Helv Phys Acta
Vol. 65 (1992)

0018-0238/92/070887-11$1.50+0.20/0

Complete sets of commuting observables and irreducible sets of observables

Abraham Lozada , G. De Sarrazin and P.L. Torres
Escuela de Física y Matemática, Facultad de Ciencias, Universidad Central de Venezuela, A. P. 20513
Caracas 1020-A, Venezuela.

(26. II. 1992, revised 25. VI. 1992)

Abstract. In this paper, some general results about the concepts of complete sets of commuting observables (CSCO) and irreducible sets of observables (ISO) are obtained. It is proved the following: (i) Any relevant observable is an essential part of some CSCO. (ii) Any relevant observable, which is a CSCO, is an essential part of some ISO. (iii) Let $\{b_\alpha\}_{\alpha\in I}$ be a CSCO, or an ISO, then, I is countable.

1. Introduction

The physical and mathematical structure of the concept of complete set of commuting observables (CSCO), introduced by Dirac [1], has been firmly established in different settings of quantum mechanics [2-10]. Also, the concept of irreducible set of observables (ISO) has been treated in the literature, for example, through the canonical commutation relations (see, e.g.,[9 ,11]), or the formulation set in [10].

The main purpose of the present paper is to obtain rigorous results about the existence of different kinds of CSCO and ISO. From the practical point of view, both concepts are important when the observables are directly measurable at the laboratory (let us call them "relevant observables"). It is therefore of great interest to study in general how to construct a CSCO or an ISO with arbitrary observables. In particular, given a "relevant observable", it is physically necessary to have a proof which asserts that this observable is part of a CSCO or of an ISO. This is the stronger motivation for undertaking the present work.

The results of this paper are collected in theorems II.4, II.7, III.4 and III.6. Theorems II.4, III.4 and III.6 have as consequences: (i) Any relevant observable is an essential part of some CSCO, (ii) Any relevant observable, which is a CSCO, is an essential part of some ISO, (iii) Let $\{b_\alpha\}_{\alpha\in I}$ be a CSCO, or an ISO, then, I is countable. Theorem II.7 concerns with the problem of comparing the Dirac's formulation of

CSCO in infinite dimensional Hilbert spaces with Jauch's definition [2]. It is shown that Dirac's formulation is a particular case of Jauch's formulation in infinite dimensional spaces.

All our results are independent of the physical system under study, and their physical interest is addressed in notes and discussions. See specially, remarks II.5, II.8, III.5 and III.8 . Only physical systems without superselection rules are considered. The set of observables of the physical systems will be taken as the set of all bounded self-adjoint operators in a Hilbert space. However, the above results are still valid in other settings of quantum mechanics. For instance, if unbounded self-adjoint operators are considered, our theorems hold true (see, Note III.9).

We shall adopt the following conventions. By R and C we stand for the real and complex numbers ($\bar{\lambda}$ is the complex conjugate of $\lambda \in C$); R^N and C^N are cartesian products of R and C respectively. By $\mathcal{H}$ we denote a complex separable Hilbert space of dimension greater than one (dim $\mathcal{H} \geq 2$), with scalar product (.,.) linear in the second variable. By $\mathcal{A} = \mathcal{L}(\mathcal{H})$ we represent the set of all bounded operators in $\mathcal{H}$, with norm $\| . \|$. The adjoint element of $a \in \mathcal{A}$ will be denoted by a^*, and $\mathcal{A}^s$ stands for the set of all self - adjoint elements of $\mathcal{A}$. The identity in $\mathcal{A}$ is denoted by e . The commutator of $a,b \in \mathcal{A}$ is represented by $[a,b]$.

Mathematical results found in [12 - 22] will be used throughout the paper. The commutant of a set $\mathfrak{B} \subset \mathcal{A}$ will be denoted by $\mathfrak{B}^c$. The Von Neumann algebra generated by a set $\mathfrak{D} = \{d_\alpha\}_{\alpha \in I} \subset \mathcal{A}$ will be indicated by $\mathfrak{R}(\mathfrak{D}) = \mathfrak{R}(\{d_\alpha\})$. An abelian Von Neumann algebra $\mathfrak{D} \subset \mathcal{A}$ is maximal abelian [12,13] iff $\mathfrak{D}^c = \mathfrak{D}$ (see, proposition 13, Part. I, Cap. I, of [12]).

