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Analysis of a Non-Resonant Maser Model

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Abstract. A system of N two-level molecules interacting by dipole coupling with one non-resonant mode of the radiation field is treated quantum mechanically. First the eigenvalue spectrum of the Hamiltonian is discussed and asymptotic expressions (for large, but finite N) for the difference of successive eigenvalues are given. The time behaviour of the mean photon number and its mean square variance is then derived for various initial states. The time evolution of these quantities strongly depends on the ratio of the coupling constant and the difference of the field energy and the transition energy of the molecules.

1. Introduction

For an understanding of the laser phenomenon, a fully quantum mechanical treatment of the laser-active molecules and the radiation field coupling them is necessary. Such a program is very ambitious. However, in order to understand the basic mechanism, it is sufficient to consider a model suggested by Dicke [1]: a) the laser-active molecules are described by two-level molecules, b) only one mode of the radiation field is of importance, c) the losses of the system are neglected. Assumption a) is justified, since in a pulsed laser only the levels of the laser transition are important, the pump levels are necessary only for the preparation of the initial state. The single-mode laser operation can today be obtained with almost any type of laser [2], and is important for many applications.

If we describe the interaction of the radiation field with the molecules by the dipole approximation and make the rotating wave approximation [3], we arrive at the Hamiltonian

$$H = \sum_{i=1}^{N} b_i^+ b_i + (1+\Delta) a^+ a + \sum_{i=1}^{N} (g_i a^+ b_i + \bar{g}_i a b_i^+). \tag{1.1}$$

Here the photon operators a, a^+ satisfy

$$[a, a^+] = 1, \tag{1.2}$$

and for the spin-flip operators b_i, b_i^+ of the molecules we have

$$[b_i, b_j] = [b_i, b_j^+] = 0, \quad i \neq j,$$

$$b_i b_i^+ + b_i^+ b_i = 1, \quad b_i^2 = 0. \tag{1.3}$$

Furthermore, considering a ring laser, which allows the selection of modes running in one direction [4], we have

$$g_i = g\, e^{i\alpha_i}, \quad g > 0, \quad \text{for all } i, \tag{1.4}$$

irrespective of the spatial distribution of the molecules in the optical resonator. In the case of a Pérot–Fabry interferometer, equation (1.4) holds if the molecules are placed at equal mode positions, or if all molecules are concentrated in a region whose dimensions are very small compared to the wavelength of the mode in consideration. It is evident that the transformation

$$b_i \to b_i\, e^{-i\alpha_i} \tag{1.5}$$

leaves equation (1.3) unchanged. Therefore we will study the Hamiltonian

$$H = \sum_{i=1}^{N} b_i^+ b_i + (1+\Delta)\, a^+ a + g \sum_{i=1}^{N} (a^+ b_i + a b_i^+). \tag{1.6}$$

In the case of dipole coupling, the coupling constant depends on the quantization volume V of the radiation field [11]

$$g = \frac{\text{const.}}{\sqrt{V}}. \tag{1.7}$$

We will consider large N with fixed N/V. Thus we have

$$g = \frac{1}{c\sqrt{N}}. \tag{1.7'}$$

In the resonant case $\Delta = 0$, this Hamiltonian has been treated by various authors. Its spectrum was investigated numerically by M. Tavis and F. Cummings [5], W. Mallory [6], D. Walls and R. Barakat [7] and G. Scharf [8]; asymptotic expressions (for large N) for the eigenvalues were given by G. Scharf [8]. The time evolution of the system was discussed by R. Bonifacio and G. Preparata [9] and G. Scharf [10] by entirely different methods.

In the present paper, we are interested in the non-resonant case $\Delta \neq 0$, and we will extend the methods of Refs. [8] and [10] to this more general situation. First we investigate the spectrum of the Hamiltonian (1.6) and compute asymptotic expressions (for large N) for the difference of successive eigenvalues (Section 2). In Sections 3 and 4 the time behaviour is studied for various initial states.

Let us consider some of the results. For the laser problem, the initial state with all N molecules in the upper level is very interesting. If at $t = 0, 2L$ additional photons are present, we will show that for large N the average photon number is given by

$$\langle n(t)\rangle = 2L + (2L+1)\,\frac{N}{P(\tau) + L + 1}, \tag{1.8}$$

where we have introduced

$$\tau = gt. \tag{1.9}$$

In equation (1.8) $P(\tau)$ denotes the function $\wp(\tau) - (E/3)$, where

$$E = N + L - \tfrac{1}{4}\Delta^2 c^2 N + 1, \tag{1.10}$$

and $\wp(\tau)$ is Weierstrass' elliptic function with the invariants

$$g_2 = \tfrac{4}{3} E^2 + 2\Delta^2 c^2 N(L+1) + 4L^2, \tag{1.11}$$

$$g_3 = -\tfrac{8}{27} E^3 + \tfrac{8}{3} EL^2 - \tfrac{2}{3}\Delta^2 c^2 NE(L+1) + N(2L+1) + \Delta^2 c^2 N(2L^2 + 2L + 1). \tag{1.12}$$

If only a few photons are present initially, $L \ll N$, the average photon number oscillates with a period

$$T = c\left(1 - \frac{\Delta^2 c^2}{4}\right)^{-1/2} \log N \tag{1.13}$$

if the laser is 'weakly detuned'

$$|\Delta| c < 2. \tag{1.14}$$

The maximal height of a pulse is found to be

$$\left\langle n\left(\frac{T}{2}\right)\right\rangle = N\left(1 - \frac{\Delta^2 c^2}{4}\right), \tag{1.15}$$

where we assumed $L \gg 1$. On the other hand, for a 'strongly detuned' laser

$$|\Delta| c > 2, \tag{1.16}$$

the period is given by

$$T = 2\pi\left(\Delta^2 - \frac{4}{c^2}\right)^{-1/2} \tag{1.17}$$

and the maximal height of a pulse is $O(1)$. We thus see that for increasing $|\Delta| c$, the height of the pulse decreases strongly, and is only $O(1)$ for $|\Delta| c = 2$; the laser action has then vanished. A similarly drastic change can be noticed considering the periods, equations (1.13) and (1.17). Such periodic behaviour of the physical quantities is true only for not too large times. In fact, we will show that in the weakly detuned case, the periodicity is destroyed after about $2L \log N$ periods, whereas in the strongly detuned case $O(N)$ oscillations are periodic.

