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Abstract. The scattering properties of the non-linear $O(3)$ model in (2+1)-D, modified by the addition of both a potential-like term and a Skyrme-like term, are considered. Most of the work is numerical. The skyrmion-scattering is found to be quasi-elastic, the skyrmions' energy density profiles remaining unscathed after collisions. In low-energy processes the skyrmions exhibit back-scattering, while at larger energies they scatter at right angles. These results confirm those obtained in previous investigations, in which a similar problem was studied for a different choice of the potential-like term.

1 Introduction

In the past few years, σ-models in low dimensions have become an increasingly important area of research, often arising as approximate models in the contexts of both particle and solid state physics. They have been used in the construction of high-T_c superconductivity and the quantum Hall effect; in two Euclidean dimensions, they appear to be the low-dimensional analogues of four-dimensional Yang-Mills theories. Moreover, they are examples of the harmonic maps studied by differential geometers and, as such, are interesting in themselves. But only very special σ-models in (2+1)-D are integrable [1], and the physically relevant Lorentz-invariant models are not amongst them; in these cases, recourse to numerical evolution must be made.

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The simplest Lorentz-invariant model in (2+1)-D is the $O(3)$ model, which involves three real scalar fields, $\boldsymbol{\phi}(x^\mu) \equiv \{\phi_a(x^\mu), a=1,2,3\}$, with the constraint that $\boldsymbol{\phi}$ lies on the unit sphere $S_2^{(\phi)}$:

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1. \tag{1.1}$$

Subject to this constraint, the Lagrangian density and the corresponding equations of motion are

$$\mathcal{L}_\sigma = \frac{1}{4}(\partial_\mu \boldsymbol{\phi}) \cdot (\partial^\mu \boldsymbol{\phi}), \tag{1.2}$$

$$\partial_t^2 \boldsymbol{\phi} = [-(\partial_t \boldsymbol{\phi})^2 + (\partial_x \boldsymbol{\phi})^2 + (\partial_y \boldsymbol{\phi})^2]\boldsymbol{\phi} + \partial_x^2 \boldsymbol{\phi} + \partial_y^2 \boldsymbol{\phi}. \tag{1.3}$$

Note that we are concerned with the model in (2+1)-D: $x^\mu \equiv (x^0, x^1, x^2) = (t, x, y)$, with the speed of light set equal to unity.

An alternative and convenient formulation of the model is in terms of one independent complex field, W, related to the fields ϕ_a via

$$W = \frac{1 - \phi_3}{\phi_1 + i\phi_2}. \tag{1.4}$$

In this formulation, the Lagrangian density and the corresponding equations of motion read (asterisk denotes complex conjugation)

$$\mathcal{L}_\sigma = \frac{\partial_\mu W \partial^\mu W^*}{(1 + |W|^2)^2} \tag{1.5}$$

and

$$\partial_t^2 W = \partial_x^2 W + \partial_y^2 W + \frac{2W^*[(\partial_t W)^2 - (\partial_x W)^2 - (\partial_y W)^2]}{1 + |W|^2}. \tag{1.6}$$

The problem is completely specified by giving the boundary conditions. As usual we take

$$\lim_{r \to \infty} \boldsymbol{\phi}(r, \theta, t) = \boldsymbol{\phi}_0(t), \tag{1.7}$$

where $\boldsymbol{\phi}_0(t)$ is independent of the polar angle θ. In (2+1)-D this condition ensures a finite potential energy, whereas in two Euclidean dimensions, i.e., when $\boldsymbol{\phi}$ is independent of time, it leads to the finiteness of the action, which is precisely the requirement for quantization in terms of path integrals. As shown by several people [2, 3], any rational function $W(z)$ or $W(z*)$, where $z = x + iy$, is a static solution of Eq. (1.6). These are the instantons of the model, and can be regarded as *static* solitons of the same model in (2+1)-D. The simplest one-soliton solution, $W = \lambda z$ (λ is a free parameter determining the size of the soliton) has been numerically studied by Leese *et.al* [4]. When viewed as an evolving structure in (2+1)-D, the soliton has been found to be unstable. Any small perturbation, either explicit or introduced by the discretization procedure, changes its size. This instability is associated with the conformal invariance of the $O(3)$ Lagrangian in two dimensions.

The $O(3)$ solitons, however, can be stabilized through a judicious introduction of a scale into the model, thereby breaking its conformal invariance. This has been done by Leese *et.al.* [5] and, using a more general field, by ourselves [6].

In the present article we present the results of soliton-like scattering in (2+1)-D obtained by applying the methods of [5] to a more general skyrmion field. In the next section we present our skyrme model and give a brief account of our previous paper [6], where the same model was considered for the case of zero-speed systems. After explaining the numerical procedure in section 3, we pass on to discuss two-skyrmion scattering in section 4. The closing section contains our conclusions.

2   Skyrme model in (2+1) dimensions

Using the W-formulation, our Skyrme model is defined by the Lorentz-invariant Lagragian density

$$
\begin{aligned}
\mathcal{L} = {} & \mathcal{L}_\sigma \\
& - 2\theta_1 \Big[\frac{(\partial_t W^* \partial_y W - \partial_t W \partial_y W^*)^2 + (\partial_t W^* \partial_x W - \partial_t W \partial_x W^*)^2}{(1+|W|^2)^4} \\
& - \frac{(\partial_x W^* \partial_y W - \partial_x W \partial_y W^*)^2}{(1+|W|^2)^4} \Big] \\
& - 4\theta_2 \frac{|W-\lambda|^8}{(1+|W|^2)^4},
\end{aligned}
\tag{2.1}
$$

where $\mathcal{L}_\sigma$ is given by Eq. (1.5) and θ_1, θ_2 are real parameters with dimensions of length squared and inverse length squared, respectively; they introduce a scale into the model, which is no longer conformal invariant. If the size of the solitons is appropriately chosen, it is energetically unfavourable for the solitons to change it. The θ_1-term is the (2+1)-D analogue of the Skyrme term, whereas the θ_2-term is a potential-like one. Unlike the former, the latter term is highly nonunique [7].

