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THE DIRAC EVOLUTION EQUATION IN THE PRESENCE OF AN ELECTROMAGNETIC WAVE

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Abstract: We consider the Dirac evolution equation in presence of a time dependent electromagnetic potential which is a solution of the homogenous wave equation with regular and compactly supported initial data. We prove a propagation property for the free Dirac Hamiltonian using the explicit form of the free propagator that we use together with an energy estimation and the finite propagation speed of the Dirac evolution we prove existence and unitarity of the wave operators associated to the couple of Dirac evolutions: the free one and the time dependent one.

1. INTRODUCTION

In this paper we study the existence and completeness of the wave operators for a Dirac Hamiltonian with a time dependent potential defined as the solution of the free Maxwell equations with regular and compactly supported initial data. This problem is of interest in connection with the study of the Dirac Quantum Field in interaction with an external electromagnetic field. It can be shown [T] that if the "one-particle" scattering matrix exists and satisfies a special property, then the scattering matrix for the quantum field also exists and can be computed by the second quantization procedure. This problem is considered in [P] but only partial results are obtained and nothing is said concerning the completeness of the "one-particle" wave operators. We restrict ourselves to the case of compactly supported electromagnetic fields and prove that the wave operators exist and are unitary.

We shall work in the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$ and we shall denote by Q_j the operator of multiplication with x_j in $\mathcal{H}$ and by D_j the operator $-i\partial/\partial x_j \equiv -i\partial_j$. We shall also use the notations: $D=-i\nabla$, $\partial_t=\partial/\partial t$. We shall denote $<\xi>=\{1+|\xi|^2\}^{1/2}$ for $\xi\in\mathbb{R}^n$ and also for n-tuples of commuting selfadjoint operators, by using the functional calculus for selfadjoint operators. We shall denote by $B(x_0,R)$ the closed ball of radius R and center x_0 and by $S(x_0,R)$ its surface. For any subset Δ in $\mathbb{R}^n$ we shall denote by χ_Δ its characteristic function and by Δ^c its complementary in $\mathbb{R}^n$. Moreover we shall denote by $\chi(|Q|<R)$ the selfadjoint operator associated to the function $\chi_{\{|\xi|<R\}}$ by the functional calculus for selfadjoint operators and similarly for "$<$" replaced by "$\leq,>,\geq$". We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and by $\mathcal{B}(\mathbb{C}^4)$ the algebra of linear operators on $\mathbb{C}^4$. For any $s\in\mathbb{R}$ we denote by $H^s(\mathbb{R}^3)$ the Sobolev space of order s on $\mathbb{R}^3$ with the norm: $\|u\|_s=\|<D>^s u\|_{L^2(\mathbb{R}^3)}$ and $\mathcal{H}^s=H^s(\mathbb{R}^3)\otimes\mathbb{C}^4$.

Let us consider a Dirac Hamiltonian describing an electron in interaction with an external electromagnetic field without sources. We shall consider the light velocity $c=1$. The electromagnetic field in the Coulomb gauge is described by a three component real vector field $A_j(x,t)$ with $j=1,2,3$ and $x\in\mathbb{R}^3$, $t\in\mathbb{R}$ that satisfies the homogeneous wave equation :

$$(1.1)\qquad \left(\frac{\partial^2}{\partial t^2}-\Delta\right)A_j=0 \qquad \text{for } j=1,2,3.$$

We consider the following type of initial data:

$$(1.2)\qquad \begin{aligned} &A_j(x,0)=a_j(x), && a_j\in C_0^\infty(\mathbb{R}^3)\\ &\frac{\partial}{\partial t}A_j(x,0)=b_j(x), && b_j\in C_0^\infty(\mathbb{R}^3)\end{aligned}$$

$$(1.3)\qquad \bigcup_{j=1,2,3}\{(\operatorname{supp} a_j)\cup(\operatorname{supp} b_j)\}\subset B(0,R).$$

It is well-known that the solution of (1.1) in $\mathbb{R}^3$ can be written in the form :

$$(1.4)\qquad A_j(x,t)=\frac{1}{4\pi}\left\{\frac{\partial}{\partial t}\left(t\int_{|y|=1}a_j(x+ty)d\sigma(y)\right)+t\int_{|y|=1}b_j(x+ty)\,d\sigma(y)\right\}.$$

Let α_j for $j=1,2,3$ and β be the Dirac matrices (complex hermitian 4 by 4 matrices) so that the Dirac Hamiltonian will be :

$$(1.5)\qquad \begin{aligned} &H(t)=-i\alpha\cdot\nabla+m\beta+e\alpha\cdot A=H_0+V(t)\\ &H_0=-i\alpha\cdot\nabla+m\beta\\ &V(t)=e\alpha\cdot A\end{aligned}$$

where $m>0$. Sometimes we shall denote the unit matrix in $\mathbb{C}^4$ by α_0 in order to simplify some notations. We shall be interested in the nonhomogeneous time evolution on $\mathcal{H}$ generated by the family $\{H(t)\}_{t\in\mathbb{R}}$ given by (1.5) i.e. the two parameter family $\{U(t,s)\}_{t,s\in\mathbb{R}}$ of unitary operators on $\mathscr{H}$, solution of the Cauchy problem :

$$\begin{aligned} &i\partial_t U(t,s) = H(t)U(t,s) \\ &U(s,s) = \mathbb{1}. \end{aligned} \tag{1.6}$$

One can easily observe that for $|t|\rightarrow\infty$, H(t) goes in norm resolvent sense [RS1] to H_0 and we would like to compare the nonhomogeneous evolution U(t,s) with the free one generated by H_0 :

$$U_0(t) = \exp\{-itH_0\}. \tag{1.7}$$

We shall define the operators:

$$\begin{aligned} &W_s(t) = U(s,t)U_0(t-s) \\ &W_s^*(t) = U_0(s-t)U(t,s) \end{aligned} \tag{1.8}$$

and we shall study the existence of their limits when $|t|\rightarrow\infty$, with respect to the strong operator topology on $\mathscr{B}(\mathscr{H})$. Our main result is that for $t\rightarrow\pm\infty$ both the above operators have limits in the strong topology . We denote these limits by:

$$\begin{aligned} &W_s^{\pm} = \underset{t\rightarrow\pm\infty}{\text{s-lim}}\ W_s(t) \\ &(W_s^{\pm})^* = \underset{t\rightarrow\pm\infty}{\text{s-lim}}\ W_s^*(t) \end{aligned} \tag{1.9}$$

for any $s\in\mathbb{R}$.

For $k\in\mathbb{R}^3$ we shall denote $v(k)\in\mathbb{R}^3$ the classical velocity corresponding to the momentum k, for the free movement :

$$v(k) = (k^2+m^2)^{-1/2}k. \tag{1.10}$$

For $K\in\mathbb{R}_+$ we denote :

$$u(K) = K(K^2+m^2)^{-1/2} \in [0,1). \tag{1.11}$$

2. THE FREE DIRAC PROPAGATOR

In studying the limits 1.9 some detailed information about the free evolution $U_0(t)$ will be needed and we dedicate this section to this problem. Let us begin by deducing an explicit form for the distribution kernel of $U_0(t)$ for any $t\neq0$.

It is well known [T] that H_0 defines a selfadjoint operator on $\mathscr{H}$, having domain $\mathscr{H}^1$ and spectrum $\sigma(H_0)=(-\infty,-m]\cup[m,\infty)$ and being essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$. Thus $U_0(t)$ defined by (1.5) is the strongly continuous unitary group generated by H_0.

