A Lorentz covariant description of the classical radiation field

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A Lorentz Covariant Description of the Classical Radiation Field

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Abstract. We propose a Lorentz covariant description of the radiation field. It is based on the observation that there exists a representation of the Lie algebra of the Lorentz group on the subspace of helicity states of the statespace of spin 1. This permits the determination of an action of the Lorentz group on the amplitudes of the field, and thereby the construction of the general Lorentz covariant solutions.

1. Introduction

The classical radiation field is described by the plane-wave solutions of the Maxwell equations. In standard notation, and in the helicity representation, they are expressed as the sums $E = E_+ + E_-$ and $B = B_+ + B_-$ of solutions of positive and negative helicity [1],

$$\vec{E}_\pm(\vec{x}, t; c_\pm) = (2\pi)^{-3/2} \int \sqrt{\frac{\omega}{2}} i(\vec{\varepsilon}_\pm(k) c_\pm(k) e^{i(\vec{k}\vec{x}-\omega t)} - \vec{\varepsilon}^{\,*}_\pm(k) c^*_\pm(k) e^{-i(\vec{k}\vec{x}-\omega t)}) d^3k$$

$$\vec{B}_\pm(\vec{x}, t; c_\pm) = (2\pi)^{-3/2} \int \sqrt{\frac{\omega}{2}} (\vec{\varepsilon}_\pm(k) c_\pm(k) e^{i(\vec{k}\vec{x}-\omega t} + \vec{\varepsilon}^{\,*}_\pm(k) c^*_\pm(k) e^{-i(\vec{k}\vec{x}-\omega t)}) d^3k$$

and such that $\vec{k} \cdot \vec{\varepsilon}_\pm(k) = 0$, $|\vec{\varepsilon}_\pm| = 1$, $\omega = |\vec{k}|$, $\vec{k} \in R^3$, $(\vec{x}, t) \in R^4$ and $c_\pm : R^3 \to C$ with $\int |c_\pm|^2 d^3k < \infty$. The general solution of the **transversality conditions** $\vec{k} \cdot \vec{\varepsilon}_\pm = 0$, are

$$\vec{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\vec{\varepsilon}_1 \pm i\vec{\varepsilon}_2)e^{i\alpha}$$

where

$$\vec{\varepsilon}_1(k) = (-\frac{k_1k_3}{\omega\sqrt{k_1^2+k_2^2}}, -\frac{k_2k_3}{\omega\sqrt{k_1^2+k_2^2}}, \frac{1}{\omega}\sqrt{k_1^2+k_2^2})$$

$$\vec{\varepsilon}_2(k) = (\frac{k_2}{\sqrt{k_1^2+k_2^2}}, \frac{-k_1}{\sqrt{k_1^2+k_2^2}}, 0)$$

and α is an undetermined constant that can be absorbed by the phase of $c_\pm$.

These solutions are not in covariant form : the transversality condition is not Lorentz invariant. One can therefore generate **new** solutions by applying Lorentz transformations on the solutions already constructed. It is however, not apriori clear how the amplitudes transform under the Lorentz group.

In the following we propose a solution to this problem. It is based on the observation that there exists a symplectic action of the group $SO(3, 2)$ on the space of helicity states (coherent states) of spin 1. To be able to apply this observation we consider the field as a function(al) on the statespace $\times$ spacetime, thus obtaining a framework in which questions about covariance are more easily discussed.

2. Spin 1 and helicity

The (quantum) observables of spin s are described by the generators S_1, S_2, S_3 of the unitary (projective) representation of the rotation group $SO(3)$ of R^3 in C^{2s+1}, the spin state space. For spin 1 the state space is thus C^3 and a representation of the spin observables is given by the selfadjoint operators

$$(S_i) = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) .$$

In this particular representation, the real and imaginary part of the state-vector transforms independently, and as "vectors", under the rotations. It is thus natural to use the notation $\vec{w} = \frac{1}{\sqrt{2}}(\vec{u} + i\vec{v})$. In terms of the real vectors $\vec{u}$ and $\vec{v}$, the spin-density then has the following form

$$\vec{s} = (\bar{w})\vec{S}(w) = \frac{1}{i}\vec{w}^* \wedge \vec{w} = \vec{u} \wedge \vec{v} .$$