We finally consider the limit $N \to \infty$. In the case of the weakly detuned laser, the function $P(\tau)$ tends to [25]

$$P(\tau) = E \sinh^{-2}(\sqrt{E}\tau), \tag{1.18}$$

and the average photon number is thus given by

$$\langle n(t)\rangle = 2L + (2L+1)\left(1 - \frac{\Delta^2 c^2}{4}\right)^{-1} \sinh^2\left(t\sqrt{\frac{1}{c^2} - \frac{\Delta^2}{4}}\right). \tag{1.19}$$

In this case the time evolution becomes aperiodic and the photon number increases to ∞. For the strongly detuned case we obtain, in a similar way,

$$\langle n(t)\rangle = 2L + (2L+1)\left(\frac{\Delta^2 c^2}{4} - 1\right)^{-1} \sin^2\left(t\sqrt{\frac{\Delta^2}{4} - \frac{1}{c^2}}\right), \tag{1.20}$$

which means that the photon number remains finite. Equations (1.19) and (1.20) agree with the corresponding results in the work of K. Hepp and E. Lieb [11].

2. The Eigenvalue Spectrum

As a first step in the analysis of the Hamiltonian (1.6) we introduce the operators

$$S_+ = \sum_{i=1}^{N} b_i^+, \quad S_- = \sum_{i=1}^{N} b_i, \quad S_3 = \tfrac{1}{2}[S_+ S_-]. \tag{2.1}$$

They obey commutation relations of angular momentum operators. Equation (1.6) now takes the form

$$H = S_3 + \tfrac{1}{2} N \mathbb{1} + (1+\Delta)\, a^+ a + g(a^+ S_- + a S_+). \tag{2.2}$$

This Hamiltonian commutes with the square of the total angular momentum $S^2 = S_3^2 + \frac{1}{2}(S_+ S_- + S_- S_+)$. Thus H does not connect states of different eigenvalues $s(s+1)$ of the square of the total angular momentum, and we need consider only subspaces corresponding to one fixed value s. In our analysis, we may assume $s = \frac{1}{2}N$; if $s < \frac{1}{2}N$ a reduction to the former case is possible[1]).

In order to reduce the problem to an algebraic form, we choose as a basis the products $|n,m\rangle$ of the eigenstates $|n\rangle$ and $|m\rangle$ of the photon number operator $a^+ a$ and the angular momentum S_3 respectively. In this basis we have

$$\begin{aligned} H|n,m\rangle &= \left(m + \frac{N}{2} + (1+\Delta)\, n\right)|n,m\rangle \\ &\quad + g\sqrt{n+1}\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right) - m(m-1)}\,|n+1, m-1\rangle \\ &\quad + g\sqrt{n}\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right) - m(m+1)}\,|n-1, m+1\rangle. \end{aligned} \tag{2.3}$$

We see from this equation, that the quantity

$$R = n + m + \frac{N}{2} \tag{2.4}$$

is preserved. Physically, this means that the total number of photons and excited molecules remains constant. Hence a further reduction of the problem is possible: we need consider only subspaces with one fixed value of R. Furthermore, without loss of generality, we may assume $R \geqslant N$ [10]. In this subspace we introduce as a basic variable the number of excited molecules

$$x = m + \frac{N}{2} = R - n, \quad x = 0, 1, \ldots, N. \tag{2.5}$$

In this representation the Hamiltonian (2.3) reads

$$\begin{aligned} H_{x'|x} &= R\delta_{x'|x} + \Delta(R-x)\,\delta_{x'|x} + g\sqrt{(R-x+1)\,(N-x+1)\,x}\,\delta_{x'|x-1} \\ &\quad + g\sqrt{(R-x)\,(N-x)\,(x+1)}\,\delta_{x'|x+1} = (R\mathbb{1} + M)_{x'|x}. \end{aligned} \tag{2.6}$$

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[1]) This is achieved by a redefinition of the quantities N and R (see below) entering in equation (2.6) [10].

The energy eigenvalues $R+\epsilon$ of the Hamiltonian H are determined by the algebraic eigenvalue problem

$$M\mathbf{e}=\epsilon\mathbf{e}, \quad \mathbf{e}=(e_0,e_1,\ldots,e_N). \tag{2.7}$$

Following the procedure in Ref. [8], we convert equation (2.7) into a difference equation

$$(\epsilon'-\Delta'(R-x))\,e'_x=(R-x+1)\,(N-x+1)\,e'_{x-1}+(x+1)\,e'_{x+1}, \tag{2.8}$$

where we have introduced

$$e'_x=e_x\binom{N}{x}^{1/2}(R-x)!^{-1/2}, \tag{2.9}$$

$$\epsilon'=\frac{\epsilon}{g}, \quad \Delta'=\frac{\Delta}{g}. \tag{2.10}$$

Equation (2.8) shows that in the resonant case $\Delta'=0$, the eigenvalues lie symmetrically around 0. For $\Delta'\neq 0$ this property does not hold any longer. Remembering the transformation (1.5), we see that it is sufficient to consider either sign of Δ'; this property will be used later in the analysis.