Let $\mathscr{F}:L^2(\mathbb{R}^n,dx)\to L^2(\mathbb{R}^n,dk)$ be the unitary operator induced by the Fourier transform :

$$(2.1)\qquad (\mathscr{F}f)(k)\equiv\hat{f}(k):=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ikx}f(x)d^nx.$$

We shall also denote by $\mathscr{F}$ the unitary operator on $\mathscr{H}$ obtained by taking the tensor product with $\mathbb{1}\in\mathscr{B}(\mathbb{C}^4)$.

We denote by δ_a the usual Dirac measure at point $a\in\mathbb{R}$, by $\delta(x=0)$ the Dirac measure at 0 in $\mathbb{R}^n$ and by $\delta(|x|=a)$ the inverse image $f^*\delta_a$ of δ_a by the function $\mathbb{R}^n\ni x\mapsto f(x)=|x|\in\mathbb{R}$. Then for $\phi\in\mathscr{S}(\mathbb{R}^n)$ we have :

$$(2.2)\qquad <\delta(|x|=a),\phi> = \int_{|x|=a}\phi(x)d\sigma_a(x) = a^{n-1}\int_{|v|=1}\phi(av)d\sigma(v)$$

where $d\sigma_r$ is the measure induced by the Lebesgue measure of $\mathbb{R}^n$ on the sphere $S(0,r)$.

Now let us study the free evolution in the representation obtained by the Fourier transform. We define:

$$(2.3)\qquad \hat{H}_0=\mathscr{F}H_0\mathscr{F}^{-1}$$

and the following applications:

$$\begin{aligned}
&\mathbb{R}^3\ni k\mapsto \hat{H}_0(k):=\alpha\cdot k+m\beta\in\mathscr{B}(\mathbb{C}^4)\\
(2.4)\qquad &\mathbb{R}^3\ni k\mapsto \mu(k):=\{k^2+m^2\}^{1/2}\in\mathbb{R}_+\\
&\mathbb{R}^3\ni k\mapsto \hat{\Pi}_\pm(k):=(2\mu(k))^{-1}(\mu(k)\mathbb{1}\pm\hat{H}_0(k))\in\mathscr{B}(\mathbb{C}^4).
\end{aligned}$$

Then $\hat{H}_0$ is the operator of multiplication with $\hat{H}_0(k)$ and we have the relation:

$$(2.5)\qquad \hat{H}_0(k)=\mu(k)\,(\hat{\Pi}_+(k)-\hat{\Pi}_-(k)).$$

In this representation the unitary operator $U_0(t)$ is given by multiplication with the following matrix valued function :

$$\begin{aligned}
(2.6)\qquad \hat{U}_0(t;k) &= \exp\{-it\hat{H}_0(k)\} = \exp\{-it\mu(k)\}\hat{\Pi}_+(k)+\exp\{it\mu(k)\}\hat{\Pi}_-(k) =\\
&= (\partial_t-i\hat{H}_0)\,\frac{\sin t\mu(k)}{\mu(k)}\,.
\end{aligned}$$

We shall make the notation:

$$(2.7)\qquad \hat{S}(t;k) := \frac{\sin t\mu(k)}{\mu(k)}.$$

Remark 2.1: For any t, $\hat{S}(t;k)$ as a function of $k\in\mathbb{R}^3$ is a bounded function of class $C^\infty(\mathbb{R}^3)$ and thus belongs to $\mathcal{S}'(\mathbb{R}^3)$. Moreover it is easy to see that it is an odd function of t.

We want to compute the inverse Fourier transform of $\hat{S}(t;.)$ by making use of the integral representation and recurrence relations for Bessel functions [WW]. We shall denote by J_N the Bessel function of order N for which we have the following recurrence relations:

$$(2.8)\qquad J_1(z) = -\frac{d}{dz} J_0(z)$$

$$(2.9)\qquad J_0(z) = (z^{-1}\frac{d}{dz})^N (z^N J_N(z)).$$

Then by denoting $r=|x|$ we have the following formula [GSh]:

$$(2.10)\qquad \mathcal{F}\{\delta(|x|=a)\}(x) = a^{n/2} r^{1-n/2} J_{n/2-1}(ar).$$

If we consider now that $n=2m+3$ and the explicit form of the Bessel functions of half-integer order, then we can get the formula [GSh] :

$$(2.11)\qquad \mathcal{F}\{(a^{-1}\frac{d}{da})^m a^{-1}\delta(|x|=a)\}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin ar}{r}.$$

If we denote by S(t,x) the inverse Fourier transform of $\hat{S}(t;k)$ with respect to $k\in\mathbb{R}^3$, then one can prove the following result.

Lemma 2.2: *For* $t>0$ *we have* :

$$S(t;x) = \sqrt{\frac{\pi}{2}}\left\{(\frac{1}{t}\ \delta(|x|=t) - m^2\chi_{\{|y|\le t\}}\frac{J_1(m\{t^2-x^2\}^{1/2})}{m\{t^2-x^2\}^{1/2}})\right\}.$$

Proof: We shall view the function :

$$(2.12)\qquad \mathbb{R}^3 \ni k \mapsto \mu(k) := \{k^2+m^2\}^{1/2} \in \mathbb{R}_+$$

as being defined on the cylinder :

$$(2.13)\qquad \{(k,\lambda)\in\mathbb{R}^3\times\mathbb{R}^2 \mid \lambda_1^2+\lambda_2^2 = m^2\}$$

in $\mathbb{R}^5$. Thus if we denote $\rho=\{k^2+\lambda^2\}^{1/2}$ we have :

(2.14) $$S(t;x) = (2\pi)^{-3/2}\frac{1}{2\pi m}\int_{|\lambda|=m} d\sigma_m(\lambda)\left(\int_{\mathbb{R}^3} e^{ikx}\,\frac{\sin t\rho}{\rho}\,d^3k\right).$$

Thus one can observe that S(t;x) is the restriction at the hyperplane: $\{(x,\omega)\in\mathbb{R}^5 \mid \omega_1=\omega_2=0\}$ of the inverse Fourier transform of the following distribution in $\mathscr{S}'(\mathbb{R}^5)$:

(2.15) $$\tilde{S}(t;k,\lambda) = \frac{1}{m}\frac{\sin t\rho}{\rho}\,\delta(|\lambda|=m).$$

If we denote by T_1 the inverse Fourier transform of $(\rho^{-1}\sin t\rho)$ and if we make use of formula (2.11) we obtain :

(2.16) $$T_1(t;x,\omega) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{t}\frac{d}{dt}\right)\frac{\delta(\{x^2+\omega^2\}^{1/2}=t)}{t}\ .$$

We denote by T_2 the inverse Fourier transform of the distribution $\delta(|\lambda|=m)$ and using formula (2.10) we obtain :

(2.17) $$T_2(t;x,\omega) = (2\pi)^{3/2} m J_0(m|\omega|)\,\delta(x=0).$$

For S we have then the relation :

(2.18) $$S(t;x) = (2\pi)^{-5/2}(T_1 * T_2)(t;x,0).$$

Let now ϕ be a function in $\mathscr{S}(\mathbb{R}^5)$; using (2.18) we have :

(2.19) $$\langle S,\phi\rangle = (2\pi)^{-5/2}\sqrt{\frac{\pi}{2}}\left(\frac{1}{t}\frac{d}{dt}\right)\frac{1}{t}\langle f^*\delta_t,\Phi\rangle$$

(2.20) $$\Phi(x,\omega) = (2\pi)^{3/2} m \int_{\mathbb{R}^2} J_0(m|\omega'|)\,\phi(x,\omega+\omega')\,d^2\omega'\ .$$