The field equation corresponding to the above Lagrangian can be cast into the form

$$
\begin{aligned}
W_{tt} = {} & W_{xx} + W_{yy} + \frac{2W^*[(W_t)^2 - (W_x)^2 - (W_y)^2]}{1+|W|^2} \\
& - \frac{4\theta_1}{(1+|W|^2)^2} [2W_{tx}^* W_t W_x + 2W_{ty}^* W_t W_y - 2W_{xy}^* W_x W_y \\
& + W_{xx}^*(W_y^2 - W_t^2) + W_{yy}^*(W_x^2 - W_t^2) - W_{tt}^*(W_x^2 + W_y^2) \\
& + W_{xx}(|W_t|^2 - |W_y|^2) + W_{yy}(|W_t|^2 - |W_x|^2) + W_{tt}(|W_x|^2 + |W_y|^2) \\
& + W_{xy}(W_x^* W_y + W_y^* W_x) - W_{tx}(W_t^* W_x + W_x^* W_t) - W_{ty}(W_t^* W_y + W_y^* W_t) \\
& + \frac{2W}{1+|W|^2}((W_t^* W_y - W_y^* W_t)^2 + (W_x^* W_t - W_t^* W_x)^2 - (W_x^* W_y - W_y^* W_x)^2)] \\
& + \frac{16\theta_2 |W-\lambda|^2}{(1+|W|^2)^3},
\end{aligned}
\tag{2.2}
$$

where the notation $W_x \equiv \partial_x W$, $W_{xx} \equiv \partial_x^2 W$, etc., has been used. It is straightforward to check that

$$W = \lambda \frac{z-a}{z-b} \qquad (2.3)$$

is a static solution of Eq. (2.2), provided the following relation holds:

$$\lambda = \frac{\sqrt[4]{2\theta_1/\theta_2}}{a-b}. \qquad (2.4)$$

This is the familiar one-instanton of the non-linear $O(3)$ model, but now λ, which characterizes its size, is no longer a free parameter: It is fixed by Eq. (2.4). The instanton, with its size thus fixed, is usually referred to as a 'skyrmion'. One can readily check that the maximum of its total energy density is given by

$$E_{max} = \varepsilon(1 + \theta_1 \varepsilon), \quad \varepsilon = 8\frac{(|\lambda|^2 + 1)^2}{|\lambda(a-b)|^2}. \qquad (2.5)$$

For the parameter values given below in section 3, Eq. (2.5) yields $Emax = 129.3$, the 'canonical size'. The numerically-obtained $Emax$ is very near this value (See Figure 1). The distance from $Emax$ to the centre of the lattice is given by $(a|\lambda|^2 + b)/(|\lambda| + 1)$.

To study scattering processes we take the field

$$W = \lambda \frac{z-a}{z-b}\frac{z+c}{z+d}, \qquad (2.6)$$

which describes two instantons. However, as there are interaction forces between the instantons, Eq. (2.6) is not a static solution of the equations of motion.

In our previous paper [6], where we studied only the case of skyrmions started off at rest, we numerically evolved Eqs. (2.3) and (2.6) and found two basic results:

- The field (2.3) remains almost perfectly static, its total energy density being practically unaltered as time elapses (see Figure 1);

- The field (2.6) shows two skyrmions shaking off some kinetic energy, thus adjusting themselves to their canonical sizes. This is in accordance to expectation, as this field is not a solution of the equations of motion. Then the skyrmions slowly move away from each other, unveiling the presence of a repulsive force between them.

These observations confirm the results obtained in [5], where the skyrmion of the model was just λz. This configuration posseses a total energy density whose maximum is positioned at the centre of the grid ($z_{max} = 0$), the analogue of Eq. (2.5) being $Emax = 8\lambda^2(1 + 8\theta_2\lambda^2)$, $\lambda = \sqrt[4]{2\theta_1/\theta_2}$. With regards to the scattering, reference [5] considered fields of the form $W = \lambda(z-a)(z+b)$ which, as their cousins (2.6), resemble two interacting instantons in an approximate manner.

3   Numerical procedure

Our simulations employed the fourth-order Runge-Kutta method, and approximated the spatial derivatives by finite differences; the Laplacian was evaluated using the nine-point formula. Worthy of note is the fact that the finite-difference expressions for the derivatives of the fields λz and $\lambda(z-a)(z+b)$, used in [5], are exact. This is no longer true for our more general model which, in this sense, is more 'perturbed'. Fortunately, this factor turned out to have no significant effect on the qualitative properties of the discretized version of our model.

All computations were performed on the workstations at Durham, on a fixed 201×201 lattice with spatial and time steps $\delta x = \delta y = 0.02$ and $\delta t = 0.005$. Every few iterations we rescaled the fields $\phi \rightarrow \phi/\sqrt{\phi.\phi}$. We also included along the boundary a narrow strip to absorb the various radiation waves, thus reducing their effects on the skyrmions via the reflections from the boundary. As time elapses, this absorption of radiation manifests itself through a small decrease of the total energy, which gradually stabilizes as the radiation waves are gradually absorbed.

