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The Yang–Mills Gauge Field Theory in the Context of a Generalized BRST–Formalism Including Translations

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Abstract. We discuss the algebraic renormalization of the Yang–Mills gauge field theory in the presence of translations. Due to the translations the algebra between Sorella's δ–operator, the exterior derivative and the BRST–operator closes. Therefore, we are able to derive an integrated parameter formula collecting in an elegant and compact way all nontrivial solutions of the descent equations.

1 Introduction

In modern physics gauge field theories are essential in describing the properties of matter and its interactions. Usually, such theories are quantized using the BRST–formalism [1]. This technique together with the Quantum Action Principle [2] allow for a fully algebraic proof of their renormalizability [3]. Indeed, one can calculate the invariant lagrangians and anomalies, corresponding to a set of field transformations, as the nontrivial solutions of the BRST consistency condition [4, 5]. The latter constitutes a cohomology problem [6] due to the nilpotency of the BRST–operator. Using the algebraic Poincaré lemma [7, 8] one arrives at a tower of descent equations.

In the case of a pure Yang–Mills gauge field theory an algebraic method for solving the descent equations has been proposed in [9]. It is based on the decomposition of the

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exterior spacetime derivative as a BRST–commutator [3],

$$[\delta, s_g] = d \ . \tag{1.1}$$

However, the algebra between the BRST–operator s_g, the exterior spacetime derivative d and the δ–operator does not close. This leads to the presence of a further operator [9],

$$\mathcal{G} = \frac{1}{2}[d, \delta] \ , \tag{1.2}$$

inducing an additional tower of descent equations, which has to be solved.

Inspired by the results of [10], we show that an incorporation of the translations into the BRST–transformations leads to the disappearance of the $\mathcal{G}$–operator. Due to this fact we are able to collect the general nontrivial solutions for the cocycles of the descent equations into an integrated parameter formula, which will be analogously derived to the well–known Chern–Simons formula [11].

The work is organized as follows. In section two the generalized BRST–formalism including translations will be introduced and applied to pure Yang–Mills gauge field theory [12]. The BRST–invariance of the action defining the tree approximation will be expressed by a Slavnov–Taylor identity [3, 12]. Then in section three we present the functional algebra obeyed by the Slavnov–Taylor operator and further functional differential operators which appear in the constraints defining the tree approximation of the theory. In section four we study the algebraic renormalization. We proof the stability of the action defining the tree approximation and the discussion of the anomaly problem will be done. In section five we show that the general nontrivial solution for the cocycles of the descent equations corresponding to the full BRST–operator s is given in terms of the general nontrivial solution for the cocycles of the descent equations in the gauge sector. Moreover, the decomposition of the exterior spacetime derivative as a BRST–commutator will be discussed. Guided by Stora's derivation of the Chern–Simons formula [11], we derive an integrated parameter formula (see also [10]), which presents an elegant and compact way to collect all the cocycles of the descent equations in the gauge sector in one expression. This integrated parameter formula will be used to compute the gauge anomaly in four dimensions.

2 The Yang–Mills Model

The classical dynamics of pure Yang–Mills gauge field theory is defined by the gauge invariant action [4]

$$S_{inv} = -\frac{1}{4} Tr \int d^4x \, F_{\mu\nu} F^{\mu\nu} \ . \tag{2.1}$$

The field strength $F_{\mu\nu}$ is related to the gauge field A_μ by the structure equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \ . \tag{2.2}$$

[3]Remark that we denote the BRST–operator in [9] by s_g, since the letter s will be reserved for the full BRST–operator including the translations.

[4]As usual, Greek indices refer to the Minkowskian space–time.

The gauge transformation of the gauge field is given by

$$\delta_\omega A_\mu = D_\mu \omega = \partial_\mu \omega - i[A_\mu, \omega] \ , \tag{2.3}$$

where D_μ denotes the covariant derivative. All fields are Lie–algebra valued, i.e. $A_\mu = A_\mu^A T^A$ and $F_{\mu\nu} = F_{\mu\nu}^A T^A$, where T^A are the generators of the gauge group $\mathbf{G}$ in the adjoint representation [5].

Moreover, the classical action (2.1) is also invariant under infinitesimal translations in the Minkowskian space–time,

$$\delta_\varepsilon A_\mu(x) = \varepsilon^\lambda \partial_\lambda A_\mu(x) \ , \tag{2.4}$$

with ε^λ as the infinitesimal global parameter of translations obeying $\partial_\mu \varepsilon^\nu = 0$.

In order to combine the gauge invariance and the global invariance under translations into a single one, we define the generalized gauge transformations according to

$$\delta := \delta_\omega + \delta_\varepsilon \ . \tag{2.5}$$

To quantize the model we introduce the corresponding nilpotent BRST–operator [1],

$$s = s_g + s_T \ , \tag{2.6}$$

where s_g and s_T are the BRST–operators corresponding to the gauge transformations and the translations, respectively. The latter obey the algebra

$$s_g^2 = s_T^2 = \{s_g, s_T\} = 0 \ . \tag{2.7}$$

Explicitely, the several BRST-transformations are given by

$$\begin{aligned} sA_\mu &= D_\mu c + \xi^\nu \partial_\nu A^\mu \ , \\ sc &= icc + \xi^\nu \partial_\nu c \ , \\ s\xi^\mu &= 0 \ , \end{aligned} \tag{2.8}$$

where c is the Lie–algebra valued gauge ghost and ξ^μ is a global ghost associated to the translations, obeying $\partial_\mu \xi^\nu = 0$.

Pure Yang–Mills gauge field theory in the tree approximation is described in the Landau gauge by the action [3, 12, 13]

$$\Gamma^{(0)} = S_{inv} + S_{gf} + S_{\Phi\Pi} + S_{ext} \ , \tag{2.9}$$

where

$$\begin{aligned} S_{inv} &= -\frac{1}{4} Tr \int d^4x \, F_{\mu\nu} F^{\mu\nu} \ , \\ S_{gf} + S_{\Phi\Pi} &= s \, Tr \int d^4x \, \bar{c} \, \partial^\mu A_\mu \ , \\ S_{ext} &= s \, Tr \int d^4x \, (-\rho^\mu A_\mu + \sigma c) \ , \end{aligned} \tag{2.10}$$

[5]Gauge group indices are denoted by capital latin letters. The matrices T^A are hermitian and traceless, and they obey $[T^A, T^B] = if^{ABC} T^C$, $Tr(T^A T^B) = \delta^{AB}$, where f^{ABC} are the real and totally antisymmetric structure constants of the gauge group.

are respectively the gauge invariant classical action, the sum of the gauge fixing term and the Faddeev–Popov term, and the source term. Before discussing S_{inv}, $S_{gf} + S_{\Phi\Pi}$, S_{ext} and explaining the fields appearing in $\Gamma^{(0)}$, we present the BRST–transformations of the remaining fields:

$$\begin{aligned} s\bar{c} &= b + \xi^\nu \partial_\nu \bar{c} \quad , \quad sb = \xi^\nu \partial_\nu b \ , \\ s\rho^\mu &= \xi^\nu \partial_\nu \rho^\mu \quad , \quad s\sigma = \xi^\nu \partial_\nu \sigma \ , \end{aligned} \tag{2.11}$$

whereby the antighost $\bar{c}$ and the multiplier field b transform as a BRST–doublet in the gauge sector. The external sources ρ^μ and σ are BRST–invariant in the gauge sector. Due to the nilpotency of the BRST–operator the sum of the gauge fixing term and the Faddeev–Popov term as well as the source term are invariant under the BRST–transformation.

The sum of the gauge fixing term and the Faddeev–Popov term becomes

$$S_{gf} + S_{\Phi\Pi} = Tr \int d^4x \, [b\,\partial^\mu A_\mu - \bar{c}\,\partial^\mu (D_\mu c)] \ . \tag{2.12}$$

In addition, the source term is given by

$$S_{ext} = Tr \int d^4x \, (\rho^\mu D_\mu c + \sigma icc) \ . \tag{2.13}$$

The partial derivative ∂_μ does not change the ghost number, whereas the BRST–operator raises the ghost number by one unit. The canonical dimension and the ghost number of the gauge field A_μ, the gauge ghost c, the antighost $\bar{c}$, the multiplier field b, the external sources ρ^μ and σ and the global translation ghost ξ^μ are collected in the following table:

	A_μ	c	$\bar{c}$	b	ρ^μ	σ	ξ^μ
dim	1	0	2	2	3	4	-1
$Q_{\Phi\Pi}$	0	1	-1	0	-1	-2	1

Table 1: Dimensions and Faddeev-Popov ghost charges of the fields.

3 The Functional Algebra

The Slavnov–Taylor identity in the gauge sector,

$$\mathcal{S}_g(\Gamma^{(0)}) = 0 \ , \tag{3.1}$$

with

$$\mathcal{S}_g(\Gamma^{(0)}) = Tr \int d^4x \left\{ \frac{\delta\Gamma^{(0)}}{\delta\rho^\mu(x)} \frac{\delta\Gamma^{(0)}}{\delta A_\mu(x)} + \frac{\delta\Gamma^{(0)}}{\delta\sigma(x)} \frac{\delta\Gamma^{(0)}}{\delta c(x)} + b(x) \frac{\delta\Gamma^{(0)}}{\delta\bar{c}(x)} \right\} \ , \tag{3.2}$$

and the Ward–identity describing the invariance under translations,

$$\mathcal{S}_T(\Gamma^{(0)}) = \xi^\mu \mathcal{P}_\mu \Gamma^{(0)} = 0 \ , \tag{3.3}$$

where the generator of the translations is given by [6]

$$\mathcal{P}_\mu = Tr \int d^4x \sum_\phi (\partial_\mu \phi(x)) \frac{\delta}{\delta \phi(x)} , \tag{3.4}$$

can be collected into a single Slavnov–Taylor identity

$$\mathcal{S}(\Gamma^{(0)}) = 0 , \tag{3.5}$$

with

$$\begin{aligned}\mathcal{S}(\Gamma^{(0)}) = Tr \int d^4x \Bigg\{ & \left[\frac{\delta\Gamma^{(0)}}{\delta\rho^\mu(x)} + \xi^\nu\partial_\nu A_\mu(x)\right] \frac{\delta\Gamma^{(0)}}{\delta A_\mu(x)} + [\xi^\nu\partial_\nu\rho^\mu(x)] \frac{\delta\Gamma^{(0)}}{\delta\rho^\mu(x)} \\ + & \left[\frac{\delta\Gamma^{(0)}}{\delta\sigma(x)} + \xi^\nu\partial_\nu c(x)\right] \frac{\delta\Gamma^{(0)}}{\delta c(x)} + [\xi^\nu\partial_\nu\sigma(x)] \frac{\delta\Gamma^{(0)}}{\delta\sigma(x)} \\ + & [b(x) + \xi^\nu\partial_\nu\bar{c}(x)] \frac{\delta\Gamma^{(0)}}{\delta\bar{c}(x)} + [\xi^\nu\partial_\nu b(x)] \frac{\delta\Gamma^{(0)}}{\delta b(x)} \Bigg\} .\end{aligned} \tag{3.6}$$

The Slavnov–Taylor identity (3.5) describes the invariance of $\Gamma^{(0)}$ under the BRST–transformations (2.8) and (2.11).