Let us introduce the following standard definitions:

I.1. Definition. An arbitrary set of observables is called a **set of compatible observables** if they commute each other.

I.2. Definition. Let $\{\lambda_k\}_{k=1}$ be a countable set (finite or infinite) with $\lambda_k \in C$; $\lambda_k \neq 0$, k = 1,2,...; $\lambda_k \neq \lambda_i$ if $k \neq i$. Let $\{p_k\}_{k=1}$ be a set of mutually orthogonal projection operators. Every operator b of the type

$$b = \sum_{k=1} \lambda_k p_k, \tag{1.1}$$

is called a **point spectrum operator** (PSO).

We know (see, [14], Cap. 8, §1) that b is self-adjoint iff $\{\lambda_k\} \subset R$; and that b is positive iff $\lambda_k > 0$, $\forall$ k. Also, $b \in \mathcal{L}(\mathcal{H})$ iff $\sup_k |\lambda_k| < \infty$, and in this case: $\| b \| = \sup_k |\lambda_k|$.

I.3. A functional calculus. From references [12-19], we can extract the following functional calculus related to $\mathfrak{R}(\mathfrak{C})$, where $\mathfrak{C} = \{c_1,...,c_n\}$ is a compatible set of observables.

The algebra $\mathfrak{R}(\mathfrak{C})$ is isometricly *-isomorphic to $L^\infty(\Omega,\nu)$, where $\Omega = S_p(c_1) x.....x\, S_p(c_n) \subset R^n$, $S_p(c)$ standing for the spectrum of c. This *-isomorphism is the unique extension of Gelfand *-

isomorphism, and ν is the unique (up to equivalence) positive, finite regular, Borel measure with support Ω. Every $x \in \mathcal{R}(\mathbb{C})$ is of the form $x = f(c_1,\dots,c_n)$ with $f \in L^{\infty}(\Omega,\nu)$. Moreover,

$$\|x\| = \| f(c_1,\dots,c_n)\| = \underset{\{\lambda_1,\dots,\lambda_n\}\in\Omega}{\text{ess. sup}} |f(\lambda_1,\dots,\lambda_n)| = \| f \|_{\infty},$$

$$S_p(x) = \{f(\lambda_1,\dots,\lambda_n) \,/\, \{\lambda_1,\dots,\lambda_n\} \in \Omega\}. \tag{1.2}$$

This paper is organized as follows. In Sec. 2, results about CSCO are obtained. In Sec. 3 the results about ISO are concentrated. The notes and discussions contain some concluding remarks and trivial corollaries from the obtained results. Also, for the sake of completeness, they include results which are mostly behind the existing theory. The notes are supposed to be more technical than the discussions.

2. Complete sets

II.1. Definition. A set $\mathbb{C} = \{c_\alpha\}_{\alpha\in I} \subset \mathcal{A}^S$ is termed a **complete set of commuting observables** (CSCO) if $\mathcal{R}(\mathbb{C})$ is maximal abelian in $\mathcal{A}$, and no proper subset of $\mathbb{C}$ generates $\mathcal{R}(\mathbb{C})$ [minimality of the set $\mathbb{C}$].

II.2. Note. Definition II.1 says that $\mathbb{C}$ is a CSCO iff

$$\mathcal{R}(\mathbb{C})^{c} = \mathcal{R}(\mathbb{C}), \tag{2.1}$$

and $\mathbb{C}$ is a minimal set. Moreover, since $\mathbb{C}$ is a self-adjoint set, we have that $\mathbb{C}$ is a CSCO iff

$$\mathbb{C}^{c} = \mathcal{R}(\mathbb{C}), \tag{2.2}$$

and $\mathbb{C}$ is a minimal set.

Definition II.1 is essentially the one given in [2]. The **minimality** assumed in II.1 for the set $\mathbb{C}$ is physically important. In the different settings found in the literature (see, e.g. [1,2,5-9]), for the concept of CSCO, this minimality is not explicitly stated but should be taken as understood.
Also, from (2.2) it follows that for a minimal set $\mathbb{C} = \{c_1,\dots,c_n\}$ we have that $\mathbb{C}$ is a CSCO iff for all $x \in \mathbb{C}^{c}$ there exists $f \in L^{\infty}(\Omega,\nu)$ such that $x = f(c_1,\dots,c_n)$ [$L^{\infty}(\Omega,\nu)$] comes from the functional calculus of $\mathcal{R}(\mathbb{C})$, already referred in Sec. 1]. Here, the minimality condition represents a notion of independent variables in the functional calculus.