We choose the parameters to have the values: $\theta_1 = 0.015006250$, $\theta_2 = 0.1250$, $a = c = 0.75$ and $b = d = 0.05$ which, according to Eq. (2.4), set $\lambda = 1$.

4   Skyrmion scattering

Our simulations look at the initial configuration

$$W = \lambda \frac{z - 0.75}{z - 0.05} \frac{z + 0.75}{z + 0.05}, \tag{4.1}$$

and evolve it for different initial velocities. First, let us consider head-on collisions. There is always an initial burst of radiation as the skyrmions regain their canonical size. At small speeds, the skyrmions approach each other, but the repulsive force between them results in their motion being reversed. In Figure 2 we present some pictures of the total energy density for skyrmions with initial speed equal to 0.2; the corresponding contour plots are shown in Figure 3.

A qualitatively similar behaviour is observed for speeds up to approximately 0.3, for which the skyrmions acquire enough kinetic energy to overcome their mutual repulsion; during their collision they form a rather complicated state and then re-emerge at 90° to the original direction of motion. The emerging skyrmions are initially shrinking but, after they have travelled some distance, they expand once more. This final state is achieved after some oscillations of the energy density. Figure 4 and Figure 5 show several total energy density pictures and contour plots for this 90°-scattering, whereas Figure 6 exhibits how the amplitude of the total energy density for the above cases varies in time.

In the case of non-zero impact parameter the results are very much as expected. With an impact parameter small enough to prevent the skyrmions from getting too quickly to the boundary, they scatter almost backwards along their original trajectories or at 90°, depending on the initial speed. Also, the larger the impact parameter, the smaller the scattering angle. A typical case is shown in Figure 7.

On comparing these results with those obtained for the configuration $W = \lambda(z-a)(z+b)$ we find there is no qualitative difference, and hence the results obtained in [5] are borne out by the general case studied in the present work.

Emax=128.5
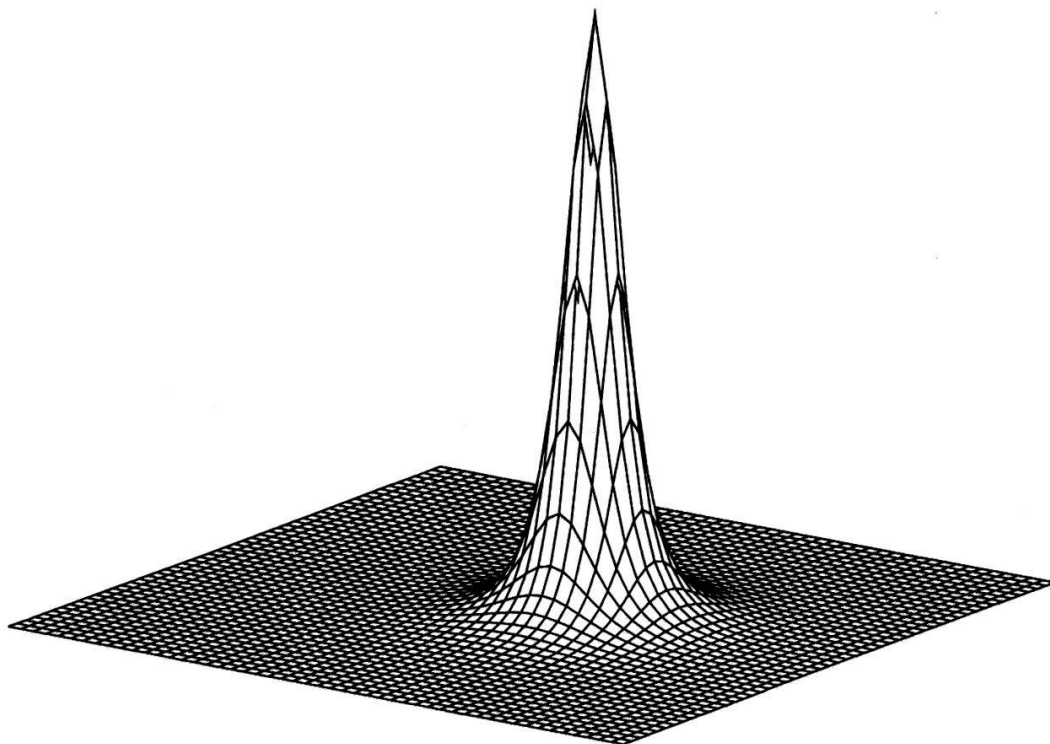
Figure 1: Typical skyrmion total energy density. This picture remains essentially unchanged althroughout the simulation process.