We now transform equation (2.8) into a differential equation using the characteristic function

$$f(z)=\sum_{x=0}^{N} e'_x z^{R-x}. \tag{2.11}$$

Thus we get

$$zf''+f'(N+1-R+\Delta' z-z^2)+f(Rz-\epsilon')=0. \tag{2.12}$$

In order to solve the eigenvalue problem, one would have to search for polynomial solutions of equation (2.12). It is more convenient, however, to perform another transformation. Defining $\tilde{y}(z)$ by

$$\tilde{y}(z)=f(z)\,z^{-l}\exp\left(-\tfrac{1}{4}z^2+\frac{\Delta'}{2}z\right), \tag{2.13}$$

where

$$2l=R-N-1, \tag{2.14}$$

we arrive at

$$-\tilde{y}''+\tilde{y}\left(\frac{z^2}{4}-z\frac{\Delta'}{2}+\frac{l(l+1)}{z^2}+\frac{\epsilon'-\Delta' l}{z}\right)=\tilde{y}\left(N+l+\tfrac{3}{2}-\frac{\Delta'^2}{4}\right). \tag{2.15}$$

This is just the Schrödinger equation for a particle moving in the potential

$$\tilde{V}(z,\epsilon')=\frac{z^2}{4}-z\frac{\Delta'}{2}+\frac{l(l+1)}{z^2}+\frac{\epsilon'-\Delta' l}{z} \tag{2.16}$$

at a fixed energy

$$E=N+l+\tfrac{3}{2}-\frac{\Delta'^2}{4}. \tag{2.17}$$

For z tending to 0, we have $\tilde{y}(z) \sim z^{l+1}$; for large z, $\tilde{y}(z)$ is proportional to $\exp -\frac{1}{4}z^2$. Thus the eigenfunctions $\tilde{y}(z)$ we are looking for are ordinary $L^2(0,\infty)$ eigenfunctions of the Schrödinger equation (2.15).

A discussion of the potential given by equation (2.16) shows that for sufficiently large positive ϵ' the minimum $\tilde{V}_{\min}$ of the potential is greater than the fixed energy (2.17), whereas for decreasing (negative) values of ϵ', the well becomes increasingly large and deep, hereby we assumed $l \geqslant 0$. Therefore the Schrödinger equation (2.15) gives an infinite sequence of possible values ϵ'. How can we pick the $N+1$ eigenvalues of the laser problem out of this infinite sequence? This question can be answered remembering the node-theorem [12], which states, that in a one-dimensional potential the eigenfunction of the nth bound state has $n-1$ nodes, if the bound states are ordered by increasing energy. On the other hand, we see from equations (2.11) and (2.13), that all $N+1$ eigenfunctions $\tilde{y}(z)$ we are looking for have not more than N nodes in the interval $(0,\infty)$. Since the potential decreases monotonously for decreasing ϵ', the energy eigenvalues of the laser Hamiltonian correspond to the $N+1$ largest values ϵ' which lead to L^2-solutions of the Schrödinger equation (2.15). Besides, all eigenvalues are non-degenerate [12].

Now we want to derive asymptotic expressions (for large, but finite N) for the difference in successive eigenvalues. For this purpose, we apply the WKB-method to the Schrödinger equation (2.15). As we will see in a moment, we have two positive classical turning points a_2 and a_1, $a_1 > a_2$. Therefore the quantization rule determining the possible eigenvalues ϵ' gives [13]

$$J_1 \equiv \int_{a_2}^{a_1} dz \sqrt{E - \frac{L^2}{z^2} - \frac{\tilde{\epsilon}'}{z} + z\frac{\Delta'}{2} - \frac{z^2}{4}} = (\tilde{n} + \tfrac{1}{2})\,\pi, \tag{2.18}$$

where we have introduced

$$\tilde{\epsilon}' = \epsilon' - \Delta' l, \tag{2.19}$$

and

$$L^2 = (l + \tfrac{1}{2})^2. \tag{2.20}$$

If we differentiate equation (2.18) with respect to $\tilde{\epsilon}'$, we arrive at

$$-\pi \frac{d\tilde{n}}{d\tilde{\epsilon}'} = J_2 \equiv \int_{a_2}^{a_1} dz(-z^4 + 2\Delta' z^3 + 4Ez^2 - 4\tilde{\epsilon}' z - 4L^2)^{-1/2}. \tag{2.21}$$

Let us discuss this equation first. The integral on the right-hand side may be reduced to the form $\mu K(m)$ [15], where $K(m)$ denotes the complete elliptic integral of the first kind. In order to perform this reduction, we first have to compute the roots of the biquadratic form

$$z^4 - 2\Delta' z^3 - 4Ez^2 + 4\tilde{\epsilon}' z + 4L^2 = 0. \tag{2.22}$$

The normal form of the associated cubic resolvent is [14]

$$y^3 + y\left(-\tfrac{1}{3}E^2 - \frac{\Delta'}{2}\tilde{\epsilon}' - L^2\right) + \tfrac{2}{27}E^3 - \tfrac{2}{3}EL^2 - \tfrac{1}{4}\tilde{\epsilon}'^2 + \tfrac{1}{6}\Delta' E\tilde{\epsilon}' - \tfrac{1}{4}\Delta'^2 L^2$$

$$\equiv y^3 + 3py + 2q = 0. \tag{2.23}$$

For the laser problem, the initial state with all molecules in the upper level, the 'fully excited' state is very interesting. We will consider this case first. This initial state lies in the subspace in consideration ($s = N/2$), and is described by $\mathbf{e}(t=0) = (0, 0, \ldots, 1)$. From equation (2.6) we get for the mean value of the energy

$$\langle H \rangle = R + \Delta(2l+1) = R + \epsilon_{\text{Laser}}, \tag{2.24}$$

and for its mean square variance

$$\langle \delta H^2 \rangle = \langle (H - \langle H \rangle)^2 \rangle = g^2 N 2(l+1). \tag{2.25}$$