A CSCO is a compatible set of observables since it generates an abelian algebra. If c belongs to any CSCO, then $c \neq \gamma e$; $\gamma \in R$, since $\mathcal{A} \neq \{o\}$, $\mathcal{A} \neq \mathcal{R}(e)$, and the stated minimality.

We have that a compatible set of observables, $\mathbb{C}$, is a CSCO iff there exists a cyclic vector for $\mathcal{R}(\mathbb{C})$ and $\mathbb{C}$ is minimal, c.f., [2,5, 6, 23]; or iff there exists a cyclic vector for the minimal set $\mathbb{C}$, c.f., [5,6,23].

Finally, we have that $\{ c \} \subset \mathcal{A}^S$ is a CSCO iff c has a simple spectrum (see, e.g., [2,4,5,14,20]); or iff c can be represented by a Jacobi matrix (see, [20], Theorems 7.13 and 7.14).

II.3. Discussion. The existence of a CSCO in $\mathcal{A}$ is obvious since for $x \in \mathcal{A}^s$ ($x \neq \gamma e$), if $\mathcal{R}(x)$ is not maximal abelian, it is contained in a maximal abelian Von Neumann Algebra [12], which, by a theorem due to Von Neumann [24], is generated by a single element of $\mathcal{A}^s$ (see, also, [22]). Additionally, through this theorem, if $\mathbb{C} = \{c_\alpha\}_{\alpha \in I}$ is a CSCO, there exists an element $c \in \mathcal{A}^s$ such that $\mathcal{R}(c) = \mathcal{R}(\mathbb{C})$.

The last statement should be contrasted with the usual examples of CSCO (see, e.g., [1,5,7,9]) in which the number of used observables in order to form a CSCO is generally greater than one (in fact, the number of observables is usually taken equal to the number of degrees of freedom). For instance, in nonrelativistic quantum mechanics, for a spinless particle in three dimensions; the set of the three components of the position operator $\{q_x, q_y, q_z\}$ is a CSCO in the usual scheme [1,5,7,9] (these observables are unbounded; but this is irrelevant for the present discussion). However, a **single** density matrix, with simple spectrum, is also a CSCO. Moreover, Von Neumann's theorem [22] establishes that there exists a single observable c such that the spectral families of q_x, q_y and q_z are functions of c . A similar discussion can be addressed for systems with an arbitrary number of degrees of freedom.

Even though there is not a priory connection between degrees of freedom and the number of observables in a CSCO, it is more natural to choose the number of observables in a CSCO to be equal to the number of degrees of freedom, since in that way the physical interpretation can be made clearer.

Although Von Neumann's theorem is a remarkable result, it is not physically very useful. In general, we have to start from an arbitrary physical observable and construct a CSCO with it.

II.4. Theorem. Let $c_1 \in \mathcal{A}^s$, with $c_1 \neq \gamma e$. If c_1 is not a CSCO, there exists a PSO, $c_2 \in \mathcal{A}^s$, such that the set $\{c_1, c_2\}$ is a CSCO.

Proof. If c_1 is a PSO, the proof is trivial. Then, let us suppose that c_1 is not a PSO.

(i) First let us show that there exists a set of mutually orthogonal projectors $\{p_n\}_{n=1}$, such that $\sum_{n=1} p_n = e$, and for each p_n there exists $\xi_n \in \mathcal{H}$ such that p_n is the projection corresponding to the closed subspace $[\mathcal{R}(c_1)\ \xi_n]$.

In fact, if $\mathfrak{D} = \mathcal{R}(c_1)$; let us form the closed subspace $[\mathfrak{D}\ \xi_1] = M_1$ where $\xi_1 \in \mathcal{H}$, and $\|\xi_1\| = 1$. Since $[\mathfrak{D}\ \xi_1] \neq \mathcal{H}$, we have that there exists $\xi_2 \in \mathcal{H}$, $\|\xi_2\| = 1$, such that $[\mathfrak{D}\ \xi_2] = M_2$ is orthogonal to M_1. Let us continue this procedure up to obtain $\mathcal{H}$ as a direct sum of subspaces M_α of this kind. These subspaces will be at most numerable since $\mathcal{H}$ is separable. Then, $\mathcal{H} = \bigoplus_{n=1} M_n$. Let p_n the projector corresponding to the closed subspace M_n. The set $\{p_n\}_{n=1}$ is the searched set of projectors.