The most prominent structural property of the theory of spin is complex linearity. Accordingly it possesses a (canonical) symplectic structure defined by the (real) two-form [2]

$$\Omega = i \sum_j dw_j^* \wedge dw_j .$$

This symplectic form is associated with the Poisson bracket

$$\{a, b\} = \frac{1}{i} \sum_j (\partial_{w_j} a \partial_{w_j^*} b - \partial_{w_j} b \partial_{w_j^*} a)$$

which can be used to define a structure of Lie algebra on the set of differentiable functions on C^3. The Lie algebra of linear operators on C^3 under commutation is injected into the Lie algebra of functions on C^3 under the Poisson bracket by the map defined by

$$A \mapsto (\bar{w})A(w) = f(A)(w^*, w) ;$$

in fact, $\{f(A), f(B)\} = (\bar{w})\frac{1}{i}[A, B](w) = f(\frac{1}{i}[A, B])$.

Definition : The vectors $\vec{w} = \frac{1}{\sqrt{2}}(\vec{u} + i\vec{v}) \in C^3$ such that $\vec{w}^2 = 0$, i.e.

$$\vec{u} \cdot \vec{v} = 0 \quad and \quad \vec{u}^2 - \vec{v}^2 = 0$$

are said to describe helicity states (i.e. states of circular polarization).

Proposition : Let (P, ω) be the symplectic manifold which in generalized spherical coordinates (ρ, τ, w, φ) is defined by $(R_+ \times \tilde{S}^3 \cup \{0\}, d\rho \wedge dw + d\tau \wedge d\varphi)$ where $\tilde{S}^3$ is the projective space associated with the three-dimensional sphere S^3. Then the map

$$c \, : \, R_+ \times \tilde{S}^3 \cup \{0\} \to C^3 \;\; ; \;\; (\rho, \tau, w, \varphi) \mapsto (w_j)$$

for

$$w_j = \frac{1}{\sqrt{2}}(u_j + iv_j) = \frac{1}{\sqrt{2}}(\varepsilon_{1j} + i\varepsilon_{2j})\sqrt{\rho}e^{iw}$$

$$(\varepsilon_{1j}) = (-\tau/\rho \cos\varphi \, , \, \tau/\rho \sin\varphi \, , \sqrt{1 - \tau^2/\rho^2}) \, ,$$

$$(\varepsilon_{2j}) = (-\sin\varphi \, , -\cos\varphi \, , 0)$$

is a symplectic embedding, i.e.

$$R_+ \times \tilde{S}^3 \cup \{0\} \simeq c(R_+ \times \tilde{S}^3 \cup \{0\}) \subset C^3 \;\; \text{and} \;\; c^*\Omega = \omega \, .$$

Moreover, $c(R_+ \times \tilde{S}^3 \cup \{0\}) \subset C^3$ is the submanifold of helicity states.

Proof : Direct computation.

Corollary : The spin density is

$$\vec{s} = \vec{u} \wedge \vec{v} = (\sqrt{\rho^2 - \tau^2}\cos\varphi \;\; , \;\; \sqrt{\rho^2 - \tau^2}\sin\varphi \;\; , \;\; \tau)$$

on the helicity states.

Proposition : Let $\vec{q} = \sqrt{\rho}\vec{u}$ and $\vec{r} = \sqrt{\rho}\vec{v}$; then, the set of functions $(q_i, \rho, r_i, s_i \, ; \, i = 1, 2, 3)$ constitute a basis for a representation of the Lie algebra $so(3, 2)$ in the Lie algebra of functions on $R_+ \times \tilde{S}^3 \cup \{0\}$ under the Poisson bracket

$$\{a, b\} = \partial_\rho a \partial_w b + \partial_\tau a \partial_\varphi b - \partial_w a \partial_\rho b - \partial_\varphi a \partial_\tau b \, .$$

Proof : The proof consists in verifying the "commutation relations"

$$\{s_i, s_j\} = \varepsilon_{ijk} s_k \, , \{s_i, r_j\} = \varepsilon_{ijk} r_k \, , \{s_i, q_j\} = \varepsilon_{ijk} q_k \, .$$

$$\{s_i, \rho\} = 0 \, , \{r_i, q_j\} = \delta_{ij}\rho \, , \{r_i, r_j\} = -\varepsilon_{ijk} s_k \, ,$$

$$\{q_i, q_j\} = -\varepsilon_{ijk} s_k \ , \{q_i, \rho\} = r_i \ , \{r_i, \rho\} = -q_i \ .$$

This is done by computation.

