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ANALYTIC COMPLETION FOR DYNAMICAL ZETA FUNCTIONS

David Ruelle **

Dedicated to Elliott Lieb

Abstract. It is possible to extend the known domain of analyticity of dynamical zeta functions, and also of resolvents of transfer operators, by use of the tube theorem. The case of piecewise monotone maps of the interval is worked out explicitly, and one recovers in a very different manner some recent results of Keller and Nowicki. In an appendix, it is shown that the study of piecewise monotone maps reduces to the Markovian case.

1. Introduction.

It is often possible to associate with dynamical systems some *dynamical zeta functions* with interesting analyticity properties. For instance, if (M, f) is a dynamical system, $\mathrm{Fix}\, f^m$ the set of fixed points for the m-th iterate of f, and $g \geq 0$ a function on M, we may introduce

$$\zeta(z, s) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k x)^s$$

and prove that ζ is holomorphic or meromorphic for (z, s) in a certain domain $D \subset \mathbb{C}^2$, under suitable assumptions on M, f, and g. For functions of several complex variables, it is however a general fact that holomorphy in a domain D $^{*)}$ may imply holomorphy in a strictly larger domain $\widehat{D}$. The extension from D to $\widehat{D}$ is called *analytic completion.* The purpose of this note is to show that by analytic completion one may derive new analytic properties of dynamical zeta functions. One can similarly extend the domain of analyticity of the *resolvent* $(1 - z\mathcal{L})^{-1}$ *of the transfer operator* $\mathcal{L}$ defined by

$$\mathcal{L}\Phi(x) = \sum_{y: fy=x} g(y)^s \Phi(y)$$

but this will not be discussed in detail here (see Remark (6) in Section 2).

The tool that we shall use is the *tube theorem* (see for instance Bochner and Martin [3]). Let D be a domain (= open connected set) in $\mathbb{C}^n$. If

$$D = \{(z_1, \ldots, z_n): (Re\, z_1, \ldots, Re\, z_n) \in \Gamma\}$$

$^{*)}$ In general, a domain D is an open connected subset of $\mathbb{C}^n$, a function $f : D \to \mathbb{C}$ is holomorphic if it has an absolutely convergent Taylor expansion (in n variables) near each point of D.

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we say that D is a tube with base Γ; the imaginary parts of the z_i are thus unrestricted. The tube theorem now asserts that *if a function is holomorphic in the tube D, it extends (uniquely) to a function holomorphic in $\widehat{D}$, where $\widehat{D}$ is the tube with base $\widehat{\Gamma}$, $\widehat{\Gamma}$ being the convex hull of Γ.*

We shall not go into any details of the study of functions of several variables, but we mention the following fact (*Hartog's main theorem*, see for instance Bochner and Martin [3]) which is important to know. *Let $\mathcal{D}$ be a domain for the variable z, and $0 < p < P \leq \infty$; if $(z,w) \to f(z,w)$ is holomorphic in $\mathcal{D} \times \{w : |w| < p\}$ and if, for each $z \in \mathcal{D}$, $w \mapsto f(z,w)$ is holomorphic in $\{w : |w| < P\}$, then $(z,w) \mapsto f(z,w)$ is holomorphic in $D \times \{w : |w| < P\}$.*

In what follows we shall give one example of application of the tube theorem, to obtain new analyticity properties of zeta functions associated with piecewise monotone maps in one dimension. In this way we shall recover some results of Keller and Nowicki [6]. This example is used to show the scope of the method, but other applications are certainly possible.

Our method gives analytic zeta functions which are not directly related to determinants of operators acting on Banach spaces. This is different from the work of Keller and Nowicki mentioned above, where the zeta functions are related to transfer operators, also studied by Young [12], which act on specific Banach spaces.

It may be worth mentioning here that analytic completion has been repeatedly used in physical applications (in quantum field theory, see for instance Streater and Wightman [11], and in statistical mechanics, see Lieb and Ruelle [7]).

2. Piecewise monotone maps.

We take X to be a compact subset of $\mathbb{R}$, and say that J is a closed interval of X if $J = X \cap [u,v]$ for suitable $u < v$. We assume that X is the union of finitely many disjoint closed intervals $J_1, \ldots, J_N$, and that $f : X \to X$ is such that $f|J_i$ is strictly monotone and satisfies the Darboux property for $i = 1, \ldots, N$. [The Darboux property means here simply that fJ_i is a closed interval of X; in particular f is continuous.]