The gauge condition is given by

$$\frac{\delta\Gamma^{(0)}}{\delta b(x)} = \partial^\mu A_\mu(x) , \tag{3.7}$$

and the global ghost equation,

$$\frac{\partial\Gamma^{(0)}}{\partial\xi^\mu} = 0 , \tag{3.8}$$

shows that the action defining the tree approximation does not depend on the global translation ghost ξ^μ.

Moreover, one can derive a local antighost equation which controls the dependence of $\Gamma^{(0)}$ on the antighost $\bar{c}$. It can be obtained by commuting the gauge condition with the Slavnov–Taylor identity (3.5):

$$\mathcal{G}(x)\Gamma^{(0)} = \left\{\frac{\delta}{\delta\bar{c}(x)} + \partial^\mu \frac{\delta}{\delta\rho^\mu(x)}\right\} \Gamma^{(0)} = 0 , \tag{3.9}$$

with the local antighost operator $\mathcal{G}(x)$ [7].

In the Landau gauge [3, 13] one can derive an integrated ghost equation which controls the dependence of $\Gamma^{(0)}$ on the ghost field c. By calculating the functional derivative of $\Gamma^{(0)}$ with respect to the ghost field c and using the gauge condition one finds [3]

$$\begin{aligned}\frac{\delta\Gamma^{(0)}}{\delta c(x)} + i\left[\bar{c}(x), \frac{\delta\Gamma^{(0)}}{\delta b(x)}\right] &= \partial_\mu \left(D^\mu\bar{c}(x) + \rho^\mu(x)\right) \\ &+ i\left[\rho^\mu(x), A_\mu(x)\right] - i\left[\sigma(x), c(x)\right] .\end{aligned} \tag{3.10}$$

[6]The sum runs over all fields ϕ of the model.

[7]Remark that the local antighost operator $\mathcal{G}(x)$ has nothing to do with the $\mathcal{G}$–operator given in (1.2).

One observes that the nonlinear terms appear in a total divergence. Therefore, an integration over spacetime will remove the nonlinear terms. This leads to the integrated ghost equation

$$\mathcal{H}\Gamma^{(0)} = \int d^4x \left\{ \frac{\delta}{\delta c(x)} + i \left[\bar{c}(x), \frac{\delta}{\delta b(x)} \right] \right\} \Gamma^{(0)} = \Delta_g \ , \tag{3.11}$$

where $\mathcal{H}$ denotes the integrated ghost operator and

$$\Delta_g = \int d^4x \ \{ i \left[\rho^\mu(x), A_\mu(x) \right] - i \left[\sigma(x), c(x) \right] \} \tag{3.12}$$

is a classical breaking, i.e. it is linear in the quantum fields.

The invariance of the action defining the tree approximation under rigid gauge transformation is expressed by the Ward–identity

$$\mathcal{W}_{rig}\Gamma^{(0)} = 0 \ , \tag{3.13}$$

where the Ward–identity operator $\mathcal{W}_{rig}$ is given by

$$\begin{aligned} \mathcal{W}_{rig} = \int d^4x \ \Bigg(& i \left[A_\mu(x), \frac{\delta}{\delta A_\mu(x)} \right] + i \left\{ \rho^\mu(x), \frac{\delta}{\delta \rho^\mu(x)} \right\} \\ & + i \left\{ c(x), \frac{\delta}{\delta c(x)} \right\} + i \left[\sigma(x), \frac{\delta}{\delta \sigma(x)} \right] \\ & + \ i \left\{ \bar{c}(x), \frac{\delta}{\delta \bar{c}(x)} \right\} + i \left[b(x), \frac{\delta}{\delta b(x)} \right] \Bigg) \ . \end{aligned} \tag{3.14}$$

The Slavnov–Taylor operator acting on an arbitrary functional $\mathcal{F}$ with even ghost charge is

$$\begin{aligned} \mathcal{S}(\mathcal{F}) = Tr \int d^4x \ \Bigg\{ & \left[\frac{\delta \mathcal{F}}{\delta \rho^\mu(x)} + \xi^\nu \partial_\nu A_\mu(x) \right] \frac{\delta \mathcal{F}}{\delta A_\mu(x)} + \left[\xi^\nu \partial_\nu \rho^\mu(x) \right] \frac{\delta \mathcal{F}}{\delta \rho^\mu(x)} \\ + \ & \left[\frac{\delta \mathcal{F}}{\delta \sigma(x)} + \xi^\nu \partial_\nu c(x) \right] \frac{\delta \mathcal{F}}{\delta c(x)} + \left[\xi^\nu \partial_\nu \sigma(x) \right] \frac{\delta \mathcal{F}}{\delta \sigma(x)} \\ + \ & \left[b(x) + \xi^\nu \partial_\nu \bar{c}(x) \right] \frac{\delta \mathcal{F}}{\delta \bar{c}(x)} + \left[\xi^\nu \partial_\nu b(x) \right] \frac{\delta \mathcal{F}}{\delta b(x)} \Bigg\} \ , \end{aligned} \tag{3.15}$$

and the linearized Slavnov–Taylor operator can be written as

$$\begin{aligned} \mathcal{S}_\mathcal{F} = Tr \int d^4x \ \Bigg\{ & \left[\frac{\delta \mathcal{F}}{\delta \rho^\mu(x)} + \xi^\nu \partial_\nu A_\mu(x) \right] \frac{\delta}{\delta A_\mu(x)} + \left[\frac{\delta \mathcal{F}}{\delta A_\mu(x)} + \xi^\nu \partial_\nu \rho^\mu(x) \right] \frac{\delta}{\delta \rho^\mu(x)} \\ + \ & \left[\frac{\delta \mathcal{F}}{\delta \sigma(x)} + \xi^\nu \partial_\nu c(x) \right] \frac{\delta}{\delta c(x)} + \left[\frac{\delta \mathcal{F}}{\delta c(x)} + \xi^\nu \partial_\nu \sigma(x) \right] \frac{\delta}{\delta \sigma(x)} \\ + \ & \left[b(x) + \xi^\nu \partial_\nu \bar{c}(x) \right] \frac{\delta}{\delta \bar{c}(x)} + \left[\xi^\nu \partial_\nu b(x) \right] \frac{\delta}{\delta b(x)} \Bigg\} \ . \end{aligned} \tag{3.16}$$

The functional algebra which is valid for any functional $\mathcal{F}$ with even ghost charge is given by the following relations:

- The nilpotency of the Slavnov–Taylor operator is contained in the two following identities:

$$\mathcal{S}_{\mathcal{F}}\left(\mathcal{S}(\mathcal{F})\right) = 0 \ , \tag{3.17}$$

$$\mathcal{S}(\mathcal{F}) = 0 \qquad \Longrightarrow \qquad \mathcal{S}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}} = 0 \ . \tag{3.18}$$

- Commuting the partial derivative with respect to the global translation ghost with the Slavnov–Taylor operator, one gets

$$\frac{\partial}{\partial \xi^\mu}\mathcal{S}(\mathcal{F}) + \mathcal{S}_{\mathcal{F}}\left(\frac{\partial \mathcal{F}}{\partial \xi^\mu}\right) = \mathcal{P}_\mu \mathcal{F} \ ,$$

$$\mathcal{P}_\mu \mathcal{S}(\mathcal{F}) - \mathcal{S}_{\mathcal{F}}(\mathcal{P}_\mu \mathcal{F}) = 0 \ . \tag{3.19}$$

- Commuting the gauge condition with the Slavnov–Taylor operator, one obtains

$$\frac{\delta}{\delta b(x)}\mathcal{S}(\mathcal{F}) - \mathcal{S}_{\mathcal{F}}\left(\frac{\delta \mathcal{F}}{\delta b(x)} - \partial^\mu A_\mu(x)\right) = \mathcal{G}(x)\mathcal{F} - \xi^\nu \partial_\nu \left(\frac{\delta \mathcal{F}}{\delta b(x)} - \partial^\mu A_\mu(x)\right) ,$$

$$\mathcal{G}(x)\mathcal{S}(\mathcal{F}) + \mathcal{S}_{\mathcal{F}}\left[\mathcal{G}(x)\mathcal{F}\right] = \xi^\nu \partial_\nu \left[\mathcal{G}(x)\mathcal{F}\right] \ . \tag{3.20}$$

- Commuting the integrated ghost operator with the Slavnov–Taylor operator, gives

$$\mathcal{H}\mathcal{S}(\mathcal{F}) + \mathcal{S}_{\mathcal{F}}(\mathcal{H}\mathcal{F} - \Delta_g) = \mathcal{W}_{rig}\mathcal{F} \ ,$$

$$\mathcal{W}_{rig}\mathcal{S}(\mathcal{F}) - \mathcal{S}_{\mathcal{F}}(\mathcal{W}_{rig}\mathcal{F}) = 0 \ . \tag{3.21}$$

4 Renormalization, Stability and Anomalies

The aim of the renormalization is to construct an extension of the theory at the tree level to all orders of perturbation theory. This extension will be described by the vertex functional,

$$\Gamma = \Gamma^{(0)} + \mathcal{O}(\hbar) \ , \tag{4.1}$$

generating the 1PI Green functions. One has to examine whether this vertex functional obeys a Slavnov–Taylor identity, being nonlinear in Γ,

$$\mathcal{S}(\Gamma) = 0 \ . \tag{4.2}$$

Since the constraints being linear in the quantum fields are renormalizable to all orders of perturbation theory, the following relations are valid [3, 13]:

- the gauge condition,

$$\frac{\delta \Gamma}{\delta b(x)} = \partial^\mu A_\mu(x) \ , \tag{4.3}$$

- the integrated ghost equation,

$$\mathcal{H}\Gamma = \Delta_g \ , \tag{4.4}$$

- the global ghost equation,

$$\frac{\partial\Gamma}{\partial\xi^\mu} = 0 \ , \tag{4.5}$$

- the local antighost equation,

$$\mathcal{G}(x)\Gamma = 0 \ , \tag{4.6}$$

- the invariance under rigid gauge transformations,

$$\mathcal{W}_{rig}\Gamma = 0 \ , \tag{4.7}$$

- the invariance under translations,

$$\mathcal{S}_T(\Gamma) = \xi^\mu \mathcal{P}_\mu \Gamma = 0 \ . \tag{4.8}$$

Within the framework of the algebraic renormalization procedure [3], based on the general grounds of power counting and locality, the discussion of the extension of the theory in the tree approximation to all orders of perturbation theory is organized according to two independent parts: First, the study of the stability of the classical action under radiative corrections. This amounts to find the invariant counterterms and to check if they all correspond to a renormalization of the free parameters of the classical theory. Second, the search for anomalies, i.e. the investigation whether the symmetries of the theory survive in the presence of radiative corrections.