We recall that in the dipole approximation the coupling constant g depends on N, equation (1.7')

$$g = \frac{1}{c\sqrt{N}}. \tag{2.26}$$

The photon number at $t = 0$ is equal to $2l + 1$. In our analysis we first assume that $l = O(1)$, i.e. initially only few photons are present; the case $l = O(N)$ will be treated in Section 4. Therefore we have

$$\epsilon_{\text{Laser}} = \Delta(2l+1) = O(1), \tag{2.27}$$

i.e.

$$\epsilon'_{\text{Laser}} = O(N^{1/2}), \tag{2.28}$$

and

$$\langle \delta H^2 \rangle = \frac{2}{c^2}(l+1) = O(1). \tag{2.29}$$

Equation (2.29) shows, that only the eigenstates with eigenvalues $\epsilon' = \epsilon'_{\text{Laser}} + \delta\epsilon'$, $\delta\epsilon' = O(N^{1/2})$ contribute strongly to our initial state.

The roots of the cubic resolvent (2.23) are found by standard methods [14]. As we have just seen, we may limit our consideration to such values of ϵ' which are $O(N^{1/2})$. In this case, all three roots of equation (2.23) are real, and for the two leading orders in N we find

$$y_1 = -\tfrac{2}{3}E - \tfrac{1}{2}\frac{\Delta' \tilde{\epsilon}'}{E} + O(N^{-1}),$$

$$y_{\substack{2\\3}} = \tfrac{1}{3}E + \tfrac{1}{4}\frac{\Delta' \tilde{\epsilon}'}{E} \pm \left[\frac{N}{E}\left(L^2 + \frac{1}{4}\frac{\tilde{\epsilon}'^2}{E}\right)\right]^{1/2} + O(N^{-1}). \tag{2.30}$$

Knowing the roots of equation (2.23), we can compute the roots of the biquadratic form (2.22) [14]

$$z_{\substack{1\\2}} = \Delta' \pm 2\left(E + \frac{\Delta'^2}{4}\right)^{1/2} - \frac{1}{2}\frac{\tilde{\epsilon}'}{E} \pm \frac{\tilde{\epsilon}' \Delta'}{4E\left(E + \frac{\Delta'^2}{4}\right)^{1/2}} + O(N^{-3/2}),$$

$$z_{\substack{3\\4}} = \frac{1}{2}\frac{\tilde{\epsilon}'}{E} \mp \left[L^2 + \frac{1}{4}\frac{\tilde{\epsilon}'^2}{E}\right]^{1/2} + O(N^{-3/2}). \tag{2.31}$$

Now it is convenient to distinguish the two cases $E > 0$ and $E < 0$. Since we are considering large values of N, we have $|\Delta|c < 2$ and $|\Delta|c > 2$ respectively [equation (2.17)].

a) $|\Delta|c < 2$. In this case the laser is weakly detuned. From equation (2.31) we have

$$z_1 > z_4 > 0 > z_1 > z_2, \tag{2.32}$$

where z_4 and z_1 are the two positive classical turning points. The integral (2.21) can therefore be written in the form

$$J_2 = \mu K(m) \tag{2.33}$$

where [15]

$$\mu \equiv 2(z_1 - z_3)^{-1/2}(z_4 - z_2)^{-1/2} = E^{-1/2}\left(1 + \frac{1}{8}\frac{\tilde{\epsilon}'\Delta'}{E^2} - \frac{1}{2E}\left(E + \frac{\Delta'^2}{4}\right)^{1/2}\left(\frac{L^2}{E} + \frac{1}{4}\frac{\tilde{\epsilon}'^2}{E^2}\right)^{1/2}\right) + O(N^{-5/2}),$$

$$m \equiv \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)} = 1 - \frac{2}{E}\left(E + \frac{\Delta'^2}{4}\right)^{1/2}\left(\frac{L^2}{E} + \frac{1}{4}\frac{\tilde{\epsilon}'^2}{E^2}\right)^{1/2} + O(N^{-2}),$$

and $K(m)$ is the complete elliptic integral of the first kind. Using for $K(m)$ the asymptotic expansion [16]

$$K(m) = \log 4(1-m)^{-1/2} + \frac{1-m}{4}(\log 4(1-m)^{-1/2} - 1) + \dots$$

we arrive at

$$-\frac{d\tilde{n}}{d\epsilon} = \frac{c}{2\pi}\left(\frac{N}{E}\right)^{1/2}\left[\log 8E - \tfrac{1}{2}\log\left(N\frac{L^2}{E} + N\frac{\tilde{\epsilon}'^2}{4E^2}\right)\right] + O\left(\frac{\log N}{N}\right). \tag{2.34}$$

Considering the leading term only, we have

$$-\frac{d\epsilon}{d\tilde{n}} = \frac{2\pi}{c}\sqrt{1 - \frac{\Delta^2 c^2}{4}}\,(\log N)^{-1} + O((\log N)^{-2}), \tag{2.35}$$

i.e. the eigenvalues in consideration are approximately equidistant:

$$\epsilon_{n-1} - \epsilon_n = \frac{2\pi}{c}\sqrt{1 - \frac{\Delta^2 c^2}{4}}\,(\log N)^{-1} + O((\log N)^{-2}). \tag{2.35'}$$

Therefore, over not too large time intervals (see below), the physically relevant quantities, e.g. the photon number, oscillate with a period

$$T = c\left(1 - \frac{\Delta^2 c^2}{4}\right)^{-1/2}\log N. \tag{2.36}$$

We see that the period T is strongly dependent on the ratio of the detuning-constant Δ and the coupling constant $g = 1/c\sqrt{N}$. In particular, assuming a fixed coupling constant $1/c$, the period is minimal in the resonant case.