(ii) Let us denote the set of all p_n , n = 1,2,...; by $\mathcal{P}$.

Since M_n reduces $\mathfrak{D}$, we have that $\mathcal{P} \subset \mathfrak{D}^c$.

Let us see that $\mathcal{R}(c_1,\mathcal{P})$ is maximal abelian. In fact, let $\xi = \sum_n 2^{-n} \xi_n \in \mathcal{H}$. We have that $\{\mathcal{D}\, \xi_n\} = \{\mathcal{D}\, p_n\, \xi\} \subset \{\mathcal{R}(c_1,\mathcal{P})\, \xi\}$, $\forall n$, which implies that (for the closure): $[\mathcal{D}\, \xi_n] \subset [\mathcal{R}(c_1,\mathcal{P})\, \xi]$, $\forall n$. Since $\mathcal{H} = \bigoplus_{n=1} M_n$, we see that ξ is a cyclic vector and hence $\mathcal{R}(c_1,\mathcal{P})$ is maximal abelian (see, e.g., [17], Corollary 2.9.4).

(iii) Finally, let $c_2 \in \mathcal{A}^s$ be defined by

$$c_2 = \sum_n 2^{-n}\, p_n. \tag{2.3}$$

Then, we have that $\mathcal{R}(c_2) = \mathcal{R}(\mathcal{P})$.

Since c_1 is not a PSO, and $\mathcal{R}(c_1,c_2)$ is maximal abelian, such c_2 (being a PSO) cannot be a CSCO. Therefore, $\{c_1,c_2\}$ is a CSCO.

II.5. Discussion. Theorem II.4 is a well known result [1,7] for the case in which c_1 is a PSO.

If c_1 is not a PSO, Theorem II.4 is a non trivial result, and does not depend on the minimality assumed in definition II.1. In fact, independently of the minimality, we have that (the PSO) c_2 cannot be a CSCO, since in this case c_1 would be an element of $R(c_2)$, and hence a PSO.

Even though our proof is not constructive, it does not mean that this result is practically useless. Indeed, theorem II.4 is important in practice since it asserts that to any observable c_1 which is not a CSCO, and that is physically relevant, we can adjoint an observable c_2, such that they form a CSCO. Furthermore, it is intuitively clear that a constructive proof of theorem II.4 will depend on the physical system, and perhaps on the concrete observable, and therefore such a proof is outside the scope of a general framework.

Although we cannot expect in a general setting to extract the physical meaning of c_2, the fact that c_2 can be supposed a PSO is of physical interest since the physical interpretation of a PSO could be easier than the one of an arbitrary observable. For instance, it could be related to a discrete symmetry or a set of discrete measurements.

Among all the possible sets which are a CSCO, the ones in which the Hamiltonian (or some bijective bounded function of it) is an element of the set are particularly notable. In fact, in this case all the elements of the set are constants of motion. For this reason, it could be desirable to start from the Hamiltonian (or some bijective bounded function of it) and to build with it a CSCO (such construction is guaranteed by this theorem).

Of course, given an observable c_1, we cannot expect the construction of a CSCO with it to be unique. For instance, the observable q_x can be completed with a PSO or with $\{q_y, q_z\}$. The non uniqueness is not a physical disadvantage and is widely used in the applications of elementary quantum mechanics.

II.6. Definition. Let $\mathfrak{C} = \{ c_i \}_{i \in I}$ be a compatible set of observables, for which exists an orthonormal basis of simultaneous eigenfunctions. Let us associate to each eigenfunction of the basis the set formed by the eigenvalues of all the elements of $\mathfrak{C}$ corresponding to the eigenfunction. If this association is one to one and if such bijection does not hold for any proper subset of $\mathfrak{C}$, we shall say that $\mathfrak{C}$ is a **CSCO in Dirac sense** (CSCOD).

Except for the explicit minimality required for $\mathfrak{C}$ and the fact that the elements of $\mathfrak{C}$ are bounded, this is the definition of CSCO introduced by Dirac (see, e.g., [1,7,9]).

Let us observe that each element of any CSCOD is a PSO and that the existence of a CSCOD in $\mathcal{A}$ is trivial.