4.1 Stability

In order to check that the action in the tree approximation is stable under radiative corrections, one perturbs it by an arbitrary integrated local functional Σ_c,

$$\Sigma = \Gamma^{(0)} + \alpha\Sigma_c \quad , \quad \alpha\Sigma_c = \mathcal{O}(\hbar) \ , \tag{4.9}$$

where α is an infinitesimal parameter with vanishing canonical dimension and vanishing ghost number. The functional Σ_c has the same quantum numbers as the action in the tree approximation[8].

One requires that the perturbed action Σ satisfies the same constraints defining the theory at the tree level, i.e. (4.2)–(4.8). The perturbation Σ_c a priori depends on all fields and on the parameter ξ^μ,

$$\Sigma_c = \Sigma_c[A_\mu, c, \bar{c}, b, \rho^\mu, \sigma](\xi^\mu) \ . \tag{4.10}$$

The gauge condition, the integrated ghost equation and the global ghost equation together with the same constraints for the action at the tree level imply

$$\frac{\delta\Sigma_c}{\delta b(x)} = 0 \quad , \quad \mathcal{H}\Sigma_c = 0 \quad , \quad \frac{\partial\Sigma_c}{\partial\xi^\mu} = 0 \ . \tag{4.11}$$

[8]One has $dim(d^4x) = -4$.

Therefore, the perturbation Σ_c does not depend on the multiplier field $b(x)$ and the global translation ghost ξ^μ, i.e.

$$\Sigma_c = \Sigma_c[A_\mu,\, c,\, \bar{c},\, \rho^\mu,\, \sigma] \ . \tag{4.12}$$

Since

$$\tilde{\mathcal{H}}\Sigma_c = \int d^4x \, \frac{\delta \Sigma_c}{\delta c(x)} = 0 \ , \tag{4.13}$$

the dependence on the ghost $c(x)$ has to be a total divergence. The local antighost equation, the invariance under rigid gauge transformations and the invariance under translations together with the same constraints for the action at the tree level imply

$$\mathcal{G}(x)\Sigma_c = 0 \quad , \quad \mathcal{W}_{rig}\Sigma_c = 0 \quad , \quad \mathcal{P}_\mu\Sigma_c = 0 \ . \tag{4.14}$$

From eqs.(4.11) and (4.14) follows

$$\begin{aligned} \tilde{\mathcal{W}}_{rig}\Sigma_c = & \int d^4x \left(i \left[A_\mu(x), \frac{\delta \Sigma_c}{\delta A_\mu(x)} \right] + i \left\{ \rho^\mu(x), \frac{\delta \Sigma_c}{\delta \rho^\mu(x)} \right\} + i \left\{ c(x), \frac{\delta \Sigma_c}{\delta c(x)} \right\} \right. \\ & + \left. i \left[\sigma(x), \frac{\delta \Sigma_c}{\delta \sigma(x)} \right] + i \left\{ \bar{c}(x), \frac{\delta \Sigma_c}{\delta \bar{c}(x)} \right\} \right) = 0 \ , \end{aligned} \tag{4.15}$$

$$\begin{aligned} \tilde{\mathcal{P}}_\mu\Sigma_c = Tr & \int d^4x \left[\partial_\mu A_\nu(x) \, \frac{\delta \Sigma_c}{\delta A_\nu(x)} + \partial_\mu \rho^\nu(x) \, \frac{\delta \Sigma_c}{\delta \rho^\nu(x)} + \partial_\mu c(x) \, \frac{\delta \Sigma_c}{\delta c(x)} \right. \\ & + \left. \partial_\mu \sigma(x) \, \frac{\delta \Sigma_c}{\delta \sigma(x)} + \partial_\mu \bar{c}(x) \, \frac{\delta \Sigma_c}{\delta \bar{c}(x)} \right] = 0 \ . \end{aligned} \tag{4.16}$$

The local antighost equation $\mathcal{G}(x)\Sigma_c = 0$ suggests the following change of field variables

$$\hat{\rho}^\mu(x) = \rho^\mu(x) + \partial^\mu \bar{c}(x) \quad , \quad \hat{\bar{c}}(x) = \bar{c}(x) \ , \tag{4.17}$$

with

$$\Sigma_c[A_\mu,\, c,\, \bar{c},\, \rho^\mu,\, \sigma] = \bar{\Sigma}_c[A_\mu,\, c,\, \hat{\bar{c}},\, \hat{\rho}^\mu,\, \sigma] \ . \tag{4.18}$$

Then the local antighost equation becomes

$$\hat{\mathcal{G}}(x)\bar{\Sigma}_c = \frac{\delta \bar{\Sigma}_c}{\delta \hat{\bar{c}}(x)} = 0 \ . \tag{4.19}$$

Therefore, the perturbation $\bar{\Sigma}_c$ depends on ρ^μ and $\bar{c}$ only through the combination $\hat{\rho}^\mu = \rho^\mu + \partial^\mu \bar{c}$, i.e.

$$\bar{\Sigma}_c = \bar{\Sigma}_c[A_\mu,\, c,\, \hat{\rho}^\mu,\, \sigma] \ . \tag{4.20}$$

Applying the Slavnov–Taylor operator to the perturbed action, one gets

$$\mathcal{S}(\Sigma) = \mathcal{S}(\Gamma^{(0)} + \alpha\Sigma_c) = \mathcal{S}(\Gamma^{(0)}) + \alpha\, \mathcal{S}_{\Gamma^{(0)}}(\Sigma_c) + \mathcal{O}(\alpha^2) \ . \tag{4.21}$$

Using the Slavnov–Taylor identity for the action in the tree approximation (3.5), the Slavnov–Taylor identity imposed to the perturbed action (4.9) translates at the first order in α into the following condition on the perturbation Σ_c:

$$\mathcal{S}_{\Gamma^{(0)}}(\Sigma_c) = 0 \ . \tag{4.22}$$

This equation is the BRST consistency condition in the ghost number sector zero. It constitutes a cohomology problem, due to the nilpotency of the linearized Slavnov–Taylor operator,

$$\mathcal{S}_{\Gamma^{(0)}}\mathcal{S}_{\Gamma^{(0)}} = 0 \ , \tag{4.23}$$

which follows from the validity of the Slavnov–Taylor identity for the action in the tree approximation (3.5), where (3.18) has been used. The solution of the BRST consistency condition in the ghost number sector zero can always be written as the sum of a trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\hat{\Sigma}$, where $\hat{\Sigma}$ has ghost number -1, and the nontrivial elements belonging to the cohomology of $\mathcal{S}_{\Gamma^{(0)}}$ in the ghost number sector zero, i.e. which cannot be written as $\mathcal{S}_{\Gamma^{(0)}}$–variations:

$$\Sigma_c = \Sigma_{ph} + \mathcal{S}_{\Gamma^{(0)}}\hat{\Sigma} \ . \tag{4.24}$$

The trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\hat{\Sigma}$ corresponds to field renormalizations which are unphysical. Using the functional derivatives of $\Gamma^{(0)}$ with respect to the gauge field A_μ, the gauge ghost c and the classical external sources ρ^μ and σ, the invariance of $\bar{\Sigma}_c$ under translations and since $\delta\bar{\Sigma}_c/\delta b(x) = 0$, the BRST consistency condition reads in the new variables

$$\begin{aligned}\mathcal{S}_{\Gamma^{(0)}}(\bar{\Sigma}_c) = Tr\int d^4x \Big[& D_\mu c(x)\,\frac{\delta\bar{\Sigma}_c}{\delta A_\mu(x)} + \left(D_\nu F^{\nu\mu}(x) + i\left\{c(x),\hat{\rho}^\mu(x)\right\}\right)\frac{\delta\bar{\Sigma}_c}{\delta\hat{\rho}^\mu(x)} \\ + \ & ic(x)c(x)\frac{\delta\bar{\Sigma}_c}{\delta c(x)} + \left(D_\mu\hat{\rho}^\mu(x) + i\left[c(x),\sigma(x)\right]\right)\frac{\delta\bar{\Sigma}_c}{\delta\sigma(x)}\Big] = 0 \ .\end{aligned} \tag{4.25}$$

Therefore, the perturbation $\bar{\Sigma}_c = \bar{\Sigma}_c[A_\mu,\, c,\, \hat{\rho}^\mu,\, \sigma]$ is an integrated local functional with canonical dimension zero and ghost number zero obeying the following set of constraints:

- the integrated ghost equation,

$$\hat{\mathcal{H}}\bar{\Sigma}_c = \int d^4x\,\frac{\delta\bar{\Sigma}_c}{\delta c(x)} = 0 \ , \tag{4.26}$$

- the invariance under rigid gauge transformations,

$$\begin{aligned}\hat{\mathcal{W}}_{rig}\bar{\Sigma}_c = \int d^4x \Bigg(& i\left[A_\mu(x),\frac{\delta\bar{\Sigma}_c}{\delta A_\mu(x)}\right] + i\left\{\hat{\rho}^\mu(x),\frac{\delta\bar{\Sigma}_c}{\delta\hat{\rho}^\mu(x)}\right\} \\ + \ & i\left\{c(x),\frac{\delta\bar{\Sigma}_c}{\delta c(x)}\right\} + i\left[\sigma(x),\frac{\delta\bar{\Sigma}_c}{\delta\sigma(x)}\right]\Bigg) = 0 \ ,\end{aligned} \tag{4.27}$$