Using theorem 6.1.2 in [H], considering the application:

$$\mathbb{R}_+\times[0,2\pi)\times\mathbb{R}^3 \ni (\rho,\xi,x) \longmapsto h(\rho,\xi,x) := (x,\{\rho^2-x^2\}^{1/2}\cos\xi,\{\rho^2-x^2\}^{1/2}\sin\xi)\in\mathbb{R}^5$$

and observing that $|\det h'(\rho,\xi,x)|=\rho$ we obtain :

(2.21) $$\langle S,\phi\rangle = (2\pi)^{-1}\sqrt{\frac{\pi}{2}}\,m\left(\frac{1}{t}\frac{d}{dt}\right)\int_{\mathbb{R}^3} d^3x\int_{\mathbb{R}^2} d^2\omega\,\phi(x,\omega)\cdot$$
$$\cdot\left(\chi_{\{|y|\le t\}}\int_{S(0,1)} J_0(m|\omega-\nu\{t^2-x^2\}^{1/2}|)\,d\sigma_1(\nu)\right).$$

Taking now into account relation (2.8) and differentiating in $\mathscr{S}'(\mathbb{R}^3)$ with respect to t we obtain the desired formula for S. ■

Proposition 2.3: *If* $U_0(t)$ *is the strongly continuous unitary group generated by* H_0 *and* $f \in \mathscr{S}(\mathbb{R}^3)$ *then we have the following formula:*

(i) $(U_0(t)\phi)(x) =$

$$\frac{1}{4\pi}\mathbb{D}\left\{\left(\frac{1}{t}\int_{S(0,1)}\phi(x-y)\,d\sigma_t(y)\right)-m^2(\text{sign } t)\int_{B(0,t)}\frac{J_1(m\{t^2-y^2\}^{1/2})}{m\{t^2-y^2\}^{1/2}}\phi(x-y)\,d^3y\right\}$$

(ii) $$(U_0(t)\phi)(x) = \frac{\text{sign } t}{4\pi}\mathbb{D}\int_{B(0,t)} d^3y\, J_0(m\{t^2-y^2\}^{1/2})\{(|y|^{-2}\phi(x-y)+|y|^2 y(\nabla_y\phi(x-y))\}$$

where $\mathbb{D}=\partial_t - i\hat{H}_0$.

Proof: The first formula results immediately from lemma 2.2 if one remarks that for t>0 the distribution kernel of $U_0(t)$ is just $(2\pi)^{-3/2}\mathbb{D}S(t;x-y)$ and it is easy to see that for t<0 a sign t appears due to remark 2.1. For the second formula one can use once more the relation 2.8 and integrate by parts in the first formula of the proposition so that the surface contribution cancels out with the first term and one gets the desired result by observing that if we put $z=\{t^2-\rho^2\}^{1/2}$ then :

(2.22) $$\frac{d}{dz} = -\frac{z}{m\rho}\frac{d}{d\rho} \quad . \quad ■$$

In the rest of this section we shall prove some propagation properties for the free evolution. First let us consider the case when at time zero the state has its Fourier transform of class $C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$.

Proposition 2.4: *Let* $f\in\mathscr{H}$ *be such that* $\hat{f}\in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$ *with* $\text{supp}\,\hat{f}\subset B(0,K)$, *then for any* $N\in\mathbb{N}$ *there is some* $C_N<\infty$, *depending on* N *and* f, *such that for* t>0 *one has*

$$\|\chi(|Q|\geq u(2K)t)U_0(t)f\| \leq \frac{C_N}{(1+t)^N} .$$

Proof: Using formula (2.6) we get:

(2.23) $(U_0(t)f)(x) =$

$$(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ikx-it\mu(k)}\hat{\Pi}_+(k)\hat{f}(x)\,d^3x+(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ikx+it\mu(k)}\hat{\Pi}_-(k)\hat{f}(x)\,d^3x$$

where $\hat{\Pi}_\pm\hat{f}\in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$ with support in B(0,K). Let us denote :

(2.24) $\theta(x,t) = |x|+t$

$$\phi_{\pm}(k;x,t) = \theta(x,t)^{-1}\{k\cdot x \pm t\mu(k)\}$$

so that the exponents in (2.23) are of the form $\theta\phi_{\pm}$. Now we observe that :

$$\nabla_k \phi_{\pm}(k;x,t) = \theta(x,t)^{-1}\{x \pm \frac{k}{\mu(k)}t\}. \tag{2.25}$$

We evidently have that $\frac{|k|}{\mu(k)} \le u(K)$, so that if $|x| \ge u(2K)t$ we shall have :

$$|(\nabla_k\phi_{\pm})(k;x,t)| \ge \frac{u(2K)-u(K)}{|x|+t}\,t \ge \frac{u(2K)-u(K)}{1+u(2K)} > 0. \tag{2.26}$$

Thus the usual non-stationary phase argument [RS3] gives us the result. ■

Now let us consider the case when at time zero the state is of class $C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$.

Proposition 2.5: *Let* $f\in\mathcal{H}$ *be such that* $f\in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$ *with* $\operatorname{supp} f\subset B(0,r)$, *then for* $t\in\mathbb{R}_+$, $u_0\in(0,1)$ *and* $x\in\mathbb{R}^3$ *with* $|x|>r+ut$ *and* $u\in[u_0,1)$ *we have that for any* $N\in\mathbb{N}$ *we can find a constant* C_N *depending only on* N *and* u_0 *but not on* f, *such that* :

$$|(U_0(t)f)(x)| \le C_N\, t\{1-u^2\}^{N/2}\|(\mathbb{1}-\Delta)^{(N+2)/2} f\|_{L^2(\mathbb{R}^3)}.$$

Proof: For $x\in\mathbb{R}^3$ and $\rho\in\mathbb{R}_+$ we shall use the notation :

$$\langle f\rangle(x;\rho) := \int_{|v|=1} f(x-\rho v)\,d\sigma_1(v) \tag{2.27}$$

so that the second point of proposition 2.3 may be written :

$$(U_0(t)f)(x) = \frac{\operatorname{sign} t}{4\pi}\,\mathbb{D}\int_0^t d\rho\, J_0(m\{t^2-\rho^2\}^{1/2})\frac{d}{d\rho}(\rho\langle f\rangle(x;\rho)). \tag{2.28}$$

If we use now the fact that $\operatorname{supp} f\subset B(0,r)$ so that $|x-\rho v|\le r$ and the condition imposed to x, we get:

$$(U_0(t)f)(x) = \frac{\operatorname{sign} t}{4\pi}\,\mathbb{D}\int_{ut}^t d\rho\, J_0(m\{t^2-\rho^2\}^{1/2})\frac{d}{d\rho}(\rho\langle f\rangle(x;\rho)). \tag{2.29}$$

Now we remind the recurrence relation (2.9) and the relation (2.22) and we make the notation $z=\{t^2-\rho^2\}^{1/2}$ in order to get:

$$(U_0(t)f)(x) = \frac{\operatorname{sign} t}{4\pi}\,\mathbb{D}\int_{ut}^t d\rho\,\left(\frac{1}{z}\frac{d}{dz}\right)^N(z^N J_N(z))\frac{d}{d\rho}(\rho\langle f\rangle(x;\rho)) = \tag{2.30}$$

$$= \frac{1}{m^{2N}}\frac{\operatorname{sign} t}{4\pi}\,\mathbb{D}\int_{ut}^t d\rho\,((z^N J_N(z))\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^N(\rho\langle f\rangle(x;\rho)).$$