Given $g : X \to \mathbb{C}$, we let

$$\operatorname{var} g = \sup \sum_1^n |g(a_i) - g(a_{i-1})|$$

where the sup is over finite families of points of X, with $a_0 < a_1 < \ldots < a_n$. We say that g is of *bounded variation* if $\operatorname{var} g < \infty$. We shall use the fact that if $g \geq 0$ and $Re\, s > 0$, then

$$\operatorname{var}(g^s) \leq \frac{|s|}{Re\, s} \operatorname{var}\left(g^{Re\, s}\right) .$$

[Indeed if $0 \leq u < v$, then

$$|v^s - u^s| \leq \int_u^v |s\, t^{s-1}| dt = \frac{|s|}{Re\, s} \int_u^v Re\, s\, .\, t^{Re\, s-1} \quad dt$$
$$= \frac{|s|}{Re\, s} \left(u^{Re\, s} - v^{Re\, s}\right) \qquad].$$

We say that $(J_1, \ldots, J_N)$ is a *generating partition* if every intersection

$$\bigcap_{n=0}^{\infty} f^{-n} J_{i(n)}$$

consists of at most one point. Define $\varepsilon : X \to \pm 1$ such that $\varepsilon \mid J_i = +1$ (resp. -1) if f is increasing (resp. decreasing) on J_i. [We may assume that no J_i is reduced to a point, or else choose $\varepsilon \mid J_i$ arbitrarily.] Write then

$$\mathrm{Fix}^- f^m = \left\{ x \in \mathrm{Fix}\, f^m : \prod_{k=0}^{m-1} \varepsilon(f^k x) = -1 \right\} .$$

Baladi and Keller [1] have studied the zeta function

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k x)$$

where f is piecewise monotone as above, associated with a generating partition, and g is of bounded variation. If

$$\theta = \lim_{m \to \infty} \big(\max_{x \in X} \prod_{k=0}^{m-1} |g(f^k x)|\big)^{1/m}$$

they show in particular that ζ *is holomorphic when* $|z| < \theta^{-1}$. Their proof uses an idea of Haydn [4] and a device called Markov extension, originating with Hofbauer, and for which there is an improved and lucid exposition in Keller and Nowicki [6]. It is possible, however to bypass the use of the Markov extension and reduce the problem to the Markovian case, as explained in the Appendix of the present paper. The proof of the results of Baladi and Keller simplifies quite a bit in the Markovian case, as shown to the author by V. Baladi. We shall also use in what follows the "negative zeta function"

$$\zeta^-(z) = \exp 2 \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}^- f^m} \prod_{k=0}^{m-1} g(f^k x) .$$

Note the factor 2 in the exponential and note also that it is not required that the partition $(J_1, \ldots, J_N)$ be generating. This negative zeta function is also holomorphic for $|z| < \theta^{-1}$ (see Baladi and Ruelle [2]. As indicated in [2] it is not necessary to assume that g is piecewise continuous as in [1], bounded variation is sufficient.)

If u is of bounded variation and $u \geq 0$, then u^α is of bounded variation when $\alpha \geq 1$. [In fact

$$\mathrm{var}\, u^\alpha \leq \alpha\big(\|u^{\alpha-1}\|_0\big)\, \mathrm{var}\, u \ .$$

Indeed, if $0 < u,\ v$, we have

$$|v^\alpha - u^\alpha| = |\int_u^v \alpha\, t^{\alpha-1} dt| \leq \alpha.\max(u^{\alpha-1}, v^{\alpha-1}).|u-v| \quad].$$

The condition that u^α be of bounded variation for all $\alpha > 0$ (used in the following theorem) is automatically satisfied if $u \geq \text{const} > 0$; it is otherwise a mild restriction on u.