Corollary : Let λ denote the action of $SO(3,1)$ on $R_+ \times \tilde{S}^3 \cup \{0\}$ generated by $(r_i, s_i \ : \ i = 1, 2, 3)$, Λ the Lorentz transformations on R^4 leaving invariant the Minkowski metric, and let

$$q^\mu = (\rho, \vec{q}) \quad \text{and} \quad s^{\mu\nu} = \begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ -r_1 & 0 & s_3 & -s_2 \\ -r_2 & -s_3 & 0 & s_1 \\ -r_3 & s_2 & -s_1 & 0 \end{pmatrix} .$$

Then

$$q^\mu(\lambda_\Lambda(\rho, \tau, w, \varphi)) = \Lambda^\mu{}_\nu q^\nu(\rho, \tau, w, \varphi)$$

$$s^{\mu\nu}(\lambda_\Lambda(\rho, \tau, w, \varphi)) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta s^{\alpha\beta}(\rho, \tau, w, \varphi)$$

i.e. q^μ and $s^{\mu\nu}$ are manifestly covariant with respect to the Lorentz transformations.

From this corollary it follows that the expressions

$$s^{\mu\nu} s_{\mu\nu} = \vec{r}^{\,2} - \vec{s}^{\,2} (= 0)$$

$$\varepsilon_{\alpha\beta\gamma\delta} s^{\alpha\beta} s^{\gamma\delta} = \vec{r} \cdot \vec{s} (= 0)$$

$$q^\mu q_\mu = \rho^2 - \vec{q}^{\,2} (= 0)$$

are invariant under the Lorentz transformations. Indices are raised and lowered by the Minkowski metric. Moreover, for this particular representation

$$s^\mu{}_\nu q^\nu = (-\vec{q} \cdot \vec{r} \, , \vec{s} \wedge \vec{q} - \rho \vec{r}) = (0, \vec{0}) \ .$$

Thus, the relations

$$\vec{s} \wedge \vec{q} - \rho \vec{r} = \vec{0} \quad \text{and} \quad \vec{q} \cdot \vec{r} = 0$$

are also invariant under the Lorentz transformations, and it follows that the orthogonality of the "vectors" $\vec{q}$, $\vec{r}$ and $\vec{s}$ is invariant. One can also show by direct computation that the Lorentz transformed $\vec{s}\,'$ of $\vec{s}$ satisfies

$$\vec{s}\,'^2 = \rho'^2$$

accordingly,

$$\vec{s}\,'^2 = \vec{q}\,'^2 = \vec{r}\,'^2 = \rho'^2 \quad \text{and} \quad \vec{s}\,' \cdot \vec{r}\,' = \vec{s}\,' \cdot \vec{q}\,' = \vec{r}\,' \cdot \vec{q}\,' = 0\,,$$

i.e. the orthonormal frame $(\vec{q}/\rho, \vec{r}/\rho, \vec{s}/\rho)$ is transformed into an orthonormal frame $(\vec{q}\,'/\rho', \vec{r}\,'/\rho', \vec{s}\,'/\rho')$ by the given action of the Lorentz group.

3. The state space of the radiation field

There exists a non-linear action of the Lorentz group $SO(3,1)$ on R^3. Formally it corresponds to the Lorentz action

$$k^\mu \mapsto \Lambda^\mu{}_\nu k^\nu$$

where $k^\mu = (|\vec{k}|, \vec{k})$, and Λ is the usual linear representation of $SO(3,1)$ on R^4. The measure d^3k/ω, $\omega(k) = |\vec{k}|$, is an invariant measure on R^3 under the Lorentz action of $SO(3,1)$.