Using now the Hölder inequality, the fact that $|J_N(z)|\leq 1$ and denoting $z_0=t\{1-u^2\}^{1/2}$ we obtain :

(2.31) $$|(U_0(t)f)(x)| \leq$$

$$\leq \frac{C}{m^{2N}} \frac{\text{sign } t}{4\pi} \mathbb{D}\,(\int_0^{z_0} z^{2N+1}dz))^{1/2}(\int_{ut}^{t} \{(\frac{1}{\rho}\frac{d}{d\rho})^N(\frac{1}{\rho}\frac{d}{d\rho})(\rho(f)(x;\rho))\}^2\rho^2 d\rho)^{1/2} \leq$$

$$\leq \frac{Cz_0^{N+1}}{m^{2N}} \frac{\text{sign } t}{4\pi} \mathbb{D}\,(\int_{ut}^{t} \{(\frac{1}{\rho}\frac{d}{d\rho})^N(\frac{1}{\rho}(f)(x;\rho)+\frac{d}{d\rho}(f)(x;\rho))\}^2\rho^2 d\rho)^{1/2}.$$

It is very easy to see by induction on N that the following formula is true :

(2.32) $$(\frac{1}{\rho}\frac{d}{d\rho})^N F(\rho) = \rho^{-N}\Sigma_{j=1}^N a_N(j)F^{(j)}(\rho)(\frac{-1}{\rho})^{N-j}$$

where $a_N(0)=0$, $a_N(N)=1$, $a_N(j)=(N-j-1)a_{N-1}(j)+a_{N-1}(j-1)$. Finally, considering also the derivatives appearing in $\mathbb{D}$ and the fact that $\rho\geq ut$ we get the desired result. ■

3. THE NONHOMOGENEOUS TIME EVOLUTION

In this section we want to define and study the solution of the Cauchy problem (1.6). We shall begin by discussing some properties of the electromagnetic potential defined by (1.1-1.2). It is well known that the solution of (1.1) in $\mathbb{R}^3$ can be written in the form (1.4).

Lemma 3.1: *The solution* (1.4) *has the following properties* :

a) *For* $t\geq R$, supp $A_j \subset \mathscr{K}_{R,t} \equiv \{x\in\mathbb{R}^3 \mid t-R\leq|x|\leq t+R\}$,

b) $\sup_{x\in\mathbb{R}^3} |A_j(x,t)| \leq \frac{C}{1+|t|}$,

c) $\|A_j(.,t)\|_{L^2(\mathbb{R}^3)} \leq C$, *for any* t.

Proof: We shall consider only the case $t>0$, the other case being quite similar. Due to formula (1.4), in order to have $A_j(x,t)\neq 0$ we must have $|x+ty|\leq R$ for some $y\in S(0,1)$. But then we have $||x|-t| \leq |x+ty| \leq |x|+t$ and point (a) follows immediatly. Now let us change variables in formula (1.4) and pass to the sphere S(0,t) in order to get:

(3.1) $$A_j(x,t) = \frac{1}{4\pi}\{ \frac{-1}{t^2} \int_{S(0,t)\cap B(x,R)} a_j(x+y)d\sigma_t(y) +$$

$$+ \frac{1}{t} \int_{S(0,t)\cap B(x,R)} (b_j(x+y)+(y\cdot\nabla)a_j(x+y))d\sigma_t(y)\}.$$

It is easy to see that for $|x|>R$ we have the relations:

$$(3.2) \qquad \mathscr{M}(S(0,t)\cap B(x,R)) = \frac{\pi t}{|x|}\left(R^2-(t-|x|)^2\right) \leq \frac{\pi R^2(R+|x|)}{|x|} \leq 2\pi R^2$$

(where $\mathscr{M}(\Delta)$ denotes the Lebesgue measure of Δ), so that one easily obtains the estimation of point (b). Moreover we see that :

$$(3.3) \qquad \mathscr{M}(\mathscr{K}_{R,t}) = \frac{4\pi}{3}\left((R+t)^3-(t-R)^3\right) = \frac{8\pi R}{3}(3R^2+t^2)$$

so that one gets the estimation of point (c) by using point (b). ■

Corollary 3.2: *We have the following estimation for the operator norm of the time dependent potential defined in* (1.5) :

$$\|(\partial_t)^k V(t)\|_{\mathscr{B}(\mathscr{H})} \leq \frac{C}{1+|t|}.$$

Let us now consider the existence of the solution of the Cauchy problem (1.6). We shall follow [P] and use the procedure described in chapter X.12 of [RS 2]. The results that we proved concerning the electromagnetic potential imply that V(t) is a hermitian bounded operator on $\mathscr{H}$, for any $t\in\mathbb{R}$ and goes to 0 for $t\to\infty$, with respect to the operator norm. Thus for any $t\in\mathbb{R}$ we can define : $H(t) = H_0 + V(t)$ as a self-adjoint operator in $\mathscr{H}$ with domain $\mathscr{H}^1$. We conclude that the problem (1.6) is well defined. For the completeness of our argument we shall review the main points of the construction in chapter X.12 of [RS 2]. Let us first define the bounded operator :

$$(3.4) \qquad \tilde{V}(t) = U_0(t)^* V(t) U_0(t).$$

Proposition 3.3: *For any* $t\in\mathbb{R}$ *the operator* $\tilde{V}(t)$ *is bounded and hermitian in* $\mathscr{H}$ *and there exists a two-parameter family of unitary operators* $\{\tilde{U}(t,s)\}_{t,s\in\mathbb{R}}$ *in* $\mathscr{H}$ *such that:*

i) the application $\mathbb{R}^2\ni(t,s)\mapsto\tilde{U}(t,s)\in\mathscr{B}(\mathscr{H})$ is strongly continuous.

ii) $\tilde{U}(r,s)\tilde{U}(s,t) = \tilde{U}(r,t)$, *for any* $r,s,t\in\mathbb{R}$.

iii) $i\partial_t\tilde{U}(t,s) = \tilde{V}(t)\tilde{U}(t,s)$, *for any* $t,s\in\mathbb{R}$ *and* $\tilde{U}(s,s)=\mathbb{1}$ *for any* $s\in\mathbb{R}$.

Proof: We want to make use of the theorem X.69 in [RS 2], so that we have to verify that the application: $\mathbb{R}\ni t\mapsto\tilde{V}(t)\in\mathscr{B}(\mathscr{H})$ is strongly continuous. Let us fix t_1,t_2 in $\mathbb{R}$, then $\tilde{V}(t_1)-\tilde{V}(t_2)=U_0(t_1)^*V(t_1)U_0(t_1)-U_0(t_2)^*V(t_2)U_0(t_2)$. But the unitary group generated by H_0 is strongly continuous and uniformly bounded so that all we have to prove is that these facts are also true for the application $\mathbb{R}\ni t\mapsto V(t)\in\mathscr{B}(\mathscr{H})$. The uniform boundedness follows from the

Corollary 3.2 and we shall prove now that it is continuous even in the operator norm topology. In fact :

$$\|V(t_1)-V(t_2)\| \leq C \max_{j=1,2,3} \| A_j(\cdot,t_1)-A_j(\cdot,t_2)\|_{L^\infty(\mathbb{R}^3)} .$$

Using now formula (3.1) and the dominated convergence theorem one proves the proposition. ∎

We shall now define the unitary operators:

(3.5) $$U(t,s) := U_0(t)\tilde{U}(t,s)U_0(s)^* \in \mathcal{B}(\mathcal{H}) .$$

Lemma 3.4: *For* $t,s\in\mathbb{R}$ *and* $f\in\mathcal{H}^1$ *we have that* $U(t,s)f\in\mathcal{H}^1$.

