimann 1975; "Decomposition of projections on orthomodular lattices" *Can. Math. Bull.* **18** 263-267.

H. Wang 1974; "From Mathematics to Philosophy" Routledge and Kegan Paul (London).

G. C. Wick, A. S. Wightman and E. P. Wigner 1952; "The intrinsic parity of elementary particles" *Phys. Rev.* **88** 101-105.