Let $\mathcal{B}(P)$ denote a Banach manifold of measurable maps

$$R^3 \to P = R_+ \times \tilde{S}^3 \cup \{0\}$$

such that $\int \rho(k) d^3k/\omega < \infty$. We denote by M_+ and M_- the Banach submanifolds of positive and negative helicity consisting of points satisfying the transversality condition

$$\vec{s}(\rho(k), \tau(k), w(k), \varphi(k)) \wedge \vec{k} = 0.$$

As is easily verified

$$M_+ = \{(\rho, \tau, w, \varphi) \in \mathcal{B}(P) | \tau(k) = \frac{k_3}{\omega}\rho(k)$$

$$\text{and } \varphi(k) = -\arcsin(\frac{k_2}{\sqrt{k_1^2 + k_2^2}})\}$$

$$M_- = \{(\rho, \tau, w, \varphi) \in \mathcal{B}(P) | \tau(k) = \frac{-k_3}{\omega}\rho(k)$$

$$\text{and } \varphi(k) = \pi - \arccos(\frac{k_1}{\sqrt{k_1^2 + k_2^2}})\}\,.$$

An action of $SO(3,1)$ on $\mathcal{B}(P)$ is defined by

$$(\Gamma_\Lambda(\rho,\tau,w,\varphi))(k) = \lambda_\Lambda(\rho,\tau,w,\varphi)(\Lambda^{-1}k)\ .$$

Since the transversality condition is invariant under the rotations, the orbit of a particular point $(\rho,\tau,w,\varphi) \in M_\pm$ is parametrized by

$$SO(3,1)/SO(3) \simeq (n \in R^4 | n_o^2 - \vec{n}^2 = 1\ ,\ n_o > 0) = H.$$

Definition : Let $L_n = L(n)$ denote the boost defined by $L(n) = \Lambda(\frac{\vec{n}}{n_o})$, where $\Lambda(\frac{\vec{n}}{n_o})$ denote a pure Lorentz transformation "for the velocity $\vec{n}/n_o$"; the subspace $\mathcal{M}_+ \cup \mathcal{M}_- \subset \mathcal{B}(P)$ such that

$$\mathcal{M}_\pm = \{\Gamma_{L_n}(m) | m \in M_\pm \ \text{ and } \ n \in H\}$$

provide a representation of the state space of the radiation field.

From the construction of $\mathcal{M}_\pm$ we see that they are diffeomorphic to

$$\tilde{\mathcal{M}}_\pm = M_\pm \times H$$

the diffeomorphism being given by

$$\tilde{\Gamma}\ :\ \tilde{\mathcal{M}} \to \mathcal{M}\ ;\ (m,n)(k) \mapsto (\Gamma_{L_n}(m))(k) = \lambda_{L_n}(m)(L_n^{-1}k)\ .$$

Accordingly, the induced action $\tilde{\lambda} = \tilde{\Gamma}^{-1} \circ \lambda \circ \tilde{\Gamma}$ of the Lorentz group on $\tilde{\mathcal{M}}$, is

$$(\tilde{\lambda}_\Lambda((m,n))(k) = (\lambda_R(m)(R^{-1}k)\ , \Lambda n)$$

where $R = R(n,\Lambda) = L^{-1}(\Lambda n)\Lambda L(n)$ is a rotation (Wigner rotation) [3].

Remark : From the Poisson bracket relations between the functions (q_i, ρ, r_i, s_i) we see that we could have chosen (q_i, s_i) to generate the Lorentz transformations on P. Then,

$$(\rho, r_i)$$

transforms as a fourvector, and

$$\begin{pmatrix} 0 & q_1 & q_2 & q_3 \\ -q_1 & 0 & s_3 & -s_2 \\ -q_2 & -s_3 & 0 & s_1 \\ -q_3 & s_2 & -s_1 & 0 \end{pmatrix}$$

transforms as a tensor. This freedom of choice is related to the existence of the two classes of solutions of the transversality condition. In fact, with this choice the $\mathcal{M}_+$ defined above will describe states of negative helicity and $\mathcal{M}_-$ those of positive helicity.

4. The field variables of the classical radiation field

The geometry of the framework suggest that $\vec{q}$ and $\vec{r}$ can be used to define amplitudes for the electric and magnetic components of the radiation field. In fact, as one varies the phase w, $\vec{q}$ and $\vec{r}$ rotates in the plane orthogonal to $\vec{s}$. By testing this idea one finds that the amplitudes $a_\mu = (v, -\vec{a})$, $e_i = f_{oi}$ and $b^i = -\frac{1}{2}\varepsilon^{ijk} f_{jk}$ of the vectorpotential and the field can be written