In equation (2.36) we considered the leading term only. The following term, being $O(\log N)$ smaller, depends on $\tilde{\epsilon}'^2$; this accounts for a disturbance of the periodic behaviour. In order to get an estimate for the aperiodicity, we compare the periods T_1 and T_2 for $\epsilon^2 = \epsilon^2_{\text{Laser}}$ and $\epsilon^2 = \epsilon^2_{\text{Laser}} + \langle \delta H^2 \rangle$ respectively. From equation (2.34) we find that after

$$\frac{1}{2}\left|\frac{T_1}{T_1 - T_2}\right| = n_0 = 2L \log N = O(\log N) \tag{2.37}$$

periods, the two considered oscillations differ by half a period; this means that the time evolution of a physical quantity is periodic only for n periods, where $n \ll n_0$.

Now let us discuss the eigenvalue spectrum in more detail. By standard transformations [15] we can express the integral (2.18) as follows

$$\begin{aligned} J_1 = \mu\left(\frac{\Delta'}{4} z_3^2 + E z_3 - \tfrac{3}{2}\tilde{\epsilon}' - \frac{2L^2}{z_3}\right) K(m) + \mu(z_4 - z_3)\left(E + \frac{\Delta'}{2} z_3\right) \Pi(a, m) \\ + \mu \frac{2L^2}{z_3 z_4}(z_4 - z_3)\, \Pi\left(a \frac{z_3}{z_4}, m\right) + \frac{\mu}{4}\Delta'(z_3 - z_4)^2 \\ \times \int_0^{\Pi/2} d\phi (1 - a \sin^2\phi)^{-2} (1 - m \sin^2\phi)^{-1/2} \end{aligned} \tag{2.38}$$

where we have introduced

$$a = \frac{z_1 - z_4}{z_1 - z_3}, \tag{2.39}$$

and $\Pi(a,m)$ denotes the complete elliptic integral of the third kind. The remaining integral in equation (2.38) can be reduced to a sum of complete elliptic integrals of the first, second and third kind [17]:

$$\begin{aligned} \int_0^{\Pi/2} d\phi (1 - a \sin^2\phi)^{-2} (1 - m \sin^2\phi)^{-1/2} = [2(1-a)(m-a)]^{-1} \\ \times [(a - m) K(m) - a E(m) + (a^2 - 2am - 2a + 3m)\, \Pi(a, m)]. \end{aligned} \tag{2.40}$$

Expanding these elliptic integrals for large N, we arrive at [16, 18]

$$\begin{aligned} J_1 = \frac{\Delta c}{2} N \sqrt{1 - \frac{\Delta^2 c^2}{4}} + \pi N \left(1 - \frac{2}{\pi} \arcsin \sqrt{\frac{1}{2}\left(1 - \frac{\Delta c}{2}\right)}\right) \\ + \frac{1}{2}\left(1 - \frac{\Delta^2 c^2}{4}\right)^{-1} \left(L^2 + \frac{c^2}{4}(\epsilon - \Delta l)^2 \left(1 - \frac{\Delta^2 c^2}{4}\right)^{-1}\right)^{1/2} \log N + O(1). \end{aligned} \tag{2.41}$$

From equations (2.24) and (2.41) we may determine the leading order of the WKB quantum number $\tilde{n}$:

$$\tilde{n} = N\left(\pi + \frac{\Delta c}{2}\sqrt{1 - \frac{\Delta^2 c^2}{4}} - 2 \arcsin \sqrt{\frac{1}{2}\left(1 - \frac{\Delta c}{2}\right)}\right)\frac{1}{\pi} \tag{2.42}$$

This equation shows where in the spectrum the eigenvalues $\epsilon = O(1)$ are situated (see Fig. 1). In the resonant case $\Delta = 0$, the middle of the spectrum, $\tilde{n} = N/2$, is important,

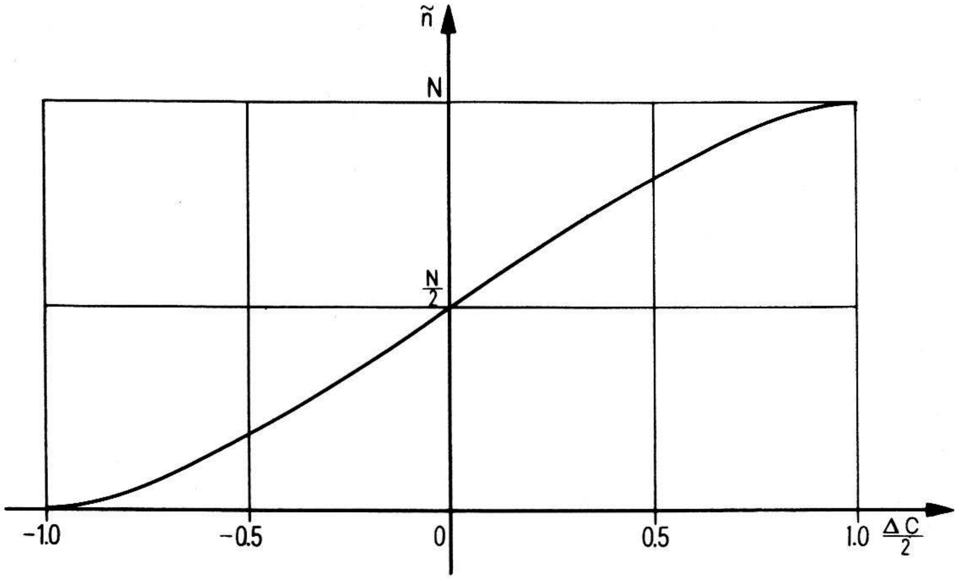

Figure 1
The WKB quantum number $\tilde{n}$ as a function of $\Delta c/2$.