2.1. Theorem. *Let the functions u_1, u_2, v_1, v_2 be of bounded variation $X \to \mathbb{C}$, with $v_1, v_2 \geq 0$. Furthermore assume that $x \mapsto |u_2(x)|^\alpha,\ v_1(x)^\alpha,\ v_2(x)^\alpha$ are of bounded variation for all $\alpha > 0$. We write*

$$g_s(x) = \begin{cases} 0 \quad \text{if} \quad u_1(x).u_2(x).v_1(x).v_2(x) = 0 \\ \frac{u_1(x)}{u_2(x)} \cdot \left(\frac{v_1(x)}{v_2(x)}\right)^s \qquad \text{otherwise.} \end{cases}$$

If the partition $(J_1, \ldots, J_N)$ is generating, define the zeta function

$$\zeta(z,s) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \text{Fix}\, f^m} \prod_{k=0}^{m-1} g_s(f^k x)\ .$$

Define also the negative zeta function

$$\zeta^-(z,s) = \exp 2 \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \text{Fix}^-\, f^m} \prod_{k=0}^{m-1} g_s(f^k x)$$

without assuming that $(J_1, \ldots, J_N)$ be generating. Write (for real σ)

$$\theta(\sigma) = \limsup_{m\to\infty} \big(\max_{x \in \text{Fix}\, f^m} |\prod_{k=0}^{m-1} g_\sigma(f^k x)|\big)^{1/m}$$

$$\theta^-(\sigma) = \limsup_{m\to\infty} \big(\max_{x \in \text{Fix}^-\, f^m} |\prod_{k=0}^{m-1} g_\sigma(f^k x)|\big)^{1/m}$$

Then $1/\zeta(z,s)$ (resp. $1/\zeta^-(z,s)$) is holomorphic for $|z|\theta(Re\, s) < 1$ (resp. $|z|\theta^-(Re\, s) < 1$).

[Note that if we write

$$\theta = \limsup_{m\to\infty} \big(\max_{x \in \text{Fix}\, f^m} |\prod_{k=0}^{m-1} \frac{u_1(f^k x)}{u_2(f^k x)}|\big)^{1/m}$$

$$\tilde{\theta} = \limsup_{m\to\infty} \big(\max_{x \in \text{Fix}\, f^m} |\prod_{k=0}^{m-1} \frac{v_1(f^k x)}{v_2(f^k x)}|\big)^{1/m}$$

we have $\theta(\sigma) \le \theta.\tilde{\theta}^{\sigma}$ for all $\sigma > 0$. Similarly for $\theta^{-}(\sigma)$.]

The case of ζ (assuming that $(J_1, \ldots, J_N)$ is generating) and the case of ζ^{-} are similar, we consider the former.

We write $u_1.\overline{u}_2 = v_4$, $u_2.\overline{u}_2 = |u_2|^2 = v_3$ (so that $\frac{u_1}{u_2} = \frac{v_4}{v_3}$ except if u_2 and v_3 vanish). Also let

$$g(x, t_1, t_2, t_3) = \begin{cases} 0 \quad \text{if} \quad \prod_1^4 v_j(x) = 0 \\ v_4(x) . \prod_1^3 v_j(x)^{t_j} \quad \text{otherwise.} \end{cases}$$

Note that we may equivalently define

$$g(x, t_1, t_2, t_3) = v_4(x). \prod_1^3 v_j(x)^{t_j}$$

if $Re\, t_j > 0$ for $j = 1, 2, 3$, and continue by analyticity. We also introduce

$$\begin{aligned} & d(z, t_1, t_2, t_3) \\ & = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k x, t_1, t_2, t_3) \end{aligned}$$

and

$$\begin{aligned} & \theta_*(Re\, t_1, Re\, t_2, Re\, t_3) \\ & = \limsup_{m \to \infty} \Big(\max_{x \in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} |g(f^k x, t_1, t_2, t_3)| \Big)^{1/m} \end{aligned}$$

$$\theta_j = \lim_{m \to \infty} \Big(\sup_{x \in X} \prod_{k=0}^{m-1} v_j(f^k x) \Big)^{1/m} \quad \text{for } j = 1, 2, 3 .$$

With these definitions we see that $d(z, t_1, t_2, t_3)$ is holomorphic when

$$N|z|\theta_*(Re\, t_1, Re\, t_2, Re\, t_3) < 1 \tag{I}$$

(because $\mathrm{card\,Fix}\, f^m \le N^m$), and also when

$$\begin{cases} |z| \prod_1^3 \theta_j^{Re\, t_j} < 1 \\ Re\, t_j > 0 \quad \text{for} \quad j = 1, 2, 3 \end{cases} \tag{II}$$

This second region of holomorphy is a consequence of the theorem of Baladi and Keller [1]. (As noted earlier, one need not assume piecewise continuity of u, u_2, v_1, v_2, see [2]; see also [2] for the corresponding property of ζ^-, where it is not needed that $(J_1, \ldots, J_N)$ be generating.)