Emax=273.1
t=0

Emax=114.4
t=1.5
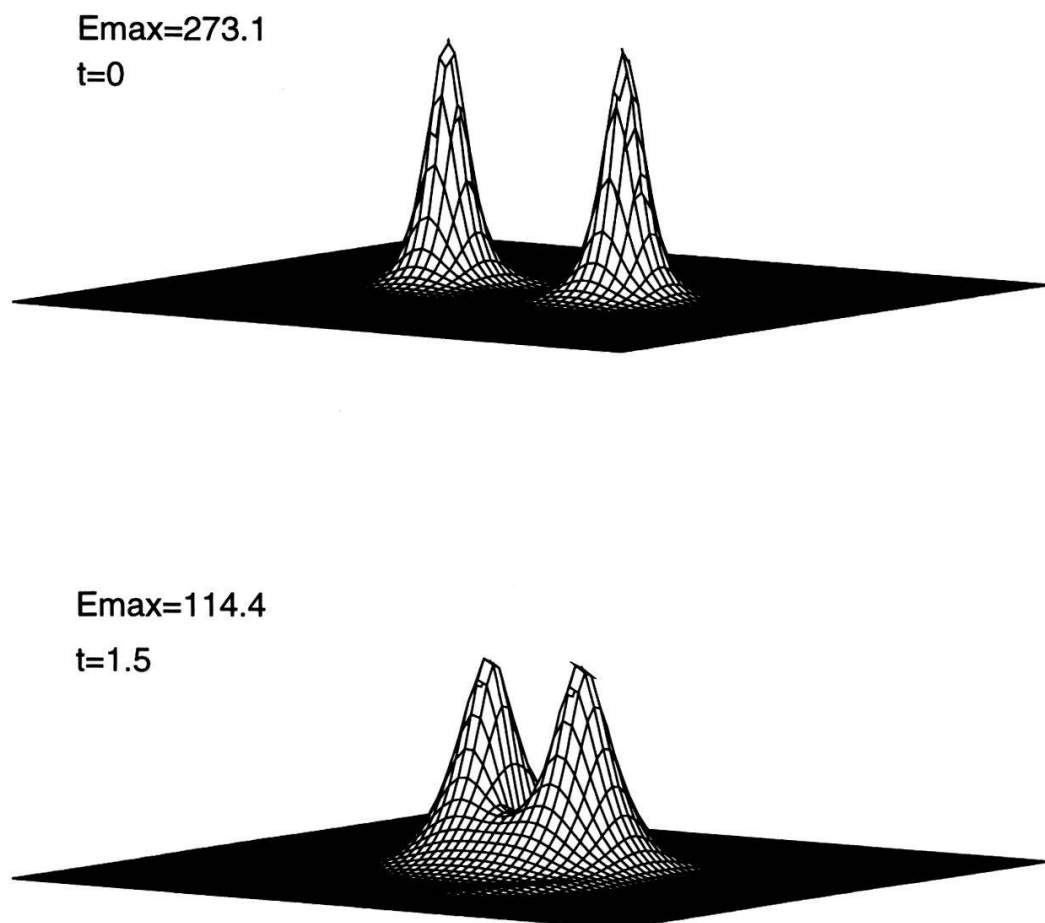
Figure 2: Total energy density pictures corresponding to the initial velocity $v = (0.2, 0.0)$.

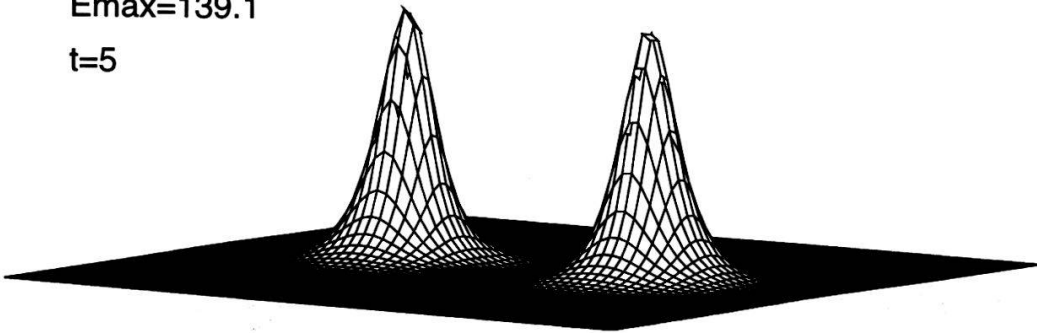Emax=134.9
t=2.5
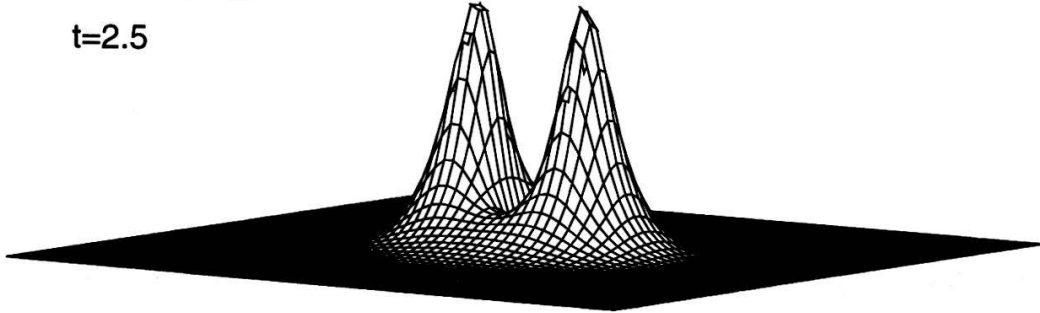
Emax=139.1
t=5

Figure 2:Continued.