II.7. Theorem. $\mathfrak{C} = \{c_1,...,c_n\}$ is a CSCOD iff $\mathfrak{C}$ is a CSCO whose elements are PSO.

Proof. (i) If $\mathfrak{C}$ is a CSCOD, each element of $\mathfrak{C}$ is a PSO, therefore $c_r = \sum_{j=1} \lambda_j^{(r)} p_j^{(r)}$, r=1,...,n. Then, to each set $(\lambda_i^{(1)},...,\lambda_j^{(r)},...,\lambda_k^{(n)})$, i,...,j,...,k = 0,1,2,.....; (taking $\lambda_0^{(r)} = 0$ if this is an eigenvalue of c_r) corresponds a unique (up to a phase) eigenfunction ϕ(i,...,j,...,k) with corresponding projector p(i,...,j,...,k). We have that $\sum_{\{i,..,j,..,k\}} p(i,...,j,...,k) = e$, and that the projectors p(i,...,j,...,k) belong to $\mathfrak{R}(\mathfrak{C})$, hence:

$$\psi = \sum_n 2^{-n} \phi_n , \tag{2.4}$$

where $\{\phi_n\}$ is a reordering of $\{\phi(i,...,j,...,k)\}$, is a cyclic vector for $\mathfrak{R}(\mathfrak{C})$. Then, $\mathfrak{R}(\mathfrak{C})$ is maximal abelian (see, e.g., [17], Corollary 2.9.4.).

(ii) Let $\mathfrak{C}$ be a CSCO, each element of $\mathfrak{C}$ being a PSO. Then, the set $\mathfrak{C}$ has obviously an orthonormal and complete set of simultaneous eigenfunctions.

Let us suppose that for some set of simultaneous eigenvalues $(\lambda_i^{(1)},...,\lambda_k^{(n)}) = \lambda$ corresponds several eigenfunctions $\phi^{(s)}$, s = 1,2,...; of the orthonormal set. Let us call $\mathcal{M}$ the closed vector subspace generated by $\{ \phi^{(s)} \}$ (dim $\mathcal{M} > 1$). $\mathcal{M}$ reduces $\mathfrak{C}$ and hence it reduces $\mathfrak{R}(\mathfrak{C})$. If $b \in \mathfrak{R}(\mathfrak{C})$, then $b = f(c_1,...,c_n)$, see, the functional calculus referred in Sec. 1, and

$$b\phi = f(\lambda)\ \phi ;\ \forall \phi \in \mathcal{M} ;\ \forall b \in \mathfrak{R}(\mathfrak{C}). \tag{2.5}$$

Let ψ be a cyclic vector of $\mathfrak{R}(\mathfrak{C})$ and ψ_1 its orthogonal projection on $\mathcal{M}$, and let $\phi_1 \in \mathcal{M}$, $\phi_1 \neq 0$, such that $(\phi_1 , \psi_1) = 0$ (such ϕ_1 exists since dim $\mathcal{M} > 1$).

Then by (2.5), $(\psi, \mathfrak{b}\phi_1) = f(\lambda)\ (\psi,\phi_1) = f(\lambda)\ (\psi_1, \phi_1) = 0$, that is $(\mathfrak{b}^* \psi, \phi_1) = 0\ \forall\ \mathfrak{b} \in \mathcal{R}(\mathfrak{C})$ and hence $\phi_1 = 0$ (since ψ is cyclic). We see, that we arrive at a contradiction if we suppose that for some λ corresponds several orthonormal eigenfunctions.

(iii) It remains to consider the minimality condition.

If $\mathfrak{C}$ is a CSCOD, any proper subset of $\mathfrak{C}$ will have for some set of simultaneous eigenvalues $\lambda \equiv (\lambda_i^{(1)},...,\lambda_k^{(n)})$ several eigenfunctions, which is incompatible with the existence of a cyclic vector [see, (ii)]. Hence no proper subset of $\mathfrak{C}$ will generate $\mathcal{R}(\mathfrak{C})$.

If $\mathfrak{C}$ is a CSCO whose elements are PSO, any proper subset of $\mathfrak{C}$ is not a CSCO, and hence, it will not have a cyclic vector. This implies that to some set of simultaneous eigenvalues λ, of any proper subset of $\mathfrak{C}$, it corresponds several eigenfunctions of the orthonormal set [since if this is not the case we can construct a cyclic vector such as (2.4)].