The regions I and II are both tubes with respect to the variables $\log z, t_1, t_2, t_3$. The basis of I is a convex cone with apex at $(-\log N, 0, 0, 0)$ and the basis of II contains an open set near $(0,0,0,0)$. The tube theorem therefore implies that $d(z, t_1, t_2, t_3)$ is holomorphic when

$$|z|\theta_*(Re\, t_1, Re\, t_2, Re\, t_3) < 1 \tag{III}$$

We take $t_1 = s,\ t_2 = -s,\ t_3 = -1$, and note that

$$g(x, s, -s, -1) = g_s(x)$$
$$\theta_*(Re\, s,\ -Re\, s, -1) = \theta(Re\, s)\ .$$

Therefore

$$1/\zeta(z, s) = d(z, s, -s, -1)$$

is holomorphic when

$$|z|\theta(Re\, s) < 1$$

proving the theorem. □

2.2. Corollary. *Let u_1, u_2 be of bounded variation $X \to \mathbb{C}$ and $x \to |u_2(x)|^\alpha$ of bounded variation for all $\alpha > 0$. Write*

$$g(x) \ = \begin{cases} 0 & \textit{if} \quad u_1(x).u_2(x) = 0 \\ \frac{u_1(x)}{u_2(x)} & \quad \textit{otherwise.} \end{cases}$$

If the partition $(J_1, \ldots, J_N)$ is generating,

$$1/\zeta(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k x)$$

is holomorphic when $|z|\theta < 1$, where

$$\theta = \limsup_{m\to\infty} \Big(\max_{x \in \mathrm{Fix}\, f^m} |\prod_{k=0}^{m-1} g(f^k x)|\Big)^{1/m}\ .$$

Similarly for ζ^-.

2.3. Corollary. *If $u_1 = u_2 = 1$ in the theorem, then $1/\zeta(z, s)$ is holomorphic for $|z|\theta^{Re\, s} < 1$, $Re\, s > 0$. In particular if $\tilde\theta < 1$, the function $1/\zeta(1, s)$ is holomorphic for $Re\, s > 0$. Similarly for ζ^-.*

This is because $\theta = 1$, hence $\theta(Re\, s) = \tilde\theta^{Re\, s}$ for $Re\, s > 0$. □

Remarks.

(1) If g_s is unbounded, it may happen that $\theta(Re\, s) = \infty$, in which case the theorem is vacuous. If $\theta(Re\, s)$ is finite, the value of the theorem is that it replaces the (easily obtained) region of holomorphy

$$N|z|\theta(Re\, s) < 1$$

by the nontrivial region

$$|z|\theta(Re\, s) < 1 \ .$$

(2) The study of piecewise monotone maps of the interval $[0,1] \subset \mathbb{R}$ is readily reduced to the situation of the theorem. [Let the monotonicity intervals be $[a_{i-1}, a_i]$ for $i = 1, \ldots, N$. If $\xi \in (0,1)$ and $k \geq 0$ is the smallest integer such that $f^k\xi \in \{a_1, \ldots, a_{N-1}\}$, replace ξ by two points $\xi_- < \xi_+$ and insert an interval of length $1/n!$ between them, obtaining a set X and a map $X \to [0,1]$. The piecewise monotone map of $[0,1]$ lifts then to a continuous map of X (see [5]). The replacement of $[0,1]$ by X produces only trivial changes in the zeta functions.]

(3) An interesting special case of (2) is when f is a piecewise monotone differentiable map $[0,1] \to [0,1]$ and $g(x) = |f'(x)|^{-1}$. Corollary 2.2 reproduces then in particular some results of Keller and Nowicki [6].