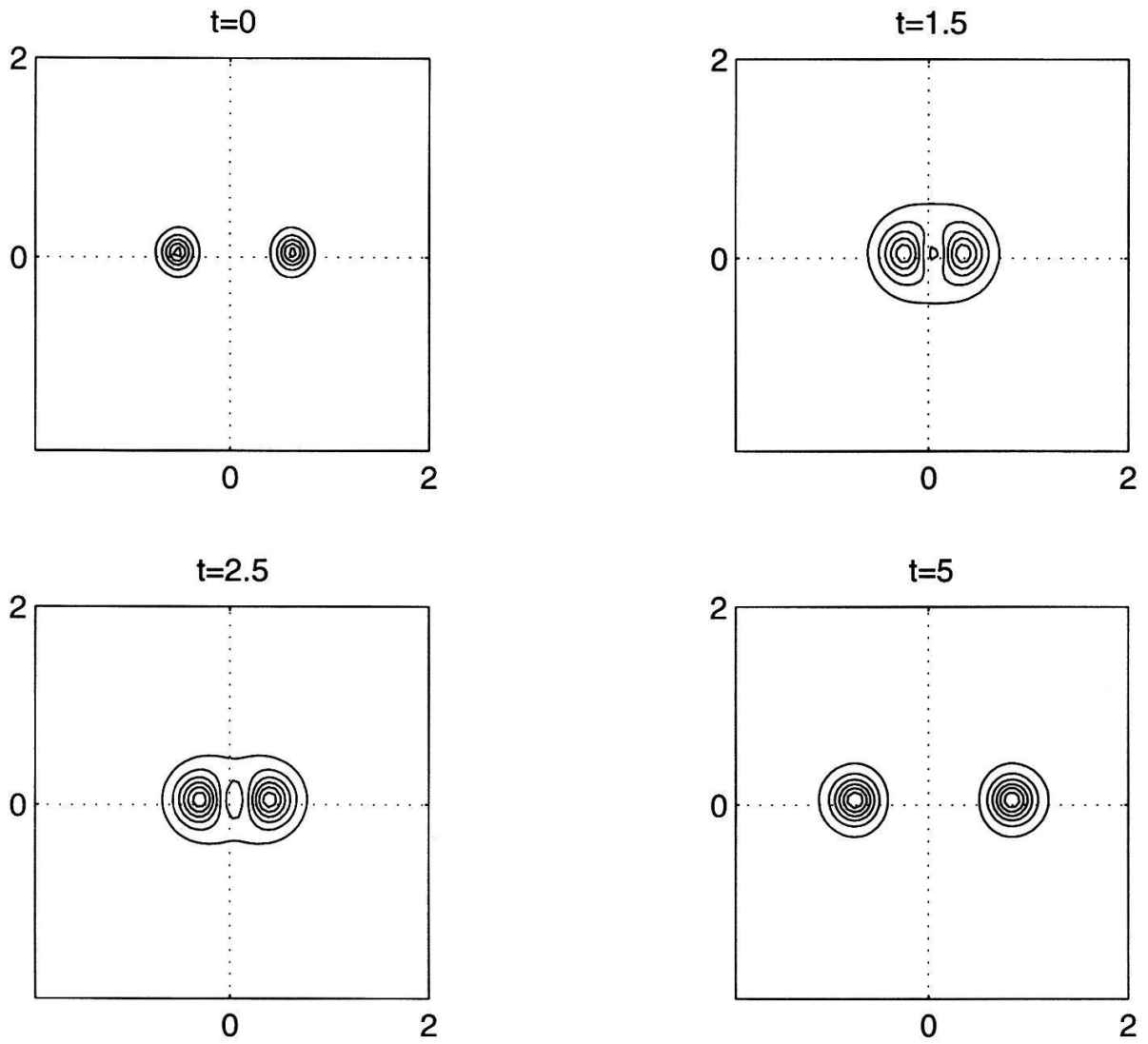Figure 3: Contour plots of the total energy density pictures shown in Figure 2.

Emax=283.7
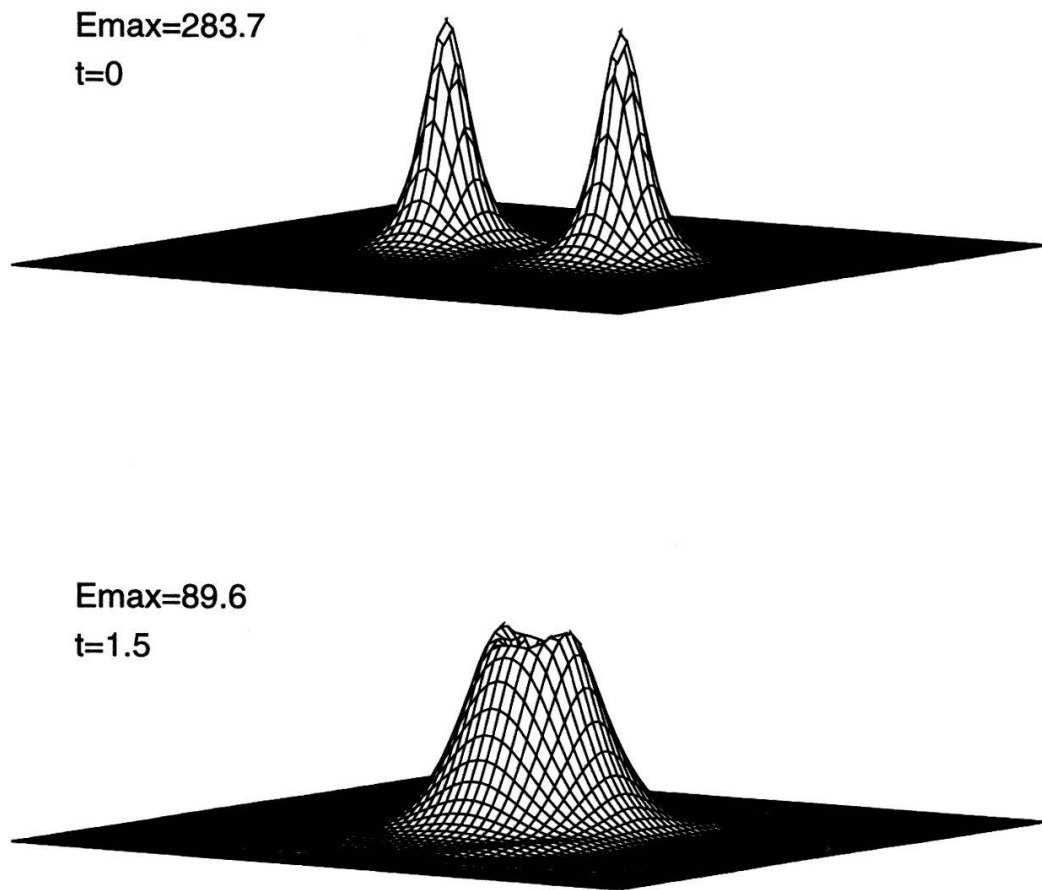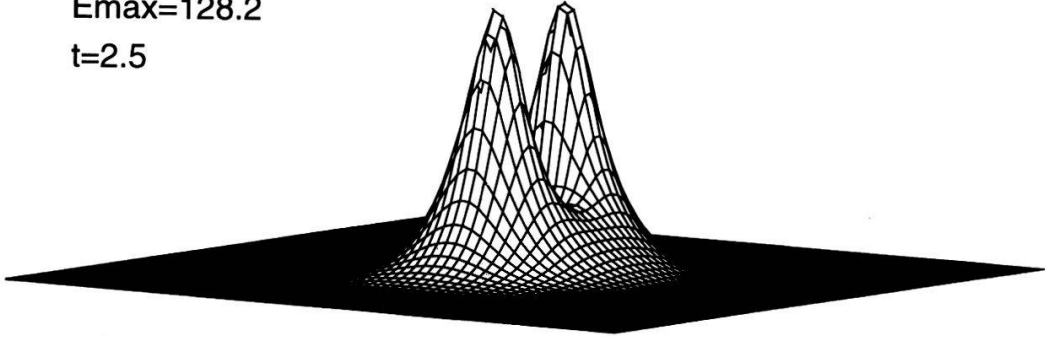t=0