Proof: Evidently D_j commutes with H_0 for any $j=1,2,3$ and one can see that $\mathcal{H}^1=\mathcal{D}(H_0)=\{f\in\mathcal{H} \mid D_j f\in\mathcal{H}$ for $j=1,2,3\}$. In the sense of distributions we have that $D_jU(t,s)f=U_0(t)D_j\tilde{U}(t,s)U_0(s)^*f$. But the unitary group generated by H_0 leaves its domain invariant, so that all we have to study is $D_j\tilde{U}(t,s)f$ for $f\in\mathcal{H}^1$ and show that it belongs to $\mathcal{H}$. Formula X.129 in [RS 2] gives :

(3.6) $$\tilde{U}(t,s)f = f+\Sigma_{k=1}^{\infty}(-i)^n\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_{n-1}}dt_n\tilde{V}(t_1)\tilde{V}(t_n)f.$$

If $f\in\mathcal{H}^1$, then $D_jV(t)f=V(t)D_jf-i(\partial_jV(t))f$ and for the derivatives of V(t) we have the estimations:

(3.7) $$\partial_jV(t) = e\Sigma_{k=1}^{3}(\alpha_j\partial_jA_k(x,t))$$

and using relation (3.2) we obtain that supp $(\partial_jA_k)\subset\mathcal{K}_{R,t}$ and $\partial_jA_k\in C_0^\infty(\mathbb{R}^3)$. Thus :

(3.8) $$\|\partial_tV(t)\|_{\mathcal{B}(\mathcal{H})} \leq \frac{C}{1+|t|} .$$

In conclusion, if $f\in\mathcal{H}^1$ then $V(t)f\in\mathcal{H}^1$ for any real t. Moreover :

$$D_j\Pi_{i=1}^{n}V(t_i) = \prod_{i=1}^{n}V(t_i)D_j+\Sigma_{l=1}^{n}\prod_{i=1}^{l-1}V(t_i)[D_j,V(t_l)]\prod_{i=l+1}^{n}V(t_i).$$

Combining all these results we obtain that $[D_j,\tilde{U}(t,s)]\in\mathcal{B}(\mathcal{H})$ and:

$$\| [D_j,\tilde{U}(t,s)]\|_{\mathcal{B}(\mathcal{H})} \leq C|t-s|e^{c|t-s|}. \quad ∎$$

Proposition 3.5: *The operators* $\{U(t,s)\}_{t,s\in\mathbb{R}}$ *form a two-parameter family of unitary operators satisfying* :

i) *the application* $\mathbb{R}^2 \ni (t,s) \mapsto U(t,s) \in \mathcal{B}(\mathcal{H})$ *is strongly continuous.*

ii) $U(r,s)U(s,t) = U(r,t)$, *for any* $r,s,t \in \mathbb{R}$.

iii) $i\partial_t U(t,s) = H(t)U(t,s)$, *for any* $t,s \in \mathbb{R}$ *and* $U(s,s) = \mathbb{1}$ *for any* $s \in \mathbb{R}$.

Proof: Take $f \in \mathcal{H}^1$, then by using lemma 3.4 and the fact that $\tilde{V}(t) \in \mathcal{B}(\mathcal{H})$ we see that

$$\partial_t U(t,s)f = \partial_t U_0(t)\tilde{U}(t,s)U_0(s)^* f = U_0(t)\{(-iH_0 - i\tilde{V}(t)\}\tilde{U}(t,s)U_0(s)^* f =$$
$$= -i\{H_0 + V(t)\}U(t,s)f = -iH(t)U(t,s)f$$

and observing that $\mathcal{D}(H) = \mathcal{D}(H_0) = \mathcal{H}^1$ one finishes the proof of the proposition. ■

In the following we prove some estimations for the evolution of the position and momentum observables.

Proposition 3.6: *For any* $f \in \mathcal{D}(\langle Q\rangle)$ *we have the following estimation* :

$$\| Q_j U(t,s)f\| \leq C\{ \| Q_j f\| + |t-s| \| f\| \}.$$

Proof: We shall follow the proof of a similar result in [T]. Let $\lambda > 0$ and let us define the bounded operator $X_{j,\lambda} = Q_j(\mathbb{1} + \lambda|Q|)^{-1} \in \mathcal{B}(\mathcal{H})$. For $f,h \in \mathcal{H}^1$ one has :

$$(h, U(t,s)^* X_{j,\lambda} U(t,s)f) = (h, X_{j,\lambda} f) + i\int_s^t (h, U(\tau,s)^* [H_0, X_{j,\lambda}]\, U(\tau,s)f)d\tau =$$

$$= (h, X_{j,\lambda} f) + \int_s^t (h, U(t,s)^* \alpha [D, X_{j,\lambda}] U(t,s)f)\, dt$$

$$[D_k, X_{j,\lambda}] = \delta_{jk}(\mathbb{1} + \lambda|Q|)^{-1} - \lambda \frac{Q_j Q_k}{|Q|(\mathbb{1} + \lambda|Q|)^2}$$

and we observe that $[D_k, X_{j,\lambda}] \in \mathcal{B}(\mathcal{H})$ and is uniformly bounded for $\lambda \to 0$. Thus if we take now $h \in \mathcal{H}$ and $f \in \mathcal{D}(\langle Q\rangle)$ we get :

$$|(h, U(t,s)^* X_{j,\lambda}\, U(t,s)f)| \leq |(h, X_{j,\lambda} f)| + C|t-s| \| f\| \| h\| .$$

Taking now the limit $\lambda \to 0$, we obtain the result. ■

In order to study the time evolution of the momentum we shall need some results concerning some multiple commutators.

Lemma 3.7: *Suppose* $f \in \mathcal{H}^1$, *then* $U(t,s)f \in \mathcal{H}^1$, *and we have the estimation* :

$$\| |D| U(t,s)f\| \leq C\{ \| |D| f\| + \ln(1 + |t-s|)\, \| f\| \}.$$

Proof: Let us define the following function of t, for s fixed:

$$\xi(t) := \| \,|D|U(t,s)f\|^2 \geq 0$$

and let us compute its derivative

$$\partial_t\xi(t) = i(f,U(t,s)^*[H(t),|D|^2]U(t,s)f) \;= i\Sigma_{j=1}^{3}(f,U(t,s)^*[V(t),D_j^2]U(t,s)f).$$

Computing the commutator and taking the module we get :

$$|\partial_t\xi(t)| \leq 2\Sigma_{j=1}^{3}\|D_jU(t,s)f\| \,\|(\partial_jV(t))U(t,s)f\| \leq C\sqrt{\xi(t)}<t>^{-1}\|f\|^2.$$

Thus $\xi(t) \leq \| \,|D|f\|^2 + C\int_s^t\sqrt{\xi(\tau)}\,\tau^{-1}d\tau$ and using now a simple form of the Gronwall lemma we obtain the result. ■

For j=1,2,3 let us denote $d=D_j$ and by induction for N≥0 the multiple commutators $V_N=[V_{N-1},d]$ that are nothing else but i^N times the N-th partial derivative of V(t) with respect of x_j .