- the invariance under translations,

$$\begin{aligned}\hat{\mathcal{P}}_\mu\bar{\Sigma}_c = Tr\int d^4x \Big[& \partial_\mu A_\nu(x)\,\frac{\delta\bar{\Sigma}_c}{\delta A_\nu(x)} + \partial_\mu\hat{\rho}^\nu(x)\,\frac{\delta\bar{\Sigma}_c}{\delta\hat{\rho}(x)} \\ + \ & \partial_\mu c(x)\,\frac{\delta\bar{\Sigma}_c}{\delta c(x)} + \partial_\mu\sigma(x)\,\frac{\delta\bar{\Sigma}_c}{\delta\sigma(x)}\Big] = 0 \ ,\end{aligned} \tag{4.28}$$

- and the BRST–consistency condition,

$$\begin{aligned}\mathcal{S}_{\Gamma^{(0)}}(\bar{\Sigma}_c) = Tr\int d^4x \Big[& c(x)\mathcal{D}(x)\bar{\Sigma}_c + D_\nu F^{\nu\mu}(x)\,\frac{\delta\bar{\Sigma}_c}{\delta\hat{\rho}^\mu(x)} \\ + \ & D_\mu\hat{\rho}^\mu(x)\,\frac{\delta\bar{\Sigma}_c}{\delta\sigma(x)}\Big] = 0 \ ,\end{aligned} \tag{4.29}$$

where the abbreviation

$$\mathcal{D}(x) = -D_\mu \frac{\delta}{\delta A_\mu(x)} + i\left\{\hat{\rho}^\mu(x), \frac{\delta}{\delta\hat{\rho}^\mu(x)}\right\} + \frac{i}{2}\left\{c(x), \frac{\delta}{\delta c(x)}\right\} + i\left[\sigma(x), \frac{\delta}{\delta\sigma(x)}\right] \tag{4.30}$$

has been used.

It will be shown in the appendix that the solution of the set of constraints (4.26)–(4.29) is given by

$$\bar{\Sigma}_c = -k\,\frac{1}{4}\,Tr\int d^4x\, F_{\mu\nu}(x)F^{\mu\nu}(x)\ , \tag{4.31}$$

which is the most general nontrivial perturbation of the action in the tree approximation. The perturbation $\bar{\Sigma}_c$ depends on the parameter k which corresponds to a possible multiplicative renormalization of the gauge coupling constant g being set in the whole work equal to one. This is an algebraic result which just shows that the nontrivial invariant counterterm (4.31) can be reabsorbed into the action at the tree level by a renormalization of its coefficient. It means that no new terms appear at the n–loop levels with $n \geq 1$. Therefore, the action in the tree approximation is stable.

4.2 Anomalies

In the following we investigate if the BRST–symmetry is preserved in the presence of radiative corrections. Indeed, the aim of the renormalization procedure is to examine if it is possible to define a vertex functional, $\Gamma = \Gamma^{(0)} + \mathcal{O}(\hbar)$, obeying as in the tree approximation the set of constraints (4.2)–(4.8). Since the operators in (4.3)–(4.8) are linear differential operators and since the breakings are linear in the quantum fields one can assume the validity of the equations (4.3)–(4.8) at the full quantum level, i.e. to all orders of perturbation theory.

Actually the program will fail because the nonlinear Slavnov–Taylor identity (4.2) will turn out to be anomalous

$$\mathcal{S}(\Gamma) = r\Delta_{AB}\ , \tag{4.32}$$

where Δ_{AB} is the anomaly to be derived in the following and r is a well–known function of order $\hbar$ of the coupling constant g, which however cannot be determined by the pure algebraic method used here. The search for the breaking Δ_{AB} of the Slavnov–Taylor identity requires some care.

The breaking is controlled by the quantum action principle, which implies

$$\mathcal{S}(\Gamma) = 0 + \hbar\,\Delta\cdot\Gamma\ , \tag{4.33}$$

where the quantum breaking $\Delta\cdot\Gamma$ is, at lowest order in $\hbar$, an integrated local functional

$$\Delta\cdot\Gamma = \Delta + \mathcal{O}(\hbar\Delta)\quad ,\quad \Delta = \int d^4x\,\Delta(x)\ , \tag{4.34}$$

with canonical dimension zero and Faddeev–Popov charge one. Thus one gets

$$\mathcal{S}(\Gamma) = 0 + \hbar\Delta + \mathcal{O}(\hbar^2) \ . \tag{4.35}$$

Applying the linear functional differential operator $\mathcal{S}_\Gamma$ and using the algebraic relation (3.17) one gets

$$0 = 0 + \hbar\, \mathcal{S}_\Gamma\Delta + \mathcal{O}(\hbar^2) \ . \tag{4.36}$$

From $\Gamma = \Gamma^{(0)} + \mathcal{O}(\hbar)$ follows

$$\mathcal{S}_\Gamma = \mathcal{S}_{\Gamma^{(0)}} + \mathcal{O}(\hbar) \ , \tag{4.37}$$

and eq.(4.36) becomes

$$0 = 0 + \hbar\, \mathcal{S}_{\Gamma^{(0)}}\Delta + \mathcal{O}(\hbar^2) \ . \tag{4.38}$$

Therefore, the integrated local functional Δ obeys the BRST consistency condition in the ghost number sector one,

$$\mathcal{S}_{\Gamma^{(0)}}\Delta = 0 \ . \tag{4.39}$$

The functional algebra (3.19)–(3.21), written for the functional Γ, the renormalized constraints (4.3)–(4.8), and the broken Slavnov–Taylor identity (4.35) lead to the following set of constraints which the breaking,

$$\Delta = \Delta[A_\mu,\, c,\, \bar{c},\, b,\, \rho^\mu,\, \sigma](\xi^\mu) \ , \tag{4.40}$$

has to obey:

$$\frac{\delta\Delta}{\delta b(x)} = 0 \ , \tag{4.41}$$

$$\mathcal{H}\Delta = 0 \ , \tag{4.42}$$

$$\frac{\partial\Delta}{\partial\xi^\mu} = 0 \ , \tag{4.43}$$

$$\mathcal{G}(x)\Delta = 0 \ , \tag{4.44}$$

$$\mathcal{W}_{rig}\Delta = 0 \ , \tag{4.45}$$

$$\mathcal{P}_\mu\Delta = 0 \ . \tag{4.46}$$

The constraints (4.41) and (4.43) imply that the breaking Δ does not depend on the multiplier field $b(x)$ and the global translation ghost ξ^μ,

$$\Delta = \Delta[A_\mu,\, c,\, \bar{c},\, \rho^\mu,\, \sigma] \ . \tag{4.47}$$

The integrated ghost equation (4.42) becomes

$$\hat{\mathcal{H}}\Delta = \int d^4x\, \frac{\delta\Delta}{\delta c(x)} = 0 \ , \tag{4.48}$$

stating that the dependence of the breaking Δ on the ghost $c(x)$ has to be a total divergence. The local antighost equation (4.44) will be transformed by the following change of field variables,

$$\hat{\rho}^\mu(x) = \rho^\mu(x) + \partial^\mu\bar{c}(x) \quad , \quad \hat{\bar{c}}(x) = \bar{c}(x) \ , \tag{4.49}$$

into the equation

$$\hat{\mathcal{G}}(x)\Delta = \frac{\delta\Delta}{\delta\hat{\bar{c}}(x)} = 0 \ . \tag{4.50}$$

This implies that the breaking Δ depends on ρ^μ and $\bar{c}$ only through the combination $\hat{\rho}^\mu = \rho^\mu + \partial^\mu \bar{c}$, i.e.

$$\Delta = \Delta[A_\mu, c, \hat{\rho}^\mu, \sigma] \ . \tag{4.51}$$

Furthermore, the breaking Δ has to be invariant under the rigid gauge transformations and translations. The BRST consistency condition,

$$\mathcal{S}_{\Gamma^{(0)}}\Delta = 0 \ , \tag{4.52}$$

constitutes a cohomology problem in the space of integrated local functionals with dimension zero and ghost number one due to the nilpotency of the linearized Slavnov–Taylor operator $\mathcal{S}_{\Gamma^{(0)}}$, i.e.

$$\mathcal{S}_{\Gamma^{(0)}}\mathcal{S}_{\Gamma^{(0)}} = 0 \ , \tag{4.53}$$

which follows from the validity of the Slavnov–Taylor identity (3.5). The solution of the BRST consistency condition (4.52) can always be written as the sum of a trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\hat{\Delta}$, where $\hat{\Delta}$ has ghost number zero, and the nontrivial elements belonging to the cohomology of $\mathcal{S}_{\Gamma^{(0)}}$ in the ghost number sector one, i.e. which cannot be written as $\mathcal{S}_{\Gamma^{(0)}}$–variations,

$$\Delta = \mathcal{S}_{\Gamma^{(0)}}\hat{\Delta} + \Delta^* \ . \tag{4.54}$$

The trivial cocycle $\mathcal{S}_{\Gamma^{(0)}}\hat{\Delta}$ can be absorbed into the vertex functional Γ as an integrated local noninvariant counterterm $-\hbar\hat{\Delta}$,

$$\Gamma \quad \longrightarrow \quad \Gamma - \hbar\hat{\Delta} \ , \tag{4.55}$$

leading to

$$\mathcal{S}(\Gamma - \hbar\hat{\Delta}) = \mathcal{S}(\Gamma) - \hbar\, \mathcal{S}_\Gamma \hat{\Delta} + \mathcal{O}(\hbar^2) = \hbar\Delta^* + \mathcal{O}(\hbar^2) \ . \tag{4.56}$$