whereas for increasing $|\Delta|$, $|\Delta|c < 2$, the eigenvalues move towards the boundary of the spectrum.

b) Now let us consider a strongly detuned laser: $|\Delta|c > 2$. We first assume $\Delta > 0$ or $\Delta' > 0$ respectively. The condition that the (fixed) energy E must be bigger than the minimum of the potential (2.16) yields

$$\epsilon^{(-)} < -|\Delta|l - |\Delta|(l+\tfrac{1}{2})\sqrt{1-\frac{4}{\Delta^2 c^2}}, \tag{2.43}$$

where the index of ϵ denotes the sign of Δ. Recalling equation (2.27), namely $\epsilon^{(-)}_{\text{Laser}} = -|\Delta|(2l+1)$, we see that $\epsilon^{(-)}_{\text{Laser}}$ lies near the upper boundary of the spectrum. Hence the eigenfunctions corresponding to the eigenvalues under consideration have only 0, 1, 2, ... nodes, which means that the use of the WKB approximation would be doubtful [13].

Therefore we now take $\Delta' > 0$ instead, without loss of generality. Equation (2.43) then reads

$$\epsilon^{(+)} > \Delta l + \Delta(l+\tfrac{1}{2})\sqrt{1-\frac{4}{\Delta^2 c^2}}, \tag{2.43'}$$

and the eigenfunctions have N, $N-1$, $N-2$, ... nodes. In this case, equation (2.31) gives

$$z_1 > z_2 > 0 > z_4 > z_3, \tag{2.44}$$

i.e. we again have two positive classical turning points.

The evaluation of the integral J_2 in equation (2.21) runs along similar lines as in the case a). We arrive at

$$\frac{d\tilde{n}}{d\epsilon} = -\frac{c}{2}\left(\frac{N}{|E|}\right)^{1/2}\left(1-\frac{3}{8}\frac{\tilde{\epsilon}'\Delta'}{E^2}\right) + O(N^{-2}). \tag{2.45}$$

Considering the leading term only, we have, remembering Δ', $\epsilon' = O(N^{1/2})$, equation (2.28),

$$\epsilon_{n-1} - \epsilon_n = \sqrt{\Delta^2 - \frac{4}{c^2}} + O(N^{-1}), \tag{2.46}$$

and the period

$$T = 2\pi \left(\Delta^2 - \frac{4}{c^2}\right)^{-1/2} \tag{2.47}$$

is independent of N, in contrast to the weakly detuned case a), where T is proportional to $\log N$, equation (2.36). The uncertainty of the period T can be estimated as above (2.37). We find a more stable periodic behaviour: now n oscillations are periodic, $n \ll n_0$, where n_0 is given by

$$n_0 = \tfrac{4}{3} N \frac{1}{|\Delta| c} \left(1 - \frac{\Delta^2 c^2}{4}\right)^2 (2L+1)^{-1/2} = O(N). \tag{2.48}$$

This result is entirely different from the case a), where only $O(\log N)$ oscillations are periodic.

3. The Time Evolution of the Photon Number and its Variance

In order to investigate the time-dependent problem, we return to equation (2.12), where we replace ϵ' by $(i/g)(\partial/\partial t)$:

$$i \frac{1}{g} \frac{\partial}{\partial t} f(z,t) = z f''(z,t) + f'(z,t)(N + 1 - R + \Delta' z - z^2) + R z f(z,t), \tag{3.1}$$

here we have introduced the time-dependent characteristic function

$$f(z,t) = \sum_{x=0}^{N} e'_x(t)\, z^{R-x}. \tag{3.2}$$

In this case, the transformation (2.13), which led to the Schrödinger equation (2.15), is not appropriate, because the operator $\partial/\partial t$ would enter in the Coulomb term. Instead we search for a transformation, yielding for $y(z,t)$ an equation of the Schrödinger type $i(\partial/\partial t) y = \mathscr{H} y$, where the operator $\mathscr{H}$ is hermitean. This is achieved by taking

$$f(z,t) = y(z,t)\, z^L \exp\left(\tfrac{1}{4} z^2 - \frac{\Delta'}{2} z\right), \tag{3.3}$$

where

$$L = \tfrac{1}{2}(R - N) = l + \tfrac{1}{2}. \tag{3.4}$$

We in fact arrive at

$$i \frac{\partial}{\partial \tau} y = -\frac{1}{2}\left(z \frac{\partial^2}{\partial z^2} y + \frac{\partial^2}{\partial z^2} z y\right) + y\left(\frac{z^3}{4} - \frac{\Delta'}{2} z^2 - z E + \frac{L^2}{z} + \Delta' l\right) = \mathscr{H} y, \tag{3.5}$$

where E is given by equation (2.17), and

$$\tau = -g t. \tag{3.6}$$

We remark that the transformation (3.3) differs from (2.13) by a factor $z^{1/2}$. Therefore the modification (2.20) of the centrifugal potential arises in a very natural way here. The term $\Delta' l$ in equation (3.5) may be dropped, because it only gives rise to a phase $\exp(-i\Delta' l\tau)$, which is of no interest.