$$v = \vec{s} \cdot \vec{k} / \sqrt{\omega\rho}$$

$$\vec{a} = (-\vec{k} \wedge \vec{r} + \omega\vec{s}) / \sqrt{\omega\rho}$$

$$\begin{aligned} \vec{e} &= -\omega^2 \vec{r} / \sqrt{\omega\rho} = -\omega\sqrt{\omega}\vec{v} = i\omega\sqrt{\omega/2}(\vec{w} - \vec{w}^*) \\ &= i\omega\sqrt{\omega/2}(\vec{\varepsilon}\sqrt{\rho}e^{iw} - \vec{\varepsilon}^*\sqrt{\rho}e^{-iw}) \,, \end{aligned}$$

$$\vec{b} = -\omega\vec{k} \wedge \vec{r} / \sqrt{\omega\rho}$$

for the states in $M_\pm$. Note that

$$\vec{\varepsilon}(m_\pm(k)) = \frac{1}{\sqrt{2}}(\vec{\varepsilon}_1(k) \pm i\vec{\varepsilon}_2(k)) \text{ for } m_\pm \in M_\pm \,.$$

On these states the above expression can be put on the following manifestly covariant form

$$a_\mu = \varepsilon_{\mu\alpha\beta\gamma} k^\alpha s^{\beta\gamma} / \sqrt{k_\mu q^\mu}$$

$$f_{\mu\nu} = (s_{\mu\alpha} k^\alpha k_\nu - s_{\nu\alpha} k^\alpha k_\mu) / \sqrt{k_\mu q^\mu}$$

which thus are valid on $\mathcal{M}_\pm$.

It is more convenient to consider the amplitudes as functions $\tilde{a}$ and $\tilde{f}$ on $\tilde{\mathcal{M}}_\pm$. Now, defining $\tilde{a}$ and $\tilde{f}$ by

$$\tilde{a}_\mu((m(L_n^{-1}k), n, L_n^{-1}k) \equiv a_\mu((\tilde{\Gamma}(m,n))(k), k)$$

$$\tilde{f}_{\mu\nu}((m(L_n^{-1}k), n, L_n^{-1}k) \equiv f_{\mu\nu}((\tilde{\Gamma}(m,n))(k), k)$$

we get

$$\tilde{a}_\mu(m(k), n, k) = L^\alpha_{n\mu} \varepsilon_{\alpha\beta\gamma\delta} k^\beta s^{\gamma\delta}(m(k)) / \sqrt{k_\mu q^\mu(m(k))}$$

$$\tilde{f}_{\mu\nu}(m(k),n,k) = L^{\alpha}_{n\mu} L^{\beta}_{n\nu}(s_{\alpha\gamma}(m(k))k^{\gamma}k_{\beta} - \\ - s_{\beta\gamma}(m(k))k^{\gamma}k_{\alpha}) / \sqrt{k_{\mu}q^{\mu}(m(k))} \, .$$

Definition : The field observables of the radiation field are represented by the functions $R^4 \times \tilde{\mathcal{M}}_{\pm} \to R$,

$$A_{\mu}(x^{\mu};\rho,\tau,w,\varphi,n) = \\ (2\pi)^{-3/2} \int \tilde{a}_{\mu}(\rho(k),\tau(k),w(k) - (L_n k)^{\mu} x_{\mu}, \varphi(k), n, k) d^3k/\omega$$

$$F_{\mu\nu}(x^{\mu};\rho,\tau,w,\varphi,n) = \\ (2\pi)^{-3/2} \int \tilde{f}_{\mu\nu}(\rho(k),\tau(k),w(k) - (L_n k)^{\mu} x_{\mu}, \varphi(k), n, k) d^3k/\omega \, .$$

This definition is justified by the preceding discussion and the following proposition.

Proposition : The field observables satisfies the "Maxwell equations"

$$F_{\mu\nu} = \partial_{x^{\nu}} A_{\mu} - \partial_{x^{\mu}} A_{\nu} \text{ and } \partial_{x^{\nu}} F^{\mu\nu} = 0 \, .$$