The construction of the explicit form of the anomaly Δ^* is explained in the final part of this subsection. The BRST consistency condition in the ghost number sector one becomes in the new variables (4.49)

$$\begin{aligned} \mathcal{S}_{\Gamma^{(0)}}\Delta = Tr \int d^4x \Bigg[& c(x)\mathcal{D}(x)\Delta + D_\nu F^{\nu\mu}(x)\, \frac{\delta\Delta}{\delta\hat{\rho}^\mu(x)} \\ + \ & D_\mu \hat{\rho}^\mu(x)\, \frac{\delta\Delta}{\delta\sigma(x)} \Bigg] = 0 \ , \end{aligned} \tag{4.57}$$

with the abbreviation (4.30). Before solving the consitency condition (4.57), i.e. deriving the anomaly Δ^*, one eliminates the dependence of Δ on the external sources $\hat{\rho}^\mu$ and σ. The most general dependence on σ which is compatible with dimension and ghost number is [3]

$$\Delta = l_1\, Tr \int d^4x\, \sigma(x)c(x)c(x)c(x) + \dots \ , \tag{4.58}$$

where l_1 is an arbitrary constant and the low dots denote the terms which are independent of σ. Using

$$\begin{aligned} \mathcal{S}_{\Gamma^{(0)}}\sigma(x) &= i\,[c(x),\sigma(x)] + \dots\ , \\ \mathcal{S}_{\Gamma^{(0)}}c(x) &= ic(x)c(x)\ , \end{aligned} \tag{4.59}$$

the consistency condition yields

$$0 = \mathcal{S}_{\Gamma^{(0)}}\Delta = -i\,l_1\,Tr\int d^4x\,\sigma(x)c^4(x) + \dots\ , \tag{4.60}$$

implying

$$l_1 = 0\ . \tag{4.61}$$

Therefore, Δ is independent of σ,

$$\frac{\delta\Delta}{\delta\sigma(x)} = 0\ . \tag{4.62}$$

The most general dependence on $\hat{\rho}^\mu$ which is compatible with dimension and ghost number is given by [3]

$$\Delta = Tr\int d^4x\,\hat{\rho}^\mu(x)R_\mu(A,c)(x) + \dots\ , \tag{4.63}$$

where now the low dots denote the terms which are independent of $\hat{\rho}^\mu$ and $R_\mu(A,c)$ is the most general polynomial depending on A_μ and c with canonical dimension one and ghost number two

$$\begin{aligned} R_\mu(A,c)(x) &= l_2\partial_\mu c(x)c(x) + l_3c(x)\partial_\mu c(x) + l_4A_\mu(x)c(x)c(x) \\ &+\ l_5c(x)A_\mu(x)c(x) + l_6c(x)c(x)A_\mu(x)\ . \end{aligned} \tag{4.64}$$

The constants l_2, l_3, l_4, l_5 and l_6 are arbitrary. In order to restrict the coefficients l_2, l_3, l_4, l_5, l_6 one uses the BRST consistency condition (4.57) and the fact that the σ–dependence has been already eliminated, (4.62). Writing only the terms depending on $\hat{\rho}^\mu$, the l.h.s of the consistency condition becomes

$$\begin{aligned} \mathcal{S}_{\Gamma^{(0)}}\Delta &= Tr\int d^4x\ \{(-l_4 - il_2)\hat{\rho}^\mu(x)\partial_\mu c(x)c(x)c(x) + l_5\hat{\rho}^\mu(x)c(x)\partial_\mu c(x)c(x) \\ &+\ (-l_6 + il_3)\hat{\rho}^\mu(x)c(x)c(x)\partial_\mu c(x) - il_5\hat{\rho}^\mu(x)c(x)A_\mu(x)c(x)c(x) \\ &+\ il_5\hat{\rho}^\mu(x)c(x)c(x)A_\mu(x)c(x)\} + \dots\ . \end{aligned} \tag{4.65}$$

The consistency condition (4.57) implies the following relations between the constants l_2, l_3, l_4, l_5, l_6:

$$l_4 = -il_2\quad,\quad l_5 = 0\quad,\quad l_6 = il_3\ , \tag{4.66}$$

and $R_\mu(A,c)$ becomes

$$\begin{aligned} R_\mu(A,c)(x) &= l_2\partial_\mu c(x)c(x) + l_3c(x)\partial_\mu c(x) - il_2A_\mu(x)c(x)c(x) \\ &+ il_3c(x)c(x)A_\mu(x)\ . \end{aligned} \tag{4.67}$$

Therefore, the most general dependence on $\hat{\rho}^\mu$ is given by

$$\begin{aligned} \Delta &= Tr\int d^4x\,\hat{\rho}^\mu(x)\,[l_2\partial_\mu c(x)c(x) + l_3c(x)\partial_\mu c(x) \\ &-\ il_2A_\mu(x)c(x)c(x) + il_3c(x)c(x)A_\mu(x)] + \dots\ . \end{aligned} \tag{4.68}$$

Moreover, it follows that Δ can be written as the $\mathcal{S}_{\Gamma^{(0)}}$–variation of

$$\hat{\Delta} = Tr \int d^4x \, \left[-l_2 \hat{\rho}^\mu(x) A_\mu(x) c(x) + l_3 \hat{\rho}^\mu(x) c(x) A_\mu(x) \right] \, , \tag{4.69}$$

up to terms independent of $\hat{\rho}^\mu$ which are denoted by the low dots,

$$\mathcal{S}_{\Gamma^{(0)}} \hat{\Delta} = \Delta + \dots \, . \tag{4.70}$$

Hence the $\hat{\rho}^\mu$–dependence of the breaking Δ is trivial and it can be absorbed into Γ according to eq.(4.55), which leads to

$$\frac{\delta \Delta}{\delta \hat{\rho}^\mu(x)} = 0 \, . \tag{4.71}$$

From eqs.(4.62) and (4.71) follow that the breaking Δ is independent of $\hat{\rho}^\mu$ and σ. Therefore, it does only depend on A_μ and c,

$$\Delta = \Delta[A_\mu, c] \, . \tag{4.72}$$

Finally, let us discuss the well–known derivation of the breaking (4.72). It will be shown that it is equal to the gauge anomaly in four dimensions. The breaking (4.72) is the general nontrivial solution of the BRST consistency condition,

$$\mathcal{S}_{\Gamma^{(0)}} \Delta = Tr \int d^4x \, \left[D_\mu c(x) \, \frac{\delta \Delta}{\delta A_\mu(x)} + i c(x) c(x) \, \frac{\delta \Delta}{\delta c(x)} \right] = 0 \, . \tag{4.73}$$

Moreover, it obeys the integrated ghost equation,

$$\int d^4x \, \frac{\delta \Delta}{\delta c(x)} = 0 \, , \tag{4.74}$$

the Ward–identity describing the invariance under rigid gauge transformations,

$$\int d^4x \, \left(i \left[A_\mu(x), \frac{\delta \Delta}{\delta A_\mu(x)} \right] + i \left\{ c(x), \frac{\delta \Delta}{\delta c(x)} \right\} \right) = 0 \, , \tag{4.75}$$

and the Ward–identity describing the invariance under translations,

$$Tr \int d^4x \, \left[\partial_\mu A_\nu(x) \frac{\delta \Delta}{\delta A_\nu(x)} + \partial_\mu c(x) \frac{\delta \Delta}{\delta c(x)} \right] = 0 \, . \tag{4.76}$$

The general nontrivial solution of the set of constraints (4.73)–(4.76) is given by the Adler–Bardeen nonabelian gauge anomaly, $\Delta = r\mathcal{A}$,

$$\mathcal{A} = Tr \int d^4x \, \epsilon^{\mu\nu\sigma\tau} \, c(x) \, \partial_\mu \left[\partial_\nu A_\sigma(x) A_\tau(x) + \frac{i}{2} A_\nu(x) A_\sigma(x) A_\tau(x) \right] \, . \tag{4.77}$$

The actual presence of the gauge anomaly depends on the nonvanishing of its coefficient r, which cannot be determined by the algebraic renormalization [3].

The explicit calculation of (4.77) is presented in [9], where an operator δ, which allows to decompose the exterior derivative,

$$[\delta, s_g] = d \;, \tag{4.78}$$

is introduced. In [9] the algebra between the operators s_g, d and δ does not close. This leads to the presence of a further operator,

$$\mathcal{G} = \frac{1}{2}[d, \delta] \;, \tag{4.79}$$

inducing an additional tower of descent equations which has to be solved in order to compute the general nontrivial solution of the descent equations [9].

In the next section an alternative algebraic method to solve the BRST consistency condition will be presented. It is based on the use of the full BRST–operator,

$$s = s_g + s_T \;, \tag{4.80}$$

including the translations, which allow for avoiding the operator $\mathcal{G}$. This will be possible due to the fact that the s–cohomology is isomorphic to that of s_g up to trivial contributions.

5 The BRST Consistency Condition

The renormalization procedure discussed before has led to find the general nontrivial solution of the BRST consistency condition,

$$s_g \Delta = 0 \;, \tag{5.1}$$

which constitutes a cohomology problem due to the nilpotency of the BRST operator in the gauge sector,

$$s_g^2 = 0 \;. \tag{5.2}$$

The integrated local functional Δ turned out to depend on the gauge field A_μ and the ghost field c only, $\Delta = \Delta[A_\mu, c]$. As proven by [4, 5], the use of the calculus of differential forms is no restriction to the generality of the solution of the BRST consistency condition (5.1). Writing

$$\Delta = \int Q_4^G \;, \tag{5.3}$$

where[9] $Q_4^G = \Delta^G(x) d^4x$ is a volume form, the BRST consistency condition (5.1) translates into the local equation

$$s_g Q_4^G + d Q_3^{G+1} = 0 \;, \tag{5.4}$$

with $d = dx^\mu \partial_\mu$ the nilpotent exterior derivative. In the ghost number sector zero ($G = 0$), a nontrivial solution for Q_4^0, $Q_4^0 \neq d\hat{Q}_3^0$, represents an invariant Lagrangian. Since c has ghost number one, Δ is only a functional of A_μ. The solution for Δ has been already given in the previous section (4.31).

[9]The upper index denotes the ghost number, whereas the lower index denotes the form degree.

In the ghost number sector one ($G = 1$), a nontrivial solution for Q^1_4, $Q^1_4 \neq s\hat{Q}^0_4 + d\hat{Q}^1_3$, represents a possible canditate for an anomaly. The local equation (5.4) becomes

$$s_g Q^1_4 + dQ^2_3 = 0 \; . \tag{5.5}$$

It is a cohomology problem with respect to s_g modulo d. Using the algebra, $s_g^2 = 0$, $d^2 = 0$, $\{s_g, d\} = 0$, and the algebraic Poincaré lemma [7] one gets the tower of descent equations,

$$\begin{aligned} s_g Q^1_4 + dQ^2_3 &= 0 \; , \\ s_g Q^2_3 + dQ^3_2 &= 0 \; , \\ s_g Q^3_2 + dQ^4_1 &= 0 \; , \\ s_g Q^4_1 + dQ^5_0 &= 0 \; , \\ s_g Q^5_0 &= 0 \; . \end{aligned} \tag{5.6}$$

The last equation of the tower is a local cohomology problem with respect to s_g.

In order to derive an alternative algebraic method for solving the descent equations (5.6), the relation between the solution of the s modulo d cohomology and the solution of the s_g modulo d cohomology has to be analyzed.