The transformations made above define an embedding operator T

$$y(\tau) = T\,\mathbf{e}(t). \tag{3.7}$$

The Hamiltonian (3.5) may thus be written in the form

$$\mathscr{H} = TM'T^{-1}, \quad M' = -\frac{1}{g}M. \tag{3.8}$$

Here T^{-1} denotes the inverse operator on the $(N+1)$-dimensional subspace of L^2 corresponding to the laser Hilbert space [10], and M is the Hamiltonian given by equation (2.6). In the matrix representation (2.5), the expectation value of the photon number operator is given by

$$\langle n(t)\rangle = (\mathbf{e}(t), n\,\mathbf{e}(t)) = \sum_{x=0}^{N} (R-x)\,|er(t)|^2. \tag{3.9}$$

If we define the photon number operator $\mathscr{A}$ of the transformed problem by

$$\mathscr{A} = TnT^{-1}, \tag{3.10}$$

equation (3.9) may be written as follows

$$\langle n(t)\rangle = (\mathbf{e}(0), T^{-1}\,e^{i\mathscr{H}\tau}\,\mathscr{A}\,e^{-i\mathscr{H}\tau}y_0), \tag{3.11}$$

where (equation (3.6))

$$\tau = -gt.$$

From equation (3.10), namely $\mathscr{A}y_0 = Tn\mathbf{e}(0)$, the explicit form of the operator $\mathscr{A}$ is found to be

$$\mathscr{A} = \tfrac{1}{2}z^2 - \frac{\Delta'}{2}z + L + z\frac{d}{dz}. \tag{3.12}$$

Obviously, this operator is not self-adjoint and therefore does not allow a direct interpretation as a physical quantity.

As initial state we again take $\mathbf{e}(t=0) = (0,0,\ldots,0,1)$, i.e. all molecules are excited. The corresponding function $y_0(z)$ (3.3) is given by

$$y_0(z) = y(z,0) = (2L!)^{-1/2}\,z^L \exp\left(-\tfrac{1}{4}z^2 + \frac{\Delta'}{2}z\right). \tag{3.13}$$

In this case, equation (3.11) shows that we only have to compute the lowest power ($\sim z^l$) of the expression $e^{i\mathscr{H}\tau}\mathscr{A}\,e^{-i\mathscr{H}\tau}y_0$. This property will be extensively used in the following analysis.

We now revert to the Schrödinger equation (3.5). Its time-dependent solutions can be expressed by

$$y(z,\tau) = e^{-i\mathscr{H}t}y(z,0) = \int_0^\infty dz'\,K(z,z',\tau)\,y(z',0), \tag{3.14}$$

where we have introduced the propagator $K(z,z',\tau)$. We will calculate this propagator by the time-dependent WKB theory. It is shown that the propagator is given by [19]

$$K(q_1,q_2,\tau) = (2\pi i)^{-1/2} D^{1/2} \exp iS(q_1,q_2,\tau), \tag{3.15}$$

where $S(q_1,q_2,\tau)$ is the action

$$S(q_1,q_2,\tau) = \int_0^\tau \mathscr{L}\, d\tau' \tag{3.16}$$

taken along the classical path from q_2 to q_1, and

$$D = -\frac{\partial^2 S(q_1,q_2,\tau)}{\partial q_1\, \partial q_2}. \tag{3.17}$$

The propagator (3.15) satisfies the Schrödinger equation (3.5) with an additional 'false term' [19]

$$\frac{1}{\sqrt{D}}\frac{\partial}{\partial q_2}\left(2q_2\frac{\partial\sqrt{D}}{\partial q_2}\right). \tag{3.18}$$

Therefore the approximation is justified if this term can be neglected in comparison with the potential $V(q_2)$, equation (3.5). We will check the consistency of this procedure once we have found the explicit form of the propagator given by equation (3.15).

As a first step we compute the action $S(q_1,q_2,\tau)$. The classical Hamiltonian function $\mathscr{H}(p,q)$ corresponding to the Hamiltonian (3.5) is

$$\mathscr{H}(p,q) = qp^2 + V(q) \tag{3.19}$$

where

$$V(q) = \tfrac{1}{4}q^3 - \frac{\Delta'}{2}q^2 - Eq + \frac{L^2}{q}. \tag{3.20}$$

Because the function $\mathscr{H}(p,q)$ does not depend on the time explicitly, we have

$$S(q_1,q_2,\tau) = \int_{q_2}^{q_1} dq \sqrt{\frac{1}{q}(\mathscr{E} - V(q))} - \mathscr{E}\tau$$

$$= S_0(q_1,q_2,\tau) - \mathscr{E}\tau, \tag{3.21}$$

where the energy $\mathscr{E}$ is determined by the equation

$$\tau = \frac{\partial S_0}{\partial \mathscr{E}}. \tag{3.22}$$

In explicit form, equation (3.22) reads

$$\tau = \int_{q_2}^{q_1} dq(-q^4 + 2\Delta' q^3 + 4Eq^2 + 4\mathscr{E}q - 4L^2)^{-1/2} \equiv \int_{q_2}^{q_1} dq(g(q))^{-1/2}. \tag{3.22'}$$

This integral, which is very similar to the previous equation (2.21), can be inverted using the formula [20]

$$\wp(\tau) = \frac{\sqrt{g(q_1)\, g(q_2)} + g(q_2)}{2(q_1 - q_2)^2} + \frac{g'(q_2)}{4(q_1 - q_2)} + \frac{g''(q_2)}{24}, \tag{3.23}$$

where $\wp(\tau)$ denotes Weierstrass' $\wp$-function with the invariants

$$g_2 = \tfrac{4}{3}E^2 - 2\Delta'\mathscr{E} + 4L^2, \tag{3.24}$$

$$g_3 = -\tfrac{8}{27}E^3 + \tfrac{2}{3}\Delta' E\mathscr{E} + \tfrac{8}{3}EL^2 + \mathscr{E}^2 + \Delta'^2 L^2. \tag{3.25}$$

We are thus led to a transcendental equation for the energy $\mathscr{E}$

$$\mathscr{E} = \left(P - \frac{q_1 q_2}{2}\right)(q_1 + q_2) + \frac{\Delta'}{2} q_1 q_2 - [(q_1+q_2)^2 - 2\Delta'(q_1+q_2)$$
$$- 4E - 4P]^{1/2}\left[-Pq_1 q_2 + \frac{q_1^2 q_2^2}{4} + L^2\right]^{1/2}, \tag{3.26}$$

where we have introduced

$$P(\tau) = \wp(\tau) - \frac{E}{3}. \tag{3.27}$$

One might try to compute the action (3.21) in a similar way, however, as we shall see, this is not necessary.