Lemma 3.8: *The following relation is true* :

$$[V,d^N] = \Sigma_{k=1}^{N}C_N^K\, d^{N-K}V_K$$

Proof: For N=1 the formula is obvious and we can proceed by induction on N. ■

Lemma 3.9: *The following relation is true* :

$$[V,d^{2N}] = \Sigma_{k=1}^{N}C_N^K\,(d^{N-K}V_Kd^N + d^NV_Kd^{N-K}).$$

Proof: For N=1 we have $[V,d^2] = d[V,d] + [V,d]d$ and we shall proceed by induction on N, by denoting :

$$\{d^{N-K}V_Kd^N\} := (d^{N-K}V_Kd^N + d^NV_Kd^{N-K}).$$

Then :

$$[V,d^{2N+2}] = [V,dd^{2N}d] = \{V_1d^{2N+1}\} + d[V,d^{2N}]d =$$

$$=\{d^NV_1d^{N+1}\} + \{[V_1,d^N]d^{N+1}\} + d\Sigma_{k=1}^{N}C_N^K\,\{d^{N-K}V_Kd^N\}\,d.$$

Using now lemma 3.8 for the second term one easily gets the result. ■

Proposition 3.10: *For* $K\in\mathbb{N}$ *and* $f\in\mathscr{H}^K$ *we have the estimation* :

$$\| \, |D|^K U(t,s)f \| \leq C\{\| \, |D|^K f\| \, (\ln(1+|t-s|))^K \| f\| \, \}.$$

Proof: We shall proceed as in lemma 3.7 and we shall consider the function :

$$\xi_K(t) := \| \, |D|^K U(t,s)f\|^2 \leq 3^{K-1} \Sigma_{j=1}^{3} (f, U(t,s)^* D_j^{2K} U(t,s)f) \leq$$

$$\leq 3^{K-1} \Sigma_{j=1}^{3} \| D_j^K U(t,s)f)\|^2.$$

If for each j=1,2,3 we denote $\zeta_K(t) = \|D_j^K U(t,s)f\|^2$, and compute its derivative we get $\partial_t \zeta_K(t) = (f, U(t,s)^* [V(t), D_j^{2K}] U(t,s)f)$ and using now lemma 3.10 we see that :

$$|\partial_t \zeta_K(t)| \leq 2\Sigma_{L=1}^{K} C_K^L \|D_j^{K-L} U(t,s)f\| \, \| \partial_j V(t)\| \, \| D_j^K U(t,s)f\| \leq$$

$$\leq C \sqrt{\zeta_K(t) \zeta_{K-1}(t)} <t>^{-1} \| f\|^2$$

so that the induction hypothesis and the Gronwall lemma can be used in order to get the desired result. ■

Our last problem in this section will be to use the method of energetic inequalities in the same way as Chernoff [C] in order to derive some propagation properties in the regions where there is no field, i.e. in the exterior of $\mathcal{K}_{R,t}$. We shall start with reviewing the inequality proven by Chernoff. In $\mathbb{C}^4$ we shall denote the scalar product by $<\cdot,\cdot>$.

Proposition 3.11: *Let* $f \in \mathcal{H}$ *and let us denote* $f(t,x) := (U(t,s)f)(x)$. *Then for any* t *in* $\mathbb{R}_+$ *we have the following estimation* :

$$\int_{B(x_0,r)} <f(t,x), f(t,x)> d^3x \leq \int_{B(x_0,r+t-s)} <f(s,x), f(s,x)> d^3x.$$

Proof: First let us take f in $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$ and let us consider the 4-component vector field :

(3.9) $$\varphi(t,x) := \{<u(t,x), \alpha_j u(t,x)>\}_{j=0,1,2,3}$$

and the following set in $\mathbb{R}^4$:

(3.10) $$\mathcal{K}(x_0,r,s,t) := \{(\tau,x) \in \mathbb{R}^4 \,|\, |x-x_0| \leq r+(t-\tau),\ s \leq \tau \leq t\}.$$

We remark that the boundary of $\mathcal{K}(x_0,r,s,t)$ is :

(3.11) $$\partial\mathcal{K}(x_0,r,s,t) = \{B(x_0,r+t-s) \times \{s\}\} \cup \{B(x_0, r) \times \{t\}\} \cup \Sigma(x_0,r,s,t)$$

(3.12) $$\Sigma(x_0,r,s,t) := \{(\tau,x) \in \mathbb{R}^4 \,|\, |x-x_0| = r+t-\tau,\ s \leq \tau \leq t\}.$$

Then it is easy to see that :

$$(\operatorname{div} \mathscr{g})(t,x) = 2\operatorname{Re}<u(t,x),\Sigma_{j=0}^{3}\alpha_j\partial_j u(t,x)>$$

$$= 2\operatorname{Re}<u(t,x),im\beta u(t,x)> = 0$$

where we denoted by ∂_0 the partial derivative with respect to t. Thus the Gauss theorem applied in $\mathscr{K}(x_0,r,s,t)$ gives :

$$0 = \int_{B(x_0,r)} <f(t,x),f(t,x)> d^3x \;-\; \int_{B(x_0,r+t-s)} <f(s,x),f(s,x)> d^3x$$

$$+ \int_{\Sigma(x_0,r,s,t)} <u(t,x),\Sigma_{j=0}^{3}\alpha_j\nu_j u(t,x)>$$

where: $\nu = (\nu_0,\nu_1,\nu_2,\nu_3)$ is the normal to $\Sigma(x_0,r,s,t)$, pointing towards the exterior and it is easy to see that it is proportional to the 4-vector: $(1,|x_j-(x_0)_j|^{-1}(x_j-(x_0)_j))$. Observing that $\Sigma_{j=1}^{3}\alpha_j\nu_j \leq \mathbb{1}$ we get the desired estimation for f in $C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$. Then we can aproach any $f\in\mathscr{H}$ by such functions and as the estimation only depends on the L^2-norm it will remain true. ■

If $R\in\mathbb{R}_+$ and $\delta>0$, we denote by $\eta_{\delta,R}$ a "regularised characteristic function" of the ball B(0,R) satisfying the following properties:

$$\eta_{\delta,R}\in C^\infty(\mathbb{R}_+),\ 0\leq\eta_{\delta,R}(r)\leq 1,\ \operatorname{supp}\eta_{\delta,R}\subset[0,R) \text{ and } \eta_{\delta,R}(r)=1 \text{ for } r\leq R-\delta.$$

Proposition 3.12: *Let* R *be defined as in section* 2 *by the support condition for the electromagnetic potential at time* 0. *Then for any* $f\in\mathscr{H}$ *and any* $t_1\geq t_2>R$, *we have the following relation*:

$$U(t_1,t_2)\chi(|Q|<t_2-R)f = U_0(t_1-t_2)\chi(|Q|<t_2-R)f.$$

Proof: Let us fix $\delta>0$ and let us denote

$$f(t) = U(t,t_2)\eta_{\delta,t_2-R}(|Q|)f,\quad \tilde{f}(t) = U_0(t-t_2)\eta_{\delta,t_2-R}(|Q|)f.$$

Then we observe that

$$f(t_2) = \tilde{f}(t_2) = \eta_{\delta,t_2-R}(|Q|)f\ ;\ \partial_t f(t) = -iH(t)f(t);\ \partial_t\tilde{f}(t) = -iH_0\tilde{f}(t).$$