5.1 The Cohomology

In order to be quite general, the BRST consistency condition corresponding to the full BRST–operator,

$$s\Xi = 0 \; , \tag{5.7}$$

will be discussed in a spacetime with N dimensions. The integrated local functional $\Xi = \int \Omega^0_N$ is assumed to depend on A_μ, c and ξ^μ,

$$\Xi = \Xi[A_\mu, c](\xi^\mu) \; , \tag{5.8}$$

where ξ^μ plays the role of a parameter. The corresponding descent equations are

$$\begin{aligned} s\Omega^0_N + d\Omega^1_{N-1} &= 0 \; , \\ &\dots \\ s\Omega^{N-1}_1 + d\Omega^N_0 &= 0 \; , \\ s\Omega^N_0 &= 0 \; . \end{aligned} \tag{5.9}$$

The cocycles Ω^k_{N-k} can be expanded as series in powers of ξ^μ. The expansion of the BRST operator as a series in powers of ξ^μ is

$$s = s_g + s_T = s_g + \xi^\mu \partial_\mu \; . \tag{5.10}$$

Introducing the counting operator

$$\mathcal{N}_\xi = \xi^\mu \frac{\partial}{\partial \xi^\mu} \; , \tag{5.11}$$

one gets

$$\Omega^k_{N-k} = \sum_{m=0}^{k} (\Omega^k_{N-k})_{(m)} \quad , \quad k = 0, \ldots, N \ , \tag{5.12}$$

with

$$\mathcal{N}_\xi (\Omega^k_{N-k})_{(m)} = m (\Omega^k_{N-k})_{(m)} \quad , \quad 0 \leq m \leq k \quad , \quad k = 0, \ldots, N \ . \tag{5.13}$$

Moreover, one has

$$s = s_{(0)} + s_{(1)} = s_g + s_T \ , \tag{5.14}$$

with $[\mathcal{N}_\xi, s_{(0)}] = 0$ and $[\mathcal{N}_\xi, s_{(1)}] = s_{(1)}$. The descent equations (5.9) become

$$\begin{aligned} &(s_{(0)} + s_{(1)}) \sum_{m=0}^{k} (\Omega^k_{N-k})_{(m)} + d \sum_{m=0}^{k+1} (\Omega^{k+1}_{N-k-1})_{(m)} = 0 \quad , \quad k = 0, \ldots, N-1 \ , \\ &(s_{(0)} + s_{(1)}) \sum_{m=0}^{N} (\Omega^N_0)_{(m)} = 0 \ . \end{aligned} \tag{5.15}$$

Arranging then the terms with respect to increasing power in ξ^μ, one obtains

$$\begin{aligned} & s_{(0)} (\Omega^0_N)_{(0)} + d (\Omega^1_{N-1})_{(0)} + s_{(1)} (\Omega^0_N)_{(0)} + d (\Omega^1_{N-1})_{(1)} = 0 \ , \\ & s_{(0)} (\Omega^k_{N-k})_{(0)} + d (\Omega^{k+1}_{N-k-1})_{(0)} \\ & \quad + \sum_{l=1}^{k} \left[s_{(0)} (\Omega^k_{N-k})_{(l)} + s_{(1)} (\Omega^k_{N-k})_{(l-1)} + d (\Omega^{k+1}_{N-k-1})_{(l)} \right] \\ & \quad + s_{(1)} (\Omega^k_{N-k})_{(k)} + d (\Omega^{k+1}_{N-k-1})_{(k+1)} = 0 \quad , \quad k = 1, \ldots, N-1 \ , \\ & s_{(0)} (\Omega^N_0)_{(0)} + \sum_{l=1}^{N} [s_{(0)} (\Omega^N_0)_{(l)} + s_{(1)} (\Omega^N_0)_{(l-1)}] = 0 \ , \end{aligned} \tag{5.16}$$

where we have taken into account that the product of $(N+1)$ translation ghosts automatically vanishes in a spacetime with N dimensions, i.e. $s_{(1)} (\Omega^N_0)_{(N)} = 0$. The basis lemma [14]

$$\begin{aligned} & \sum_m P_{(m)} \equiv 0 \qquad \mathcal{N}_\xi P_{(m)} = m P_{(m)} \qquad m = 0, \ldots, N \\ & \quad \Leftrightarrow \qquad P_{(m)} \equiv 0 \qquad m = 0, \ldots, N \ , \end{aligned} \tag{5.17}$$

states that a series in powers of ξ^μ is identically to zero if and only if each coefficient is equal to zero. Therefore, the descent equations (5.9) devide into a set of descent equations according to the power of ξ^μ.

Since $s_{(0)} = s_g$, the descent equations in the ξ^μ–sector 0,

$$\begin{aligned} & s_{(0)} (\Omega^0_N)_{(0)} + d (\Omega^1_{N-1})_{(0)} = 0 \ , \\ & s_{(0)} (\Omega^k_{N-k})_{(0)} + d (\Omega^{k+1}_{N-k-1})_{(0)} = 0 \quad , \quad k = 1, \ldots, N-1 \ , \\ & s_{(0)} (\Omega^N_0)_{(0)} = 0 \ , \end{aligned} \tag{5.18}$$

coincide with the descent equations of the gauge sector,

$$\begin{aligned} & s_g Q^0_N + d Q^1_{N-1} = 0 \ , \\ & s_g Q^k_{N-k} + d Q^{k+1}_{N-k-1} = 0 \quad , \quad k = 1, \ldots, N-1 \ , \\ & s_g Q^N_0 = 0 \ , \end{aligned} \tag{5.19}$$

and one has

$$(\Omega^k_{N-k})_{(0)} = Q^k_{N-k} \quad , \quad k = 0, \ldots, N \; . \tag{5.20}$$

A nontrivial solution for Q^0_N represents an invariant Lagrangian in a spacetime with N dimensions, whereas a nontrivial solution for Q^1_{N-1} represents a candidate for an anomaly in a spacetime with $N-1$ dimensions. Generally, the descent equations in the ξ^μ–sector l can be collected as follows:

$$\begin{aligned}
&s_{(1)}(\Omega^{l-1}_{N-l+1})_{(l-1)} + d(\Omega^l_{N-l})_{(l)} = 0 \; , \\
&s_{(0)}(\Omega^k_{N-k})_{(l)} + s_{(1)}(\Omega^k_{N-k})_{(l-1)} + d(\Omega^{k+1}_{N-k-1})_{(l)} = 0 \quad , \quad k = l, \ldots, N-1 \; , \\
&s_{(0)}(\Omega^N_0)_{(l)} + s_{(1)}(\Omega^N_0)_{(l-1)} = 0 \; .
\end{aligned} \tag{5.21}$$

In order to solve the descent equations in the ξ^μ–sectors $l \geq 1$ one defines the operator

$$i_\xi(dx^\mu) = \xi^\mu \; , \tag{5.22}$$

obeying the following algebraic relations:

$$[i_\xi, d] = s_{(1)} \quad , \quad [i_\xi, s_{(1)}] = 0 \; . \tag{5.23}$$

We continue with the discussion of the descent equations in the ξ^μ–sector 1. Using the algebraic relation (5.23), the first descent equation in this sector,

$$s_{(1)}(\Omega^0_N)_{(0)} + d(\Omega^1_{N-1})_{(1)} = 0, \tag{5.24}$$

becomes

$$d\left[(\Omega^1_{N-1})_{(1)} - i_\xi(\Omega^0_N)_{(0)}\right] = 0 \; . \tag{5.25}$$

Since the d cohomology is trivial [3, 8], one obtains the result

$$(\Omega^1_{N-1})_{(1)} = i_\xi(\Omega^0_N)_{(0)} = i_\xi Q^0_N \; . \tag{5.26}$$

Analogously, the next $N-1$ descent equations in the ξ^μ–sector 1 lead to

$$(\Omega^k_{N-k})_{(1)} = i_\xi(\Omega^{k-1}_{N-k+1})_{(0)} = i_\xi Q^{k-1}_{N-k+1} \quad , \quad k = 2, \ldots, N \; . \tag{5.27}$$

One can show that the last descent equation in this sector is then automatically fulfilled

$$s_{(0)}(\Omega^N_0)_{(1)} + s_{(1)}(\Omega^N_0)_{(0)} = 0 \; . \tag{5.28}$$

In the same manner one finds that the general nontrivial solution for the cocycles of the descent equations (5.9) can be expressed in terms of the general nontrivial solution for the cocycles of the descent equations in the gauge sector (5.19) as follows

$$\Omega^k_{N-k} = \sum_{m=0}^{k} (\Omega^k_{N-k})_{(m)} = \sum_{m=0}^{k} \frac{1}{m!} (i_\xi)^m Q^{k-m}_{N-k+m} \quad , \quad k = 0, \ldots, N \; . \tag{5.29}$$

In particular the solution of the last descent equation in (5.9) becomes

$$\Omega^N_0 = \sum_{m=0}^{N} \frac{1}{m!} (i_\xi)^m Q^{N-m}_m \; . \tag{5.30}$$

Since Ω_0^N collects all cocycles Q_m^{N-m}, $m = 0, 1, \ldots, N$, of the descent equations in the gauge sector, one gets the following main result: *The general nontrivial solution of the descent equations in the gauge sector is equivalent to the general nontrivial solution of the local equation*

$$s\Omega_0^N = 0 \ , \tag{5.31}$$

up to trivial terms.

Since Ω_0^N can be represented by an integrated parameter formula, as it will be shown in the following, it is easier to solve the local equation (5.31) instead of solving the tower of descent equations in the gauge sector.

5.2 The Decomposition

In the preceding subsection it has been shown in (5.29) that the general nontrivial solution for the cocycles Ω_{N-k}^k of the descent equations (5.9) can be expressed in terms of the general nontrivial solution for the cocycles Q_{N-k}^k of the descent equations in the gauge sector (5.19).