II.8. Note. This theorem has been proved by Jauch [2], for the finite dimensional case. It shows that definition II.1 is a natural generalization of the concept of CSCO introduced by Dirac [1]. As a matter of fact, if $\mathfrak{C}$ is a compatible set of observables, each one being a PSO, a characteristic of $\mathcal{R}(\mathfrak{C})$ is that all its elements are PSO. However, in general (if dim $\mathcal{H} = \infty$), for an arbitrary compatible set of observables, $\bar{\mathfrak{C}}$, $\mathcal{R}(\bar{\mathfrak{C}})$ has elements which are not PSO. Then, if dim $\mathcal{H} = \infty$, the set of all CSCOD is a proper subset of the set of all CSCO.

3. Irreducible sets

III.1. Definition. A set $\mathcal{N} \equiv \{\mathfrak{a}_\alpha\}_{\alpha\in I} \subset \mathcal{A}^s$ is called an **irreducible set of observables** (ISO), if $\mathcal{R}(\mathcal{N}) = \mathcal{A}$, and no proper subset of $\mathcal{N}$ generates $\mathcal{A}$ (minimality of the set $\mathcal{N}$).

III.2. Note. If $\mathfrak{a}$ belongs to any ISO, then $\mathfrak{a} \neq \gamma\, \mathfrak{e}$; $\gamma \in R$, since $\mathcal{A} \neq \{\mathfrak{o}\}$, $\mathcal{A} \neq \mathcal{R}(\mathfrak{e})$, and the stated minimality.

Every ISO has at least two elements. Moreover, in every ISO there exist at least two noncommuting elements. The above comes from the fact that $\mathcal{A}$ is not abelian.

Let $\mathcal{N} \subset \mathcal{A}^s$. Then $\mathcal{N}$ is an ISO iff $\mathcal{N}^c = \mathcal{R}(\mathfrak{e})$, and $\mathcal{N}$ minimal (this is, essentially, Schur's lemma). Also, if $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are two CSCO, then $\{\mathfrak{c}_1,\mathfrak{c}_2\}$ is an ISO iff $\mathcal{R}(\mathfrak{c}_1) \cap \mathcal{R}(\mathfrak{c}_2) = \mathcal{R}(\mathfrak{e})$.

There exist $\mathfrak{a}_1,\mathfrak{a}_2 \in \mathcal{A}^s$ such that $\mathcal{N} \equiv \{\mathfrak{a}_1,\mathfrak{a}_2\}$ is an ISO. This is the content of theorem 2.2. of [22].

III.3. Note. Let $\mathcal{N} \equiv \{\mathfrak{c}_1,\mathfrak{c}_2\} \subset \mathcal{A}^s$, with dim $\mathcal{H} = 2$. Then, it is trivial to show that $\mathcal{N}$ is an ISO iff $[\mathfrak{c}_1,\mathfrak{c}_2] \neq \mathfrak{o}$, and that automatically $\{\mathfrak{c}_1\}$ and $\{\mathfrak{c}_2\}$ are CSCO.

Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two CSCO's, with $[\mathfrak{c}_1,\mathfrak{c}_2] \neq \mathfrak{o}$. Then, if dim $\mathcal{H} \geq 3$, $\{\mathfrak{c}_1,\mathfrak{c}_2\}$ is not necessarily an ISO. Let us give a counterexample for $\mathcal{A} = \mathcal{L}\ (C^3)$:

$$c_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} ; \quad c_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \tag{3.1}$$

Let $\mathcal{N} = \{a_1, a_2\}$ be an ISO. Then, if dim $\mathcal{H} \geq 3$, each element of $\mathcal{N}$ is not necessarily a CSCO. Let us give a counterexample for $\mathcal{A} = \mathcal{L}(C^3)$:

$$a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad a_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \tag{3.2}$$

III.4. Theorem. Let $\{c_1\}$ be a CSCO. Then, there exists a CSCO, $\{c_2\}$, such that $\{c_1, c_2\}$ is an ISO.

Proof. Let us represet c_1 by a Jacobi matrix [20]

$$c_1 = \begin{bmatrix} \mu_1 & \lambda_1 & 0 & . & . & . & . \\ \bar{\lambda}_1 & \mu_2 & \lambda_2 & . & . & . & . \\ 0 & \bar{\lambda}_2 & \mu_3 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} , \tag{3.3}$$

where, μ_n is real and $\lambda_n \neq 0$, $n = 1,2,3.....$.