Emax=89.6
t=1.5

Figure 4: Scattering at 90 degrees for the case $v = (0.3, 0.0)$.
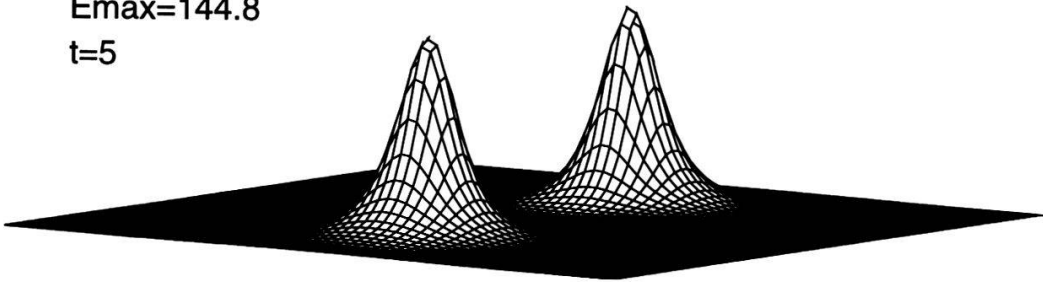
Emax=128.2
t=2.5

Emax=144.8
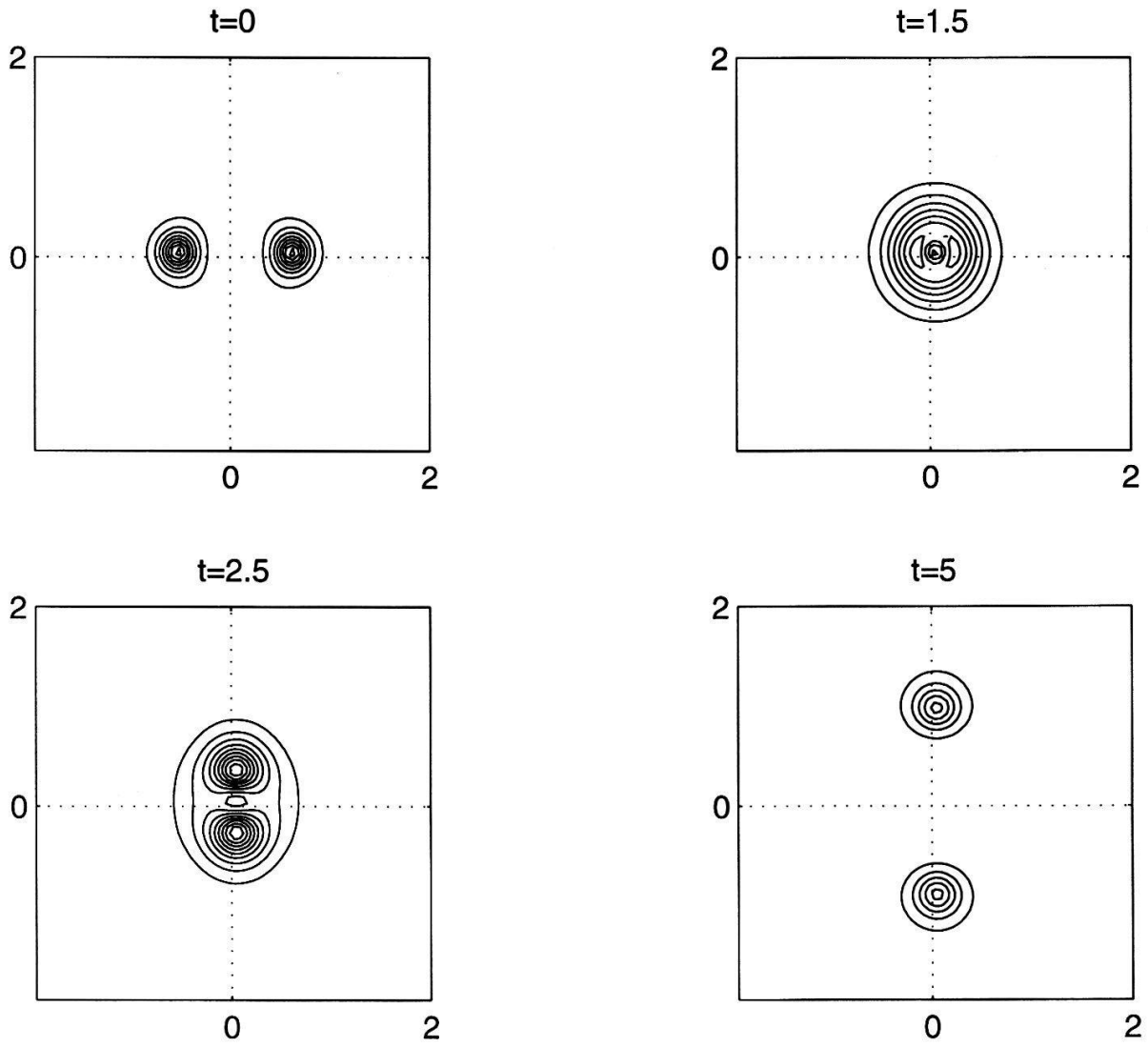t=5

Figure 4:Continued.

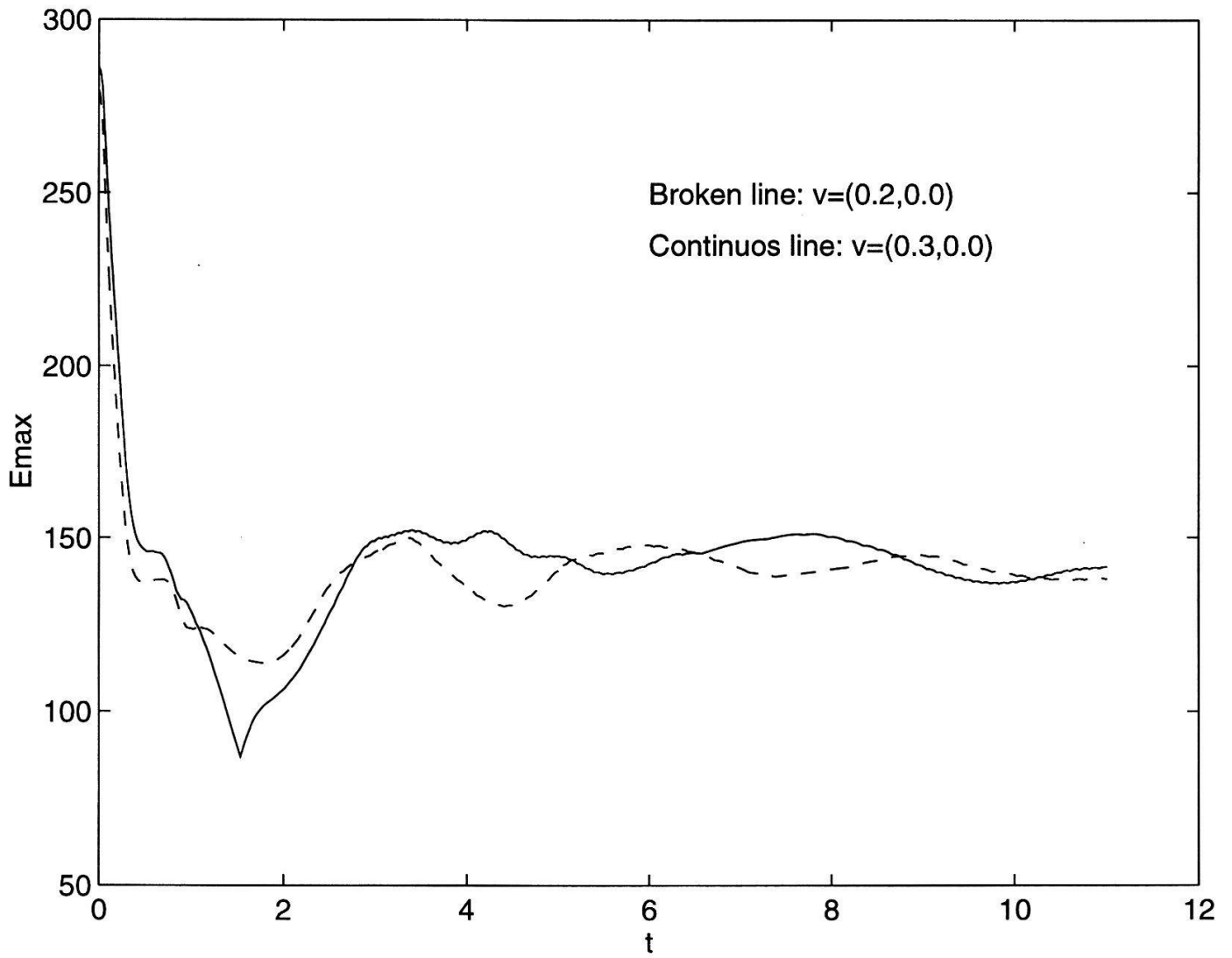Figure 5: Contour plots for the scattering of Figure 4.