Defining the operator δ as

$$\delta = dx^\mu \frac{\partial}{\partial \xi^\mu} \ , \tag{5.32}$$

the exterior derivative can be decomposed as a BRST commutator,

$$[\delta, s] = d \ . \tag{5.33}$$

Moreover, since the operator δ and the exterior derivative d commute, $[\delta, d] = 0$, an operator $\mathcal{G} = \frac{1}{2}[d, \delta]$ is absent, in contrary to [9]. Therefore, the algebra between the BRST operator, the exterior derivative and the δ–operator is closed,

$$[\delta, s] = d \quad , \quad [\delta, d] = 0 \ . \tag{5.34}$$

Using the counting operators

$$\mathcal{N}_{dx} = dx^\mu \frac{\partial}{\partial (dx^\mu)} \quad , \quad \mathcal{N}_\xi = \xi^\mu \frac{\partial}{\partial \xi^\mu} \ , \tag{5.35}$$

one obtains

$$[\delta, i_\xi] = \mathcal{N}_{dx} - \mathcal{N}_\xi \ , \tag{5.36}$$

or generally,

$$[\delta, (i_\xi)^m] = \sum_{l=0}^{m-1} (i_\xi)^l \mathcal{N}_{dx} (i_\xi)^{m-l-1} - \sum_{l=0}^{m-1} (i_\xi)^l \mathcal{N}_\xi (i_\xi)^{m-l-1} \ . \tag{5.37}$$

The application of δ to Ω_{N-k}^k gives

$$\delta\Omega_{N-k}^k = \sum_{m=0}^{k} \frac{1}{m!} \delta (i_\xi)^m Q_{N-k+m}^{k-m}$$

$$= (N-k+1)\sum_{p=0}^{k-1}\frac{1}{p!}\,(i_\xi)^p Q^{k-1-p}_{N-(k-1)+p}$$
$$= (N-k+1)\Omega^{k-1}_{N-(k-1)}\ . \tag{5.38}$$

Therefore, one obtains the recursive relation

$$\Omega^k_{N-k} = \frac{1}{N-k}\,\delta\Omega^{k+1}_{N-k-1} \quad , \quad k=0,\ldots,N-1\ . \tag{5.39}$$

The solution of the recursive relation is given by

$$\Omega^k_{N-k} = \frac{1}{(N-k)!}\,\delta^{N-k}\Omega^N_0\ , \tag{5.40}$$

which is valid for $k=0,\ldots,N$.

5.3 An Integrated Parameter Formula

In this subsection an integrated parameter formula for the general nontrivial solution of the local equation $s\Omega^N_0 = 0$ will be derived in a spacetime of dimension $N = 2k-1$. Expanding Ω^{2k-1}_0 as a series in powers of ξ^μ,

$$\Omega^{2k-1}_0 = Q^{2k-1}_0 + i_\xi Q^{2k-2}_1 + \ldots + \frac{1}{(2k-2)!}\,(i_\xi)^{2k-2}Q^1_{2k-2}$$
$$+\ \frac{1}{(2k-1)!}\,(i_\xi)^{2k-1}Q^0_{2k-1}\ , \tag{5.41}$$

and reducing the dimension of spacetime by one, the gauge anomaly in $2k-2$ dimensions will be recovered as Q^1_{2k-2}. Moreover, it will be shown that Q^0_{2k-1} is the Chern–Simons term in $2k-1$ dimensions.

In order to find the integrated parameter formula for Ω^{2k-1}_0 we use the calculus of forms. The 1–form gauge field and the associated 2–form field strength are given by

$$A = A_\mu dx^\mu \quad , \quad F = \frac{1}{2}\,F_{\mu\nu}\,dx^\mu dx^\nu = dA - iAA\ . \tag{5.42}$$

The Bianchi–identity reads

$$DF = dF - i[A,F] = 0\ , \tag{5.43}$$

where $D = dx^\mu D_\mu$ denotes the covariant exterior derivative with respect to the 1–form gauge field. The BRST–transformation of the 1–form gauge field becomes

$$sA = -Dc + \xi^\nu\partial_\nu A = -dc + i\{A,c\} + \xi^\nu\partial_\nu A$$
$$= -D(c+i_\xi A) + i_\xi F\ . \tag{5.44}$$

Introducing the shifted gauge ghost

$$\hat{c} = c + i_\xi A\ , \tag{5.45}$$

the BRST–transformations of the 1–form gauge field and the shifted gauge ghost are

$$
\begin{aligned}
sA &= -D\hat{c} + i_\xi F \ , \\
s\hat{c} &= i\hat{c}\hat{c} + \hat{F} \ ,
\end{aligned}
\tag{5.46}
$$

with the ghost field strength

$$
\hat{F} = \frac{1}{2} i_\xi i_\xi F \ , \tag{5.47}
$$

transforming according to

$$
s\hat{F} = i[\hat{c}, \hat{F}] \ . \tag{5.48}
$$

Defining a generalized covariant BRST–operator

$$
\hat{S} = s - i\hat{c} \ , \tag{5.49}
$$

one gets

$$
\hat{S}\hat{F} = s\hat{F} - i[\hat{c}, \hat{F}] = 0 \ . \tag{5.50}
$$

It follows a remarkable correspondence, which will be revealed by the following summary. This correspondence is peculiar to the full BRST–operator s including the translations.

- The 1–form gauge field corresponds to the shifted gauge ghost,

$$
A = A_\mu dx^\mu \quad , \quad \hat{c} = c + i_\xi A \ . \tag{5.51}
$$

- The 2–form field strength corresponds to the ghost field strength,

$$
\begin{aligned}
F &= \frac{1}{2} F_{\mu\nu}\, dx^\mu dx^\nu = dA - iAA \ , \\
\hat{F} &= \frac{1}{2} F_{\mu\nu}\, \xi^\mu \xi^\nu = s\hat{c} - i\hat{c}\hat{c} \ .
\end{aligned}
\tag{5.52}
$$

- The covariant exterior derivative corresponds to the generalized covariant BRST–operator,

$$
D = d - iA \quad , \quad \hat{S} = s - i\hat{c} \ . \tag{5.53}
$$

- The Bianchi–identity corresponds to the generalized covariant BRST–transformation of the ghost field strength,

$$
DF = dF - i[A, F] = 0 \quad , \quad \hat{S}\hat{F} = s\hat{F} - i[\hat{c}, \hat{F}] = 0 \ . \tag{5.54}
$$

The generalized covariant BRST–algebra is now given by

$$
i\hat{S}^2 = \hat{F} \quad , \quad iD^2 = F \quad , \quad i\{\hat{S}, D\} = i_\xi F \ . \tag{5.55}
$$

Guided by the well–known derivation of the Chern–Simons formula [11],

$$
(CS)^0_{2k-1} = k\, Tr \int_0^1 dt\, AF^{k-1}(t) \ , \tag{5.56}
$$

with

$$F(t) = dA(t) - iA(t)A(t) \quad , \quad A(t) = tA \quad , \quad 0 \leq t \leq 1 \; , \tag{5.57}$$

and using the presented correspondence it will be easy to derive an integrated parameter formula for the general nontrivial solution of the local equation

$$s\Omega_0^{2k-1} = 0 \; . \tag{5.58}$$

We introduce the interpolating shifted gauge ghost

$$\hat{c}(t) = t\hat{c} \quad , \quad 0 \leq t \leq 1 \; , \tag{5.59}$$

with $\hat{c}(0) = 0$ and $\hat{c}(1) = \hat{c}$, and the associated ghost field strength

$$\hat{F}(t) \;\; = \;\; s\hat{c}(t) - i\hat{c}(t)\hat{c}(t) \; , \tag{5.60}$$

with $\hat{F}(0) = 0$ and $\hat{F}(1) = \hat{F}$. Defining an interpolating generalized covariant BRST–operator

$$\hat{S}_t = s - i\hat{c}(t) \; , \tag{5.61}$$

with $\hat{S}_0 = s$ and $\hat{S}_1 = \hat{S}$, one gets the following identities

$$\frac{d\hat{F}(t)}{dt} = \hat{S}_t\hat{c} \quad , \quad \hat{S}_t\hat{F}(t) = 0 \; . \tag{5.62}$$

Therefore, in a spacetime with $2k$ dimensions one has

$$\begin{aligned} Tr(\hat{F}^k) &= Tr\left(\hat{F}^k(1) - \hat{F}^k(0)\right) = Tr\int_0^1 dt\,\frac{d}{dt}\,\hat{F}^k(t) \\ &= k\,Tr\int_0^1 dt\,\frac{d\hat{F}(t)}{dt}\,\hat{F}^{k-1}(t) = k\,Tr\int_0^1 dt\,(S_t\hat{c})\hat{F}^{k-1}(t) \\ &= s\left(k\,Tr\int_0^1 dt\,\hat{c}\,\hat{F}^{k-1}(t)\right) \; . \end{aligned} \tag{5.63}$$

Using the nilpotency of the BRST–operator and the fact that $Tr(\hat{F}^k) \neq 0$ in a spacetime with $2k$ dimensions, one concludes that $k\,Tr\int_0^1 dt\,\hat{c}\,\hat{F}^{k-1}(t)$ is nontrivial. Since $Tr(\hat{F}^k)$ contains the product of $2k$ fermionic translation ghosts ξ^μ, it follows:

In a spacetime with $2k-1$ *dimensions the general nontrivial solution of the local equation* $s\Omega_0^{2k-1} = 0$ *can be represented by the integrated parameter formula*

$$\Omega_0^{2k-1} = k\,Tr\int_0^1 dt\,\hat{c}\,\hat{F}^{k-1}(t) \; . \tag{5.64}$$

Compare (5.64) with (5.56) using the correspondence (5.51) and (5.52).

Expanding Ω_0^{2k-1} as a series in powers of ξ^μ,

$$\Omega_0^{2k-1} = Q_0^{2k-1} + i_\xi Q_1^{2k-2} + \ldots + \frac{1}{(2k-1)!}\,(i_\xi)^{2k-1}Q_{2k-1}^0 \; , \tag{5.65}$$

one gets the nontrivial solutions for the cocycles Q_l^{2k-1-l} of the descent equations in the gauge sector.