Let us define the matrix a by

$$a = \begin{bmatrix} \frac{\mu_1}{2} & \lambda_1 & 0 & . & . & . \\ 0 & \frac{\mu_2}{2} & \lambda_2 & . & . & . \\ 0 & 0 & \frac{\mu_3}{2} & . & . & . \\ . & . & . & . & . & . \end{bmatrix} ; \quad a^* = \begin{bmatrix} \frac{\mu_1}{2} & 0 & 0 & . & . & . \\ \bar{\lambda}_1 & \frac{\mu_2}{2} & 0 & . & . & . \\ 0 & \bar{\lambda}_2 & \frac{\mu_3}{2} & . & . & . \\ . & . & . & . & . & . \end{bmatrix} . \tag{3.4}$$

It is direct to verify, from (3.4), that the commutant of a and a^* is $\mathcal{R}(e)$. Then, $\mathcal{R}(a, a^*) = \mathcal{A}$.

We have that $c_1 = a + a^*$, and we can define $c_2 = i(a^* - a)$. Then, $\mathcal{R}(c_1,c_2) = \mathcal{A}$, i.e., $\{c_1,c_2\}$ is an ISO.

Finally, since c_2 is represented by a Jacobi matrix, we have that $\{c_2\}$ is a CSCO (see, [20], Theorem 7.14).

III.5. Discussion. A well known example that "evokes" theorem III.4, in the standard formulation of quantum mechanics [10], is the set $\mathcal{N} = \{p, q\}$ where p and q are the momentum and position operators for a spinless particle in one dimension (see, the solution of the canonical commutation relations given in matrix form in [25]). Also, because of the way composite quantum physical systems are formulated, it follows, from theorem III.4, that we can construct other ISO's by considering tensor products of Hilbert spaces.

From theorem III.4 it follows that there exists an ISO, $\mathcal{N} = \{c_1,c_2\}$, where each element of $\mathcal{N}$ is a CSCO. Furthermore each element of $\mathcal{N}$ can be chosen positive and of trace class (for example, a density operator). In fact, taking $\mu_n = 0, n=1,2,\ldots$, and $\sum_n |\lambda_n| < \infty$, we get the result. Let us observe that there exist many sets $\mathcal{N}$ of this type.

The generation of $\mathcal{A}$, wich contains many operators that are not PSO (if dim $\mathcal{H} = \infty$), by two PSO (even density operators), should be contrasted with the fact that the abelian algebra generated by a CSCO which is not a CSCOD, cannot be generated only in terms of PSO.

III.6. Theorem. Let $\mathcal{D} = \{d_\alpha\}_{\alpha\in I}$ be a set of observables. Then, there exists a countable subset of $\mathcal{D}$ which generates $\mathcal{R}(\mathcal{D})$.

Proof. If

$$d_\alpha = \int \lambda \, dp^{(\alpha)}(\lambda), \quad \alpha \in I, \ -\infty < \lambda < \infty, \tag{3.5}$$

is the spectral representation of d_α, we have that $\mathcal{R}(\mathcal{D}) = \mathcal{R}(\{p^{(\alpha)}(\lambda)\})$.

Let $\mathcal{B} = \{a \in \mathcal{R}(\mathcal{D}) \mid \|a\| \leq 1\}$ be the unit ball of $\mathcal{R}(\mathcal{D})$. We have that $p^{(\alpha)}(\lambda) \in \mathcal{B}$; $\forall\ \alpha \in I$, $-\infty < \lambda < \infty$. Since $\mathcal{H}$ is separable, the strong operator topology on $\mathcal{B}$ is separably metrizable (see, e.g., [22]); and hence, the set $\mathcal{E} = \{p^{(\alpha)}(\lambda) \mid \alpha \in I, \ -\infty < \lambda < \infty\}$ is also separably metrizable. Then, there exists a countable subset of $\mathcal{E}$ which is dense in $\mathcal{E}$ (endowed with the strong operator topology). Let us denote this countable subset by $\{p^{(n)}(i)\}$. Since $\mathcal{E} \subset \{p^{(n)}(i)\}^{cc}$ and $\mathcal{R}(\mathcal{E}) = \mathcal{R}(\mathcal{D})$, we have that $\{p^{(n)}(i)\}^{cc} = \mathcal{R}(\mathcal{D})$.