(4) Let again f be a piecewise monotone differentiable map $[0,1] \to [0,1]$. Instead of a zeta function we may sometimes define a determinant $\mathrm{Det}(1 - z\mathcal{L})$ where $\mathcal{L}$ is a transfer operator defined by

$$\mathcal{L}\Phi(x) = \sum_{y:fy=x} g(y)\Phi(y) \ .$$

Using Remark (2) we assume that we may apply the theorem with $v_1/v_2 = |f'|^{-1}$ and $u_1/u_2 = g$ or g/f'. We further assume that

$$|\prod_{k=0}^{m-1} f'(f^k x)| > 1$$

whenever $x \in \mathrm{Fix}\, f^m$, m integer > 0, and that

$$\tilde{\theta} = \limsup_{m\to\infty} \big(\max_{x\in\mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} |f'(f^k x)|^{-1}\big)^{1/m} < 1$$

$$\theta^* = \limsup_{m\to\infty} \left(\max_{x\in\mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} \left|\frac{g(f^k x)}{f'(f^k x)}\right|\right)^{1/m} < \infty \ .$$

We may then define

$$\operatorname{Det}(1 - z\mathcal{L}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \, Tr\mathcal{L}^m$$

$$= \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \frac{\prod_{k=0}^{m-1} g(f^k x)}{|1 - \prod_{k=0}^{m-1} f'(f^k x)|}$$

$$= \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \frac{\prod_{k=0}^{m-1} \left(g(f^k x)/|f'(f^k x)|\right)}{1 - \prod_{k=0}^{m-1} f'(f^k x)^{-1}}$$

$$= \exp - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in \operatorname{Fix} f^m} \prod_{k=0}^{m-1} \left(\frac{g}{|f'|f'^n}\right)(f^k x) \, .$$

Since

$$\left| \sum_{x \in \operatorname{Fix} f^m} \prod_{k=0}^{m-1} \left(\frac{g}{|f'|f'^n}\right)(f^k x) \right| \lesssim \left(N\theta^* \tilde{\theta}^n\right)^m$$

we may write

$$\operatorname{Det}(1 - z\mathcal{L}) = \prod_{n=0}^{\infty} \tilde{\zeta}_n(z)^{-1} \tag{$*$}$$

where $\tilde{\zeta}_n$ is constructed with

$$\tilde{g}_n = g|f'|^{-1} . (f')^{-n}$$

and $(*)$ converges when $|z|\theta^* < 1$.

(5) In his beautiful study of the thermodynamic formalism for the Gauss map, Mayer [8], [9] has obtained meromorphic extension of zeta functions by a method totally different from that of analytic completion discussed here.

(6) Analytic completion can also be applied to the resolvant $(1 - z\mathcal{L})^{-1}$ of the transfer operator $\mathcal{L}$. We bypass the consideration of Banach spaces and consider $\left[(1 - z\mathcal{L})^{-1}\Phi\right](x)$ for suitable Φ and given x as a function of z. This method can be applied for instance to the case of rational maps of the Riemann sphere (see [10]).

A. Appendix. Reducing the study of piecewise monotone maps to the Markovian case.

Let X be a compact subset of $\mathbb{R}$. We assume that X is the union of finitely many disjoint closed intervals $J_1, \ldots, J_N$, and that $f : X \to X$ is such that $f|J_i$ is strictly monotone and satisfies the Darboux property (i.e. fJ_i is an interval) for $i = 1, \ldots, N$.

The partition $(J_1, \ldots, J_N)$ is said to be *generating* if every intersection

$$\bigcap_{n=0}^{\infty} f^{-n} J_{i(n)}$$

consists of at most one point.

The total variation of $g : X \to \mathbb{C}$ is denoted by $\operatorname{var} g$, and g is said to be of *bounded variation* if $\operatorname{var} g < \infty$.

A.1. Proposition. *Let $X, f, (J_1, \ldots, J_N)$, g be as above, with $(J_1, \ldots, J_N)$ generating and g of bounded variation. There exist then $\widehat{X}$, $\widehat{f}$, $(\widehat{J}_1, \ldots, \widehat{J}_N)$, $\widehat{g}$ with the same properties, such that X may be identified with a closed subset of $\widehat{X}$, that $f = \widehat{f}|X$, $g = \widehat{g}|X$ and $(\widehat{J}_1, \ldots, \widehat{J}_N)$ is a Markov partition for $\widehat{f}$. Furthermore, if $\xi \notin X$, then $\widehat{g}(\widehat{f}^n \xi) = 0$ for some $n \geq 0$.*