Proof : The proposition is verified by direct computation. We note that the amplitudes are

$$\tilde{v} = (n_o \vec{s} \cdot \vec{k} - \vec{n} \cdot (\vec{k} \wedge \vec{r}) + \omega \vec{n} \cdot \vec{s}) / \sqrt{\omega\rho}$$

$$\vec{\tilde{a}} = (-\vec{k} \wedge \vec{r} + \omega \vec{s} - \frac{1}{n_o + 1}(\vec{n} \cdot (\vec{k} \wedge \vec{r}) - \omega \vec{s} \cdot \vec{n})\vec{n} + \vec{k} \cdot \vec{s}\vec{n}) / \sqrt{\omega\rho}$$

$$\vec{\tilde{e}} = \omega(\omega(-n_o \vec{r} + \frac{1}{n_o + 1}\vec{r} \cdot \vec{n}\vec{n}) - \vec{n} \wedge (\vec{k} \wedge \vec{r})) / \sqrt{\omega\rho}$$

$$\vec{\tilde{b}} = \omega(-n_o \vec{k} \wedge \vec{r} + \frac{1}{n_o + 1}(\vec{k} \wedge \vec{r}) \cdot \vec{n}\vec{n} + \omega \vec{n} \wedge \vec{r}) / \sqrt{\omega\rho}$$

when we take into account the transversality condition. We note that $\partial_w \vec{r} = \vec{q}$. This follows from the Poisson bracket $\{\vec{r}, \rho\} = -\vec{q}$.

Let $\vec{W} = \frac{1}{\sqrt{2\rho}}(-\vec{k} \wedge \vec{r} - i\omega\vec{r})$, then

$$\omega\vec{r}(.,w+\alpha,.) = \frac{1}{i}\sqrt{\rho/2}(\vec{W}(.,w,.)e^{i\alpha} - \vec{W}^*(.,w,.)e^{-i\alpha}),$$

$$\vec{k} \wedge \vec{r}(.,w+\alpha,.) = -\sqrt{\rho/2}(\vec{W}(.,w,.)e^{i\alpha} + \vec{W}^*(.,w,.)e^{-i\alpha}),$$

and

$$\begin{aligned}\vec{e} &= i\sqrt{\omega/2}(\vec{W}\,'(\ldots)(k)e^{-i(L_n k)\cdot x} - \vec{W}\,'^*(\ldots)(k)e^{i(L_n k)\cdot x}) \\ &= i\sqrt{\omega(L_n^{-1}k')/2}(\vec{W}\,'(\ldots)(L_n^{-1}k')e^{-ik'\cdot x} \\ &\quad - \vec{W}\,'^*(\ldots)(L_n^{-1}k')e^{ik'\cdot x})\end{aligned}$$

$$\begin{aligned}\vec{b} &= \sqrt{\omega/2}(\vec{W}\,'(\ldots)(k)e^{-i(L_n k)\cdot x} + \vec{W}\,'^*(\ldots)(k)e^{i(L_n k)\cdot x}) \\ &= \sqrt{\omega(L_n^{-1}k')/2}(\vec{W}\,'(\ldots)(L_n^{-1}k')e^{-ik'\cdot x} \\ &\quad + \vec{W}\,'^*(\ldots)(L_n^{-1}k')e^{ik'\cdot x})\end{aligned}$$

where $\omega(k) = |\vec{k}|$, $k' = L_n k$ and $\vec{W}\,' = n_o\vec{W} - \frac{1}{n_o+1}\vec{W}\cdot\vec{n}\vec{n} + i\vec{n}\wedge\vec{W}$.

Using this complex representation we can determine the energy and momentum of the field by standard computations, applying the relation $|\vec{W}\,'|^2 = \rho(n_o\omega + \vec{k}\cdot\vec{n})^2 = \rho\omega'^2$, i.e.

$$\begin{aligned}P^o(\ldots) &= \int \frac{1}{2}[\vec{B}(x^\mu;\ldots)^2 + \vec{D}(x^\mu;\ldots)^2]d^3x \\ &= \int \omega(L_n^{-1}k')\rho(L_n^{-1}k')d^3k' = \int L^o_{n\mu}k^\mu\rho(k)d^3k\,.\end{aligned}$$

Similarly, since $\frac{1}{i}(\vec{W}\,'\wedge\vec{W}\,'^*)^j = L^o_{n\nu}k^\nu L^j_{n\mu}k^\mu\rho$,

$$\begin{aligned}P^j(\ldots) &= \int (\vec{D}(x^\mu;\ldots)\wedge\vec{B}(x^\mu;\ldots))^j d^3x \\ &= \int k'^j(L_n^{-1}k')(\rho(L_n^{-1}k')/\omega'(L_n^{-1}k'))d^3k' \\ &= \int L^j_{n\nu}k^\nu\rho(k)d^3k\,.\end{aligned}$$

We have thus proved the following proposition.