Now we shall prove that in fact we have the following equality :

$$U(t,t_2)\eta_{\delta,t_2-R}(|Q|)f = \chi(|Q|\leq t-R)f(t).$$

In fact if $|x_0|>t-R$, let $\rho>0$ be such that $B(x_0,\rho)\subset B(0,t-R)^c$. Then according to proposition 3.11 we have that :

$$\int_{B(x_0,\rho)} <f(t,x),f(t,x)>d^3x \le \int_{B(x_0,\rho+t+t_2)} <f(t_2,x),f(t_2,x)> d^3x = 0$$

due to the fact that $B(x_0,\rho+t-t_2)\subset B(0,t_2-R)^c\subset \{\text{supp } f(t_2)\}^c$. Thus we conclude that :

$$\partial_t f(t) = -iH(t)f(t) = -i H_0 f(t) - iV(t)\chi(|Q|\le t-R)f(t) = -iH_0 f(t)$$

and due to the unicity of the solution of the Cauchy problem we have that for any $t\ge t_2$ the equality: $f(t)=\tilde{f}(t)$ is true. Then one can approach the function $\chi(|Q|<t_2-R)$ by functions $\eta_{\delta,t_2-R}(|Q|)$ with $\delta\to 0$ and use the continuity of $U(t_1,t_2)$ in order to get the result. ■

Proposition 3.13: *Let* R *be defined as in section* 2 *by the support condition for the electromagnetic potential at time* 0. *Then for any* $f\in\mathscr{H}$ *and any* $t_1\ge t_2\ge R$, *we have the following relation* :

$$\chi(|Q|>t_1+R)U(t_1,t_2)f = \chi(|Q|>t_1+R)U_0(t_1-t_2)f$$

Proof: Let us define as in the proof of the above proposition

$$f(t) = U(t,t_2)f,\ \tilde{f}(t)=U_0(t-t_2)f,\ g(t)=f(t)-\tilde{f}(t) \quad \text{such that } g(t_2)=0$$

and its time derivative is given by :

$$\partial_t g(t) = -iH(t)f(t)+iH_0\tilde{f}(t) = -iH_0 g(t) - iV(t)f(t)$$

and let us use once more the method used by Chernoff, by constructing the 4-component vector field $\mathcal{g}(t,x)$ in (3.9) associated now to the function g(t,x). We choose an arbitrary point x_0 in $B(0,t_1+R)^c$ and $\rho\in\mathbb{R}_+$ such that $B(x_0,\rho)\subset B(0,t_1+R)^c$ and we consider again the set $\mathscr{K}(x_0,\rho,t_2,t_1)$ in 3.10. We observe that $B(x_0,\rho+t_1-t_2)\subset B(0,t_2+R)^c$ and :

$$(\mathrm{div}\,\mathcal{g})(t,x) = 2\mathrm{Re} <g(t,x),\Sigma_{j=0}^{3}\alpha_j\partial_j g(t,x)>$$

$$= 2\mathrm{Re} <g(t,x),im\beta\, g(t,x)> + 2\mathrm{Re} < g(t,x),(-iV(t))f(t,x)>.$$

But we see that in $\mathscr{K}(x_0,\rho,t_2,t_1)$, due to the choice of x_0 and ρ, we have $V(t)=0$ and thus we obtain that :

$$\int_{B(x_0,\rho)} <g(t_1,x),g(t_1,x)>d^3x \le \int_{B(x_0,\rho+t_1-t_2)} <g(t_2,x),g(t_2,x)>d^3x=0.$$

In conclusion $g(t_1,x)=\chi(|Q|\le t_1+R)g(t_1,x)$ and thus $\chi(|Q|>t_1+R)g(t_1,x)=0$. ■

4. THE WAVE OPERATORS

In this section we shall study the asymptotic behaviour of the nonhomogeneous evolution U(t,s) when $t \to \pm\infty$ and s is fixed. There are two serious reasons to suppose that asymptotically this evolution should be equivalent to the free one, $U_0(t-s)$. First one has that the potential goes to zero in the norm-operator topology but the difficulty comes from the fact that the time decay of the potential and of all of its derivatives (with respect to time and space variables) decay only as (t^{-1}) and thus are not integrable. But secondly, one has that the potential leaves any compact set in $\mathbb{R}^3$ travelling with the velocity 1, the velocity of light, while for any finite energy the electron travels with a velocity v(k), see formula (1.10), that is strictly less then 1, so that after some finite time it will remain outside the region of the electromagnetic potential and thus will evoluate freely. The main point will thus be to use propositions 3.12 and 3.13 in order to prove that rigorously. However the time decay of the potential is also needed in showing that the kinetic energy of the particle cannot increase to fast, as one can see from proposition 3.10.

Let us consider the operators $W_s(t) = U(s,t)U_0(t-s)$, their duals and their limits defined in (1.9). If one supposes that these limits exist, one can see that for different values of $s \in \mathbb{R}$ one has the following relation:

$$(4.1) \qquad W^{\pm}_{s_1} = U(s_1,s_2)W^{\pm}_{s_2}\, U_0(s_2-s_1)$$

so that it is enough to consider only a fixed value of s that we shall take to be 0.

We shall begin with the problem of the existence of the limits (1.9). This problem is quite simple and has been solved in [P], but we shall give here a new proof that does not use the uniform bound with respect to time for the L^2-norm of V(t), that may no longer be true if one wants to extend these results to more general initial conditions for the electromagnetic potential.

Proposition 4.1: *The following limits* $W_0^{\pm} f = \lim_{t\to\pm\infty} W_0(t)f$ *exist for any* f *in* $\mathcal{H}$.

Proof: Let us first remind the definitions of $W_0(t)$ in (1.8): $W_0(t) = U(0,t)U_0(t)$. We shall use the Cook method and for some fixed $f \in \mathcal{H}$ we define the application:

$$\mathbb{R} \ni t \longmapsto f(t) := W_0(t)f \in \mathcal{H}.$$

Let us suppose first that $f \in \mathcal{H}^1$ and moreover that its Fourier transform $\hat{f} \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. Then the above application is differentiable and if we compute its derivative we get: $\partial_t f(t) = \partial_t W_0(t) f = iU(0,t)V(t)U_0(t)f$ so that:

$$\|f(t)\| \leq \|f\| + \int_0^t \| U(0,\tau)V(\tau)U_0(\tau)f \| d\tau.$$

Now let us suppose that supp $\hat{f} \subset B(0,K)$ and let us decompose the norm above as follows :

$$\|f(t)\| \leq \|f\| + \int_0^t \|\chi(|Q|<u(2K)\tau)V(\tau)U_0(\tau)f\|\, d\tau +$$

$$+ \int_0^t \|\chi(|Q|\geq u(2K)\tau)V(\tau)U_0(\tau)f\|\, d\tau\, .$$

For the first integral one observes that for $\tau>\tau_0$, where $\tau_0=(1-u(2K))^{-1}R$, one has that $\chi_{(|x|<u(2K)\tau)}V(\tau)=0$ so that in the first integral we can integrate only up to τ_0. For the second integral we can use proposition 2.4 and conclude that it has a finite limit for $t\to\infty$. For a general $f\in\mathscr{H}$ one can now approach it by elements with Fourier transform in $C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$ and use then the fact that $W_0(t)$ is an isometry in order to prove the existence of the limit. ■

Our next problem is to prove that the strong limits in (1.9) are in fact unitary in $\mathscr{H}$.

Theorem 4.2 *The isometric operators* $W_0^{\pm}$ *are unitary in* $\mathscr{H}$.

Proof: In order to prove this result we shall prove that the adjoints $W_0(t)^*$ also converge for $t\to\pm\infty$ with respect to the strong topology. In order to do that we shall use the properties of $U(t,s)$ that we proved in section 3 and we shall show that the function $f(t)=W_0(t)^*f$ has a Cauchy property for $t\to\pm\infty$. Thus let us fix some $f\in\mathscr{H}$.