Let $\varepsilon(i) = \pm 1$ depending on whether f is increasing or decreasing on J_i (if J_i is reduced to a point, or empty, make an arbitrary choice). We define

$$\widehat{X} = \{\xi : \mathbb{N} \to \{1, \ldots, N\}\}$$

with the topology of pointwise convergence. If $\xi(k) = \xi'(k)$ for $k < n$ and $\xi(n) < \xi'(n)$ we write $\xi < \xi'$ if $\prod_0^{n-1} \varepsilon(\xi(k)) = 1$, and $\xi > \xi'$ if $\prod_1^{n-1} \varepsilon(\xi(k)) = -1$. This makes $\widehat{X}$ into an ordered Cantor set, that can be embedded in $\mathbb{R}$. We also define $\widehat{f}\xi$ by

$$(\widehat{f}\xi)(n) = \xi(n+1)$$

i.e., $\widehat{f}$ is the shift, and

$$\widehat{J}_i = \left\{\xi \in \widehat{X} : \xi(0) = i\right\} .$$

In particular $(\widehat{J}_1, \ldots, \widehat{J}_N)$ is a Markov partition for $\widehat{f}$, and $\widehat{f}$ is monotone on each $\widehat{J}_i$. We define now $j : X \to \widehat{X}$ by

$$(jx)(n) = i \Leftrightarrow f^n x \in J_i .$$

This map is injective because we have assumed $(J_1, \ldots, J_N)$ generating, it is order preserving by our definition of the order on $\widehat{X}$ and we have $\widehat{f} \circ j = j \circ f$. We may thus use j to identify X with a subset of $\widehat{X}$, and we have then $f = \widehat{f}|X$.

There remains to define $\widehat{g}$ with the properties announced. Let

$$u_i = \min J_i \qquad , \qquad v_i = \max J_i$$

where J_i is now considered as a subset of $\widehat{J}_i$. We may extend $g \mid J_i$ to a function $\widehat{g}$ on $[u_i, v_i]$ such that

$$\mathrm{var}\,(\widehat{g} \mid [u_i, v_i]) = \mathrm{var}(g \mid J_i)\ .$$

[Take for instance $\widehat{g}(w) = g(x)$ where $x = \max\,\{y \in J_i : y \le w\}$.] We complete the definition of $\widehat{g}$ by setting it equal to 0 on the complement of the intervals $[u_i, v_i]$. The function $\widehat{g}$ is thus of bounded variation on $\widehat{X}$, and $\widehat{g} \mid X = g$. The following lemma concludes the proof of the proposition. □

A.2. Lemma. *If ξ is such that*

$$\widehat{f}^n \xi \in \left[u_{\xi(n)}, v_{\xi(n)}\right] \ \textit{for all}\ n \ge 0,\ \textit{then}\ \xi \in X\ .$$

If we define

$$J_n^*(\xi) = \bigcap_{k=0}^{n} \widehat{f}^{-k}\left[u_{\xi(k)}, v_{\xi(k)}\right]$$

then $J_n^*(\xi) \ni \xi$, hence $J_n^*(\xi) \ne \emptyset$ for all $n \ge 0$. We have $J_0^*(\xi) = \left[u_{\xi(0)}, v_{\xi(0)}\right]$ and, for $n > 0$,

$$J_n^*(\xi) = \left[u_{\xi(0)}, v_{\xi(0)}\right] \cap \widehat{f}^{-1} J_{n-1}^*(f\xi)\ .$$

By induction on n we shall show that $J_n^*(\xi)$ is a subinterval of $\widehat{J}_{\xi(0)}$, of the form $[u_n^*, v_n^*]$, with u_n^*, $v_n^* \in X$. Note that $\widehat{f}\left[u_{\xi(0)}, v_{\xi(0)}\right] \cap J_{n-1}^*(f\xi)$ is a nonempty intersection of subintervals of $\widehat{J}_{\xi(0)}$ with endpoints in X, and is therefore again a subinterval of $\widehat{J}_{\xi(0)}$ with endpoints in X. Since $f \mid J_{\xi(0)}$ has the Darboux property, we see that $J_n^* = [u_n^*, v_n^*]$, with u_n^*, $v_n^* \in X$ as announced. The nonempty intervals $J_n^*(\xi) \cap X$ form a decreasing sequence, and their intersection contains some $\xi^* \in X$. We have thus

$$\{\xi\} = \bigcap_{n \ge 0} J_n^*(\xi) \supset \bigcap_{n \ge 0} (J_n^*(\xi) \cap X) \ni \xi^*$$

hence $\xi \in X$, proving the lemma. □

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