Finally, for the sake of clarity let us discuss in detail the special case $k=3$. The tower of the descent equations in the gauge sector is then given by

$$\begin{aligned} s_g Q_5^0 + dQ_4^1 &= 0 \ , \\ s_g Q_4^1 + dQ_3^2 &= 0 \ , \\ s_g Q_3^2 + dQ_2^3 &= 0 \ , \\ s_g Q_2^3 + dQ_1^4 &= 0 \ , \\ s_g Q_1^4 + dQ_0^5 &= 0 \ , \\ s_g Q_0^5 &= 0 \ . \end{aligned} \tag{5.66}$$

In order to get the nontrivial solutions for the cocycles Q_{5-k}^k, one expands the integrated parameter formula (5.64) as a series in powers of ξ^μ,

$$\begin{aligned} \Omega_0^5 &= 3\,Tr \int_0^1 dt\, \hat{c}\hat{F}(t)\hat{F}(t) \\ &= \frac{1}{4}\,Tr\left[(c+i_\xi A)\, i_\xi i_\xi F\, i_\xi i_\xi F\right] + \frac{i}{4}\,Tr\left[(c+i_\xi A)^3\, i_\xi i_\xi F\right] \\ &\quad - \frac{1}{10}\,Tr\left[(c+i_\xi A)^5\right] \ . \end{aligned} \tag{5.67}$$

After some calculations one obtains

$$\begin{aligned} \Omega_0^5 &= -\frac{1}{10}\,Tr(ccccc) \\ &\quad + i_\xi\left[Tr\left(-\frac{1}{2}\,ccccA\right)\right] \\ &\quad + \frac{1}{2}\,i_\xi i_\xi\left[\frac{1}{2}Tr(icccF - cccAA - ccAcA)\right] \\ &\quad + \frac{1}{6}\,i_\xi i_\xi i_\xi\left[Tr\left(\frac{i}{2}\,(ccAF + cAcF + AccF) - \frac{1}{2}\,(ccAAA + cAcAA)\right)\right] \\ &\quad + \frac{1}{24}\,i_\xi i_\xi i_\xi i_\xi\left[Tr\left(cFF + \frac{i}{2}\,(cAAF + AcAF + AAcF) - \frac{1}{2}\,cAAAA\right)\right] \\ &\quad + \frac{1}{120}\,i_\xi i_\xi i_\xi i_\xi i_\xi\left[Tr\left(AFF + \frac{i}{2}\,AAAF - \frac{1}{10}\,AAAAA\right)\right] \ . \end{aligned} \tag{5.68}$$

Since

$$\begin{aligned} \Omega_0^5 &= Q_0^5 + i_\xi Q_1^4 + \frac{1}{2}\,i_\xi i_\xi Q_2^3 + \frac{1}{6}\,i_\xi i_\xi i_\xi Q_3^2 \\ &\quad + \frac{1}{24}\,i_\xi i_\xi i_\xi i_\xi Q_4^1 + \frac{1}{120}\,i_\xi i_\xi i_\xi i_\xi i_\xi Q_5^0 \ , \end{aligned} \tag{5.69}$$

one concludes that the nontrivial solutions for the cocycles of the descent equations in the gauge sector are given by

$$Q_0^5 = -\frac{1}{10}\,Tr(ccccc) \ ,$$

$$
\begin{aligned}
Q_1^4 &= -\frac{1}{2} Tr(ccccA) , \\
Q_2^3 &= \frac{1}{2} Tr(icccF - cccAA - ccAcA) , \\
Q_3^2 &= \frac{1}{2} Tr[i(ccAF + cAcF + AccF) - (ccAAA + cAcAA)] , \\
Q_4^1 &= Tr\left[cFF + \frac{i}{2}(cAAF + cAFA + cFAA) - \frac{1}{2} cAAAA\right] , \\
Q_5^0 &= Tr\left(AFF + \frac{i}{2} AAAF - \frac{1}{10} AAAAA\right) . \qquad (5.70)
\end{aligned}
$$

One sees that Q_5^0 is the Chern–Simons term in five dimensions and Q_4^1 is the gauge anomaly in four dimensions [9].

Appendix: The Most General Counterterm

In this appendix the set of constraints (4.26)–(4.29) will be solved leading to the result (4.31). One starts with the most general perturbation $\bar{\Sigma}_c = \bar{\Sigma}_c[A_\mu,\, c,\, \hat{\rho}^\mu,\, \sigma]$ which has canonical dimension zero, ghost number zero and which is invariant under the parity transformation,

$$
\begin{aligned}
\bar{\Sigma}_c = Tr \int d^4x\, [& k_1 \partial_\mu A_\nu(x) \partial^\mu A^\nu(x) + k_2 \partial_\mu A_\nu(x) \partial^\nu A^\mu(x) \\
+ & k_3 \partial_\mu A_\nu(x) A^\mu(x) A^\nu(x) + k_4 \partial_\mu A_\nu(x) A^\nu(x) A^\mu(x) \\
+ & k_5 A_\mu(x) A_\nu(x) A^\mu(x) A^\nu(x) + k_6 A_\mu(x) A_\nu(x) A^\nu(x) A^\mu(x) \\
+ & k_7 \hat{\rho}^\mu(x) \partial_\mu c(x) + k_8 \hat{\rho}^\mu(x) A_\mu(x) c(x) \\
+ & k_9 \hat{\rho}^\mu(x) c(x) A_\mu(x) + k_{10} \sigma(x) c(x) c(x)] . \qquad (A.1)
\end{aligned}
$$

The functional derivative of $\bar{\Sigma}_c$ with respect to $c(x)$ is given by

$$
\frac{\delta \bar{\Sigma}_c}{\delta c(x)} = k_7 \partial_\mu \hat{\rho}^\mu(x) - k_8 \hat{\rho}^\mu(x) A_\mu(x) - k_9 A_\mu(x) \hat{\rho}^\mu(x) + k_{10} \left[c(x), \sigma(x)\right] . \qquad (A.2)
$$

From the integrated ghost equation (4.26) follows that the functional derivative of $\bar{\Sigma}_c$ with respect to $c(x)$ has to be a total divergence. Therefore, one has

$$
k_8 = k_9 = k_{10} = 0 , \qquad (A.3)
$$

and

$$
\frac{\delta \bar{\Sigma}_c}{\delta c(x)} = k_7 \partial_\mu \hat{\rho}^\mu(x) . \qquad (A.4)
$$

The functional derivatives of $\bar{\Sigma}_c$ with respect to $\hat{\rho}^\mu(x)$ and $\sigma(x)$ are then

$$
\frac{\delta \bar{\Sigma}_c}{\delta \hat{\rho}^\mu(x)} = k_7 \partial_\mu c(x) \quad , \quad \frac{\delta \bar{\Sigma}_c}{\delta \sigma(x)} = 0 . \qquad (A.5)
$$

Considering only the $\hat{\rho}^\mu$–dependence one gets

$$
\bar{\Sigma}_c = Tr \int d^4x\, [k_7 \hat{\rho}^\mu(x) \partial_\mu c(x) + \ldots] , \qquad (A.6)
$$

where the low dots denote the terms which are independent of $\hat{\rho}^\mu$. Using

$$\mathcal{S}_{\Gamma^{(0)}}\hat{\rho}^\mu(x) = i\,\{c(x), \hat{\rho}^\mu(x)\} + \dots \ , \tag{A.7}$$

$$\mathcal{S}_{\Gamma^{(0)}}A_\mu(x) = D_\mu c(x) \ , \tag{A.8}$$

the $\mathcal{S}_{\Gamma^{(0)}}$–variation of

$$\hat{\Sigma} = -k_7\, Tr \int d^4x\, \hat{\rho}^\mu(x) A_\mu(x) \tag{A.9}$$

becomes

$$\mathcal{S}_{\Gamma^{(0)}}\hat{\Sigma} = k_7\, Tr \int d^4x\, \hat{\rho}^\mu(x)\partial_\mu c(x) + \dots \tag{A.10}$$

and one gets

$$\mathcal{S}_{\Gamma^{(0)}}\hat{\Sigma} = \bar{\Sigma}_c + \dots \ . \tag{A.11}$$

Hence the most general $\hat{\rho}^\mu$–dependence of the perturbation $\bar{\Sigma}_c$ is trivial. In order to find the most general nontrivial expression for $\bar{\Sigma}_c$ one can thus assume

$$\frac{\delta\bar{\Sigma}_c}{\delta\hat{\rho}^\mu(x)} = 0 \ , \tag{A.12}$$

which fixes k_7 as

$$k_7 = 0 \ , \tag{A.13}$$

and therefore one also has

$$\frac{\delta\bar{\Sigma}_c}{\delta c(x)} = 0 \ . \tag{A.14}$$

It remains

$$\begin{aligned}\bar{\Sigma}_c = Tr \int d^4x\, [&\partial_\mu A_\nu(x)\,(k_1\partial^\mu A^\nu(x) + k_2\partial^\nu A^\mu(x) \\ &+ \ k_3 A^\mu(x)A^\nu(x) + k_4 A^\nu(x)A^\mu(x)) \\ &+ \ A_\mu(x)A_\nu(x)\,(k_5 A^\mu(x)A^\nu(x) + k_6 A^\nu(x)A^\mu(x))] \ .\end{aligned} \tag{A.15}$$

The BRST–consistency condition becomes

$$\mathcal{S}_{\Gamma^{(0)}}(\bar{\Sigma}_c) = Tr \int d^4x\, c(x)\mathcal{D}(x)\bar{\Sigma}_c = 0 \ , \tag{A.16}$$

leading to the following set of constraints

$$\begin{aligned} k_1 + k_2 = 0 \quad &, \quad k_3 + k_4 = 0 \quad , \quad k_3 - 2ik_2 = 0 \ , \\ k_3 + 2ik_1 = 0 \quad &, \quad k_4 + 2ik_2 = 0 \quad , \quad k_4 - 2ik_1 = 0 \ , \\ ik_4 - ik_3 - 4k_5 = 0 \quad &, \quad 2k_6 - ik_3 = 0 \quad , \quad 2k_6 + ik_4 = 0 \ , \end{aligned} \tag{A.17}$$

which has the solution

$$\begin{aligned} k_2 = -k_1 \quad &, \quad k_3 = -2ik_1 \quad , \quad k_4 = 2ik_1 \ , \\ k_5 = -k_1 \quad &, \quad k_6 = k_1 \ . \end{aligned} \tag{A.18}$$

Setting

$$k_1 = -\frac{1}{2}\,k \ , \tag{A.19}$$

one gets for the most general perturbation $\bar{\Sigma}_c$ of the action in the tree approximation

$$\bar{\Sigma}_c = -k\,\frac{1}{4}\,Tr\int d^4x\,F_{\mu\nu}(x)F^{\mu\nu}(x)\ . \tag{A.20}$$

Using

$$\frac{\delta\bar{\Sigma}_c}{\delta A_\mu(x)} = k\,D_\nu F^{\nu\mu}(x)\ , \tag{A.21}$$

it easy to show that the perturbation (A.20) also obeys the Ward–identity describing the invariance under rigid gauge transformations,

$$\hat{\mathcal{W}}_{rig}\bar{\Sigma}_c = \int d^4x\,i\left[A_\mu(x), \frac{\delta\bar{\Sigma}_c}{\delta A_\mu(x)}\right] = 0\ , \tag{A.22}$$

and the Ward–identity describing the invariance under translations,

$$\hat{\mathcal{P}}_\mu\bar{\Sigma}_c = Tr\int d^4x\,\partial_\mu A_\nu(x)\,\frac{\delta\bar{\Sigma}_c}{\delta A_\nu(x)} = 0\ . \tag{A.23}$$

Thus $\bar{\Sigma}_c$ given by eq.(A.20) is the general solution of the set of constraints (4.26)–(4.29).

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