1) First let us prove the following estimation concerning the evolution "in front of the electromagnetic potential". For any $\varepsilon>0$, there is some finite $\tau_0(\varepsilon,f)$ such that for any $t\geq\tau_0(\varepsilon,f)$ we have the estimation :

(4.2) $\|\chi(|Q|>R+t)U(t,0)f\| < \varepsilon.$

In fact by using proposition 3.13 with $t_1=t$ and $t_2=R$ and denoting $f_0=U(R,0)f$ we see that one has the relation:

(4.3) $\chi(|Q|>R+t)U(t,0)f = \chi(|Q|>R+t)U(t,R)f_0 = \chi(|Q|>R+t)U_0(t,R)f_0.$

Now let us choose $g\in\mathscr{H}$ such that $\hat{g}\in C_0^\infty(\mathbb{R}^3)\otimes\mathbb{C}^4$ and:

(4.4) $\|f_0-g\| < \varepsilon/2.$

Then using proposition 2.4 we can conclude that for t large enough:

(4.5) $\left\|\chi(|Q|>R+t)U_0(t,R)f_0\right\| < \varepsilon/2$

and the estimations (4.4) and (4.5) imply now (4.2).

2) For $\tau \geq \tau_0(\varepsilon,f)$ let us consider the following decomposition of $U(\tau,0)f \equiv f_\tau$:

$$f_\tau = \eta_{\delta/2,R+\delta+\tau}(|Q|)f_\tau + \{\mathbb{1} - \eta_{\delta/2,R+\delta+\tau}(|Q|)\}f_\tau . \tag{4.6}$$

Due to (4.2) the second term is small uniformly in $\tau \geq \tau_0(\varepsilon,f)$. We shall prove now that for any $\varepsilon>0$, there exist $\tau_\varepsilon \geq \tau_0(\varepsilon,f)$ and $\gamma \in (0,1)$ such that if we denote $t_\varepsilon = \tau_\varepsilon + \tau_\varepsilon^\gamma$ we have the following estimation :

$$\| \chi(|Q|>t_\varepsilon - R - \delta/2)U(t_\varepsilon,\tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)f_{\tau_\varepsilon}\| < \varepsilon. \tag{4.7}$$

We shall start with the Duhamel formula:

$$U(t,\tau)g = U_0(t-\tau)g - \int_0^t U_0(t-\sigma)V(\sigma)U(\sigma,\tau)g\, d\sigma. \tag{4.8}$$

Using this formula we get for $t \geq \tau$:

$$\|\chi(|Q|>t-R-d/2)U(t,\tau)h_{\delta/2,R+\delta+\tau}(|Q|)f_\tau\| \leq \tag{4.9}$$

$$\leq \|\chi(|Q|>t-R-\delta/2)U_0(t-\tau)\eta_{\delta/2,R+\delta+\tau}(|Q|)f_\tau\| + \int_\tau^t \|V(\sigma)\|_{\mathcal{B}(\mathcal{H})}\, d\sigma \|f_\tau\|$$

and for the integral we see that due to corollary 3.2 one has:

$$\int_\tau^t \|V(\sigma)\|_{\mathcal{B}(\mathcal{H})}\, d\sigma \|f_\tau\| \leq C\|f\| \ln(1+\tau^{-1+\gamma}). \tag{4.10}$$

In order to estimate the first term on the right-hand side of the above inequality let us define v_ε by the following condition:

$$t_\varepsilon - R - \delta/2 = R + \tau_\varepsilon + \delta/2 + v_\varepsilon(t_\varepsilon - \tau_\varepsilon) \tag{4.11}$$

so that $v_\varepsilon = 1 - (2R+\delta)\tau_\varepsilon^{-\gamma}$ is the minimal velocity needed to get from supp $\eta_{\delta/2,R+\delta+\tau_\varepsilon}$ at time τ_ε to supp $\chi_{(|x|>t_\varepsilon - R - \delta/2)}$ at time t_ε. We remark that we can suppose $1-v_\varepsilon$ as small as we like by taking τ_ε large enough. Thus we can apply now proposition 2.5 for the first term and get that for any N we have :

$$|(\chi(|Q|>t_\varepsilon - R - \delta/2)U_0(t_\varepsilon - \tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)f_{\tau_\varepsilon})(x)| \leq \tag{4.12}$$

$$\leq C_{N,\delta}\, \tau_\varepsilon^{-(N-2)\gamma/2} \| (\mathbb{1} - \Delta)^{(N+2)/2} f\|_{L^2(\mathbb{R}^3)}.$$

Using now proposition 3.10 for the norm of the derivatives of $U(\tau_\varepsilon,0)f(x)$ and the fact that due to proposition 3.11 the volume of the support of the function:

$$\chi(|Q|>t_\varepsilon - R - \delta/2)U_0(t_\varepsilon - \tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau\varepsilon}(|Q|)f_{\tau_\varepsilon}$$

is of the order of $\tau_\varepsilon^{2\gamma}$ we finally obtain :

$$\| \chi(|Q|>t_\varepsilon - R - \delta/2)U_0(t_\varepsilon - \tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau\varepsilon}(|Q|)f_{\tau_\varepsilon}\| <$$
$$< C_{N,\delta}\,\tau_\varepsilon^{1-(N-2)/\gamma/2}\,(\ln(1+\tau_\varepsilon))^{(N+2)/2}$$

so that we can choose $N>\gamma^{-1}+2$ in order to prove (4.7).

3) Let us now use the estimations (4.2) and (4.7) in order to prove a Cauchy property for $f(t)=W_0(t)^*f$ as $t\to\infty$. Let $\varepsilon>0$ and $t_2\geq t_1\geq t_\varepsilon=\tau_\varepsilon+\tau_\varepsilon^\gamma$ with $\tau_\varepsilon\geq\tau_0(\varepsilon,f)$. Then we have :

$$
\begin{aligned}
(4.13)\qquad W_0(t_1)^*f-W_0(t_2)^*f &= U_0(t_1)U(t_1,\tau_\varepsilon)\{\mathbb{1}-\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)\}f_{\tau_\varepsilon}-\\
&-U_0(t_2)U(t_2,\tau_\varepsilon)\{\mathbb{1}-\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)\}f_{\tau_\varepsilon}+\\
&+U_0(t_1)U(t_1,t_\varepsilon)\chi(|Q|>t_\varepsilon-R-\delta/2)U(t_\varepsilon,\tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)f_{\tau_\varepsilon}-\\
&-U_0(t_2)U(t_2,t_\varepsilon)\chi(|Q|>t_\varepsilon-R-\delta/2)U(t_\varepsilon,\tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)f_{\tau_\varepsilon}+\\
&+U_0(t_1)U(t_1,t_\varepsilon)\chi(|Q|\leq t_\varepsilon-R-\delta/2)U(t_\varepsilon,\tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)f_{\tau_\varepsilon}-\\
&-U_0(t_2)U(t_2,t_\varepsilon)\chi(|Q|\leq t_\varepsilon-R-\delta/2)U(t_\varepsilon,\tau_\varepsilon)\eta_{\delta/2,R+\delta+\tau_\varepsilon}(|Q|)f_{\tau_\varepsilon}.
\end{aligned}
$$

Using (4.2) for the first two terms, (4.7) for the third and fourth terms and observing that due to proposition 3.12 the last two terms cancel out, one obtains:

$$(4.14)\qquad \| W_0(t_1)^*f-W_0(t_2)^*f\| \leq 4\varepsilon$$

and thus the following limit exists: $\lim_{t\to\infty} W_0(t)^*f$ and is equal to $W_0^{+*}f$ and thus the operator W_0^+ is unitary in $\mathcal{H}$. Evidently the whole analysis may be repeated for $t\to-\infty$ and thus one gets a similar result for W_0^-. ■

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