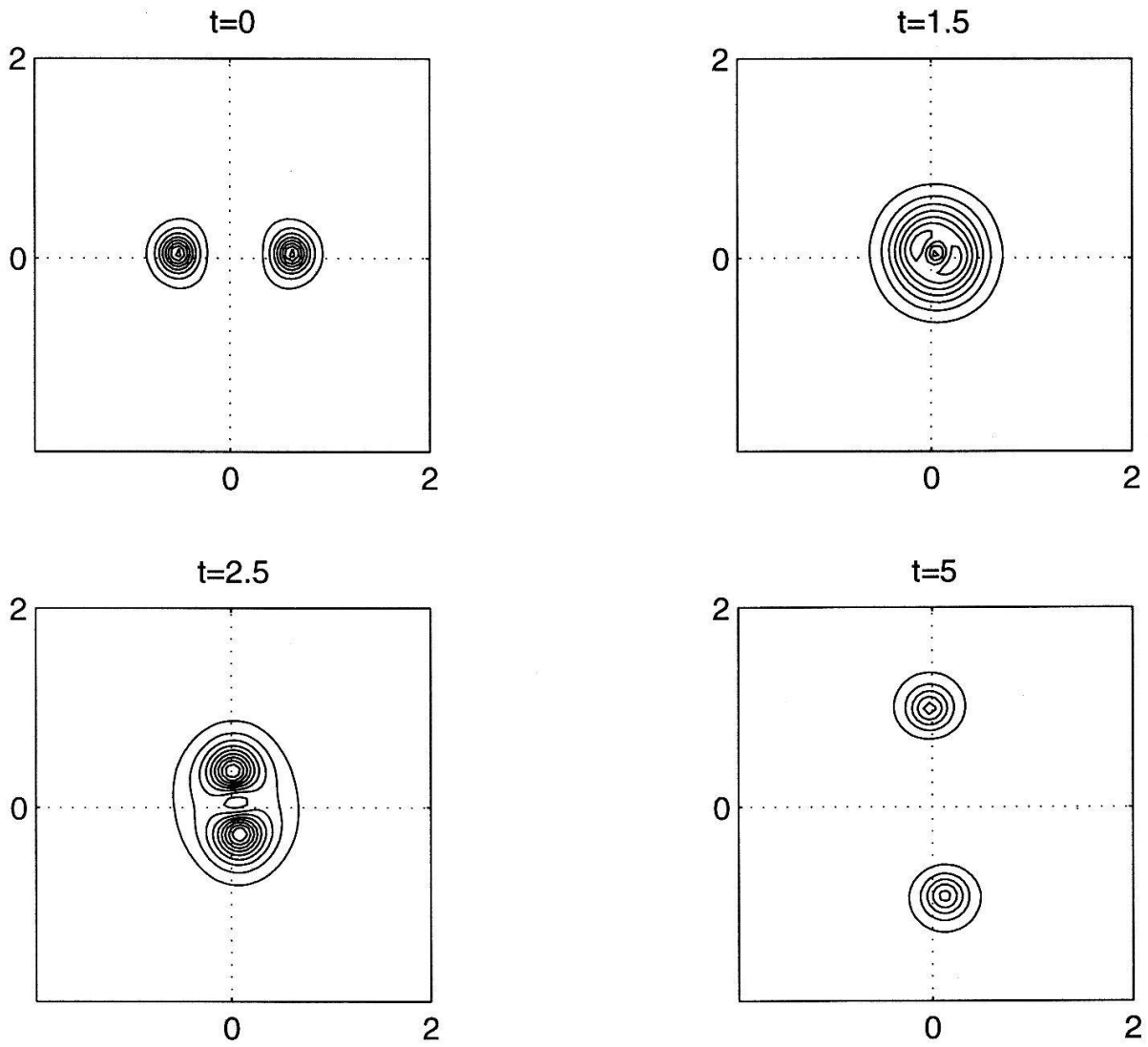Figure 6: Maximum of total energy density *vs.* time.

Figure 7: Non-zero impact parameter collision. The initial velocity is $v = (0.3, 0.01)$.

5   Conclusions

We have performed a numerical study of the scattering properties of the general two-instanton (-skyrmion) field configuration of our version of the Skyrme model in (2+1)-D, confirming the results previously found in other version of the model where, unlike the case studied in this paper, the finite-difference expressions for the derivatives of the fields enjoyed the numerically convenient feature of being exact. This factor, however, did not affect the qualitative scattering properties of the model under study.

Although our field configurations resemble two instantons in an approximate manner - the model is not integrable-, they exhibit a clear soliton-like behaviour. All radiation effects are small and the skyrmions' total energy density profiles are preserved during the scattering process. They may get distorted, but always recover when the distance between the skyrmions becomes large enough. Their interaction is of a repulsive nature, at least at large distances, the interaction being more difficult to asses when the skyrmions are close together. For head-on collisions there is a velocity below which the skyrmions bounce back, and above which they scatter at ninety degrees. There is a resemblance both with the properties of kinks in the ϕ^4 model, which has a critical velocity, and with monopole scattering at $90°$.

Acknowledgements

References

[1] Ward R. S. (1988) *Nonlinearity* **1** 671.

[2] Belavin A. A. and Poliakov A. M. (1975) *JETP letters* **22** 245.

[3] Woo G. (1977) *J. Math. Phys.* **18** 1264.

[4] Leese R. A., Peyrard M. and Zakrzewski W. J. (1990) *Nonlinearity* **3** 387.

[5] Leese R. A., Peyrard M. and Zakrzewski W. J. (1990) *Nonlinearity* **3** 773.

[6] Cova R. J. and Zakrzewski W. J. (Minsk 1994) *World Scientific* Proc. Quantum systems, new trends and methods. Also (1995) *hep-th* 9504100.

[7] Azcárraga J. A., Rashid M. S. and Zakrzewski W. J. (1991) *J.Math.Phys.* **32** 1921.