Since each projector of the countable set $\{p^{(n)}(i)\}$ is a spectral projection of some observable $d_\alpha \in \mathcal{D}$, we can extract a countable subset of $\mathcal{D}$ which generates $\mathcal{R}(\mathcal{D})$.

III.7. Corollary.

(i) Let $\{c_\alpha\}_{\alpha\in I}$ be a CSCO. Then, the set I is at most countable.

(ii) Let $\{a_\alpha\}_{\alpha\in I}$ be an ISO. Then the set I is at most countable.

Proof. The numerability of the set I follows from theorem III.6 and the minimality stated in the definitions of CSCO and ISO.

III.8. Discussion. From corollary III.7, we see that the extension of Jauch and Misra [4] results to the case of an uncountable set I, performed by Dormale and Gautrin [6], although interesting from the mathematical point of view, is unnecessary for the concept of CSCO (since a minimality assumption is physically important).

III.9. Note. In the present paper, the concepts of CSCO and ISO, and the obtained results, have been formulated in terms of bounded operators acting on a Hilbert space. However, it is straightforward that all definitions and results can be formulated in an abstract C^* algebraic setting (c.f. [11]).

Also, theorems II.4, II.7, III.4 and III.6 can be reobtained if the set of all self-adjoint operators (bounded and unbounded) is used as the set of all observables; as it is direct from our proofs and all the available literature [4-6,9,14,23]. The main technical point being that $\mathfrak{R}(a)$ is well - defined even for a self - adjoint unbounded operator a.

Acknowledgment

The authors are grateful to the referee for his suggestions that led to an improvement in the writing of the paper.

REFERENCES

[1] P.A.M. Dirac, **The Principles of Quantum Mechanics** (Oxford at the Clarendon Press 1970)
[2] J.M. Jauch, Helv. Phys. Acta **33**, 711 (1960)
[3] J.M. Jauch and B. Misra, Helv. Phys. Acta **34**, 699 (1960)
[4] J.M. Jauch and B. Misra, Helv. Phys. Acta **38**, 30 (1965)
[5] E. Prugovecki, Can. J. Phys. **47**, 1083 (1969)
[6] B.M. de Dormale and H. - F. Gautrin, J. Math. Phys. **16**, 2328 (1975)
[7] A. Messiah, **Quantum Mechanics** Vol. I (Amsterdam: North-Holland 1970)
[8] J. M. Jauch, **Foundations of Quantum Mechanics** (MA:Addison-Wesley Reading 1968)
[9] E. Prugovecki,**Quantum Mechanics in Hilbert Space** (New York: Academic Press 1981)
[10] T.F. Jordan, **Linear Operators for Quantum Mechanic** (New York:J. Wiley & Sons1969)
[11] G.G. Emch, **Algebraic Methods in Statistical Mechanics and Quantum Field Theory** (New York : Wiley - Interscience 1972)
[12] J. Dixmier, **Von Neumann Algebras** (Amsterdam :North-Holland 1981)
[13] M.A. Naimark, **Normed Algebras** (the Netherlands :Wolters - Noordhoff 1972)
[14] A. Plesner, **Spectral Theory of Linear Operators** Vol. II (New York : Frederick Ungar 1969)
[15] F. Riesz and B. SZ - Nagy, **Functional Analysis** (New York : Frederick Ungar 1978)
[16] J. Dixmier, **C* - Algebras** (Amsterdam : North - Holland 1977)
[17] S. Sakai, **C* - Algebras and W* - Algebras** (New York : Springer 1971)
[18] N. Bourbaki, **Intégration** (Paris : Hermann 1973)
[19] R. Douglas, **Banach Algebras Techniques in Operator Theory** (New York : Academic Press1972)
[20] M.H. Stone, **Linear Transformations in Hilbert Space** (Providence : Am. Math. Soc. 1932)

[21] F. Gantmacher, **The Theory of Matrices** Vol. I (New York : Chelsea 1977)
[22] T. Saito, **Lectures on Operator Algebras** ed A. Dold and B. Eckmann (Berlin : Springer Lecture Notes in Mathematics Vol. 247 1972) p. 436
[23] C.R. de Oliveira, J. Math. Phys. **31**, 2406 (1990)
[24] J. Von Neumann, Math. Ann. **102**, 370 (1929)
[25] M. Born and P. Jordan, Z. Phys. **34**, 858 (1925)