Proposition : The energy and momentum of the radiation field are given by

$$P^\mu(\rho,\tau,w,\varphi,n) = \int L^\mu_{n\nu}k^\nu\rho(k)d^3k\,.$$

The action of the Lorentz group on the observables is induced by the action on the states. Thus, the vector-potential A transforms according to

$$
\begin{aligned}
&A_\mu(x^\mu;\rho,\tau,w,\varphi,n) \mapsto A_\mu(x^\mu;\lambda_\Lambda(\rho,\tau,w,\varphi,n)) \\
&= \int (L_{\Lambda n}\varepsilon)_{\mu\alpha\beta\gamma} k^\alpha s^{\beta\gamma}(\lambda_R(.,w(R^{-1}k) - (L_n^{-1}k)\cdot x,.))/\sqrt{\omega\rho(R^{-1}k)}d^3k/\omega \\
&= \int (L_{\Lambda n}R\varepsilon)_{\mu\alpha\beta\gamma} (R^{-1}k)^\alpha s^{\beta\gamma}(.,w(R^{-1}k) - (L_n^{-1}k)\cdot x,.))/\sqrt{\omega\rho(R^{-1}k)}d^3k/\omega \\
&= \int (\Lambda L_n\varepsilon)_{\mu\alpha\beta\gamma} k^\alpha s^{\beta\gamma}(\rho(k),\tau(k),w(k) - (L_n k)\cdot(\Lambda^{-1}x),\varphi(k))/\sqrt{\omega\rho(k)}d^3k/\omega \\
&= \quad (\Lambda A)_\mu((\Lambda^{-1}x)^\mu\,;\rho,\tau,w,\varphi,n)
\end{aligned}
$$

where $R = R(n;\Lambda) = L^{-1}(\Lambda n)\Lambda L(n)$. For $F_{\mu\nu}$ and P^μ we similarly get

$$P^\mu(\ldots) \mapsto \Lambda^\mu{}_\nu P^\nu(\ldots)$$

$$F_{\mu\nu}(x^\mu;\ldots) \mapsto \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}(\Lambda^{-1\mu}{}_\nu x^\nu;\ldots)$$

5. Discussion

So far we have considered separately the positive and negative helicity solutions. In general, we must consider superpositions of these. This has no further consequences, since the positive and negative helicity amplitudes transform independently of each other under the given action of the Lorentz group. Moreover, using that $\vec{W}'^2 = \vec{W}'^{*2} = 0$ one can show that

$$
\begin{aligned}
&P^\mu(\rho_+,\tau_+,w_+,\varphi_+,\rho_-,\tau_-,w_-,\varphi_-,n) = \\
&= \int (L_n k)^\mu(\rho_+(k) + \rho_-(k))d^3k\,.
\end{aligned}
$$

By performing a Lorentz transformation $\Lambda(-\frac{\vec{n}}{n_o})$ on the general solution $F_{\mu\nu}$ (or by choosing the state-variable $\vec{n} = \vec{0}$), we obtain the solution(s) given in the introduction, and which is to be interpreted as the description of the field produced and detected in the laboratory frame of reference. The general solution thus describe a field produced in a frame of reference moving with velocity $\vec{n}/n_o$ relative to the frame of reference where it is detected (and described). We see that the formulaes take into account the Doppler effect due to the motion of the source.

The theory we have exposed is thus essentially an Einstein relativistic formulation of the "standard" classical radiation theory. Its translation into a quantum theory is straightforward. This theory is obtained, in its most simple and direct formulation, by the substitutions $c^* = \sqrt{\rho}e^{-iw} \rightarrow a^\dagger$ and $c = \sqrt{\rho}e^{iw} \rightarrow a$, where $a^\dagger$ and a denote the creation and annihilation operators. An evident application of the theory is to study the radiation from moving atoms.

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References

[1] J.J. Sakurai. *Advanced Quantum Mechanics*, Addison-Wesley (Reading, Mass. 1967).

[2] R. Abraham and J.E. Marsden. *Foundations of Mechanics*,, Benjamin-Cummings (Reading, Mass. 1978).

[3] E.P. Wigner. Ann. Math. **40**, 149 (1939).