Let us consider equation (3.11) again. As we have already pointed out, the average photon number is determined by the coefficient of the lowest power $\sim q_1^L$ of the expression $e^{i\mathscr{H}\tau}\mathscr{A}\, e^{-i\mathscr{H}\tau} y_0$:

$$\langle n(t)\rangle = \lim_{q_1\to 0} q_1^{-L}\int_0^\infty dq_2 \int_0^\infty dq_3\, K(q_1, q_2, -\tau)$$
$$\times\left[\tfrac{1}{2}q_2^2 - \frac{\Delta'}{2}q_2 + L + q_2\frac{\partial}{\partial q_2}\right] K(q_2, q_3, \tau)\, y_0(q_3)\sqrt{(2L)!}. \tag{3.28}$$

After an integration by parts, we have

$$\langle n(t)\rangle = \lim_{q_1\to 0} q_1^{-L}\int_0^\infty dq_2 \int_0^\infty dq_3 \left[\tfrac{1}{2}q_2^2 - \frac{\Delta'}{2}q_2 + L - 1 - q_2\frac{\partial}{\partial q_2}\right]$$
$$\times K(q_1, q_2, -\tau)\, K(q_2, q_3, \tau)\, y_0(q_3)\sqrt{(2L)!}, \tag{3.29}$$

where the differentiation operates on the first propagator. Because we need consider this propagator only for $q_1 \to 0$, the analysis will now be considerably simpler.

In order to evaluate expression (3.29), we differentiate the propagator $K(q_1, q_2, \tau)$ (3.15)

$$-q_2\frac{\partial}{\partial q_2}K(q_1, q_2, -\tau) = \left(iq_2\left(-\tfrac{1}{4}q_2^2 + \frac{\Delta'}{2}q_2 + E + \frac{\mathscr{E}}{q_2} - \frac{L^2}{q_2^2}\right)^{1/2}\right.$$
$$\left. - q_2\frac{1}{2D}\frac{\partial D}{\partial q_2}\right) K(q_1, q_2, -\tau). \tag{3.30}$$

In the limit $q_1 \to 0$, we get for the bracket in equation (3.30), using equation (3.23),

$$iq_2\left(-\tfrac{1}{4}q_2^2+\frac{\Delta'}{2}\,q_2+E+\frac{\mathscr{E}_0}{q_2}-\frac{L^2}{q_2^2}\right)^{1/2}-q_2\frac{1}{2D_0}\frac{\partial D_0}{\partial q_2}$$

$$=iq_2\left(P+E-\tfrac{1}{4}q_2^2+\frac{\Delta'}{2}q_2\right)^{1/2}-L-q_2\frac{1}{2D_0}\frac{\partial D_0}{\partial q_2}, \qquad (3.31)$$

where we have introduced (equation (3.26))

$$\mathscr{E}_0=\lim_{q_1\to 0}\mathscr{E}=Pq_2+2iL\left(P+E-\tfrac{1}{4}q_2^2+\frac{\Delta'}{2}q_2\right)^{1/2}. \qquad (3.32)$$

From equation (3.21) we find

$$D_0=\lim_{q_1\to 0}D=\frac{1}{2iL}\frac{\partial\mathscr{E}_0}{\partial q_2}. \qquad (3.33)$$

Now it is very convenient to introduce the energy $\mathscr{E}_0$ as a variable instead of the coordinate q_2. This may be done using a formula resulting from equation (3.22′) [20]

$$q_2=q_1+\frac{\sqrt{g(q_1)}\,\wp'+\tfrac{1}{2}g'(q_1)\left[\wp-\tfrac{1}{24}g''(q_1)\right]+\tfrac{1}{24}g(q_1)\,g'''(q_1)}{2\left[\wp-\tfrac{1}{24}g''(q_1)\right]^2-\tfrac{1}{48}g(q_1)\,g^{\mathrm{IV}}(q_1)}, \qquad (3.34)$$

which in our case yields ($q_1 \to 0$)

$$q_2=\frac{1}{P^2-L^2}(iLP'+\mathscr{E}_0P-L^2\Delta'). \qquad (3.35)$$

Here P' denotes the function P differentiated with respect to τ [21]:

$$P'=\wp'=(4\wp^3-g_2\,\wp-g_3)^{1/2}$$

$$=(4(P^2-L^2)\,(E+P)+2\Delta'\,\mathscr{E}_0P-\mathscr{E}_0^2-\Delta'^2L^2)^{1/2}. \qquad (3.36)$$

Recalling equation (3.31), the bracket (corresponding to the photon number operator) in equation (3.29) may be written as

$$-1-\frac{1}{2}\frac{1}{P+L}\frac{1}{P^2-L^2}\left[-4L(P^2-L^2)\,(E+P)-\Delta'^2L^2(P-L)\right.$$

$$+i\Delta'LP'(P-L)+\mathscr{E}_0(iP'(P-L)+\Delta'(P-L)^2)$$

$$\left.-\mathscr{E}_0^2(P-L)\right]-q_2\frac{1}{2D_0}\frac{\partial D_0}{\partial q_2}. \qquad (3.37)$$

In this equation, the energy $\mathscr{E}_0$ appears explicitly as well as in the invariants g_2 and g_3, equations (3.24) and (3.25). We now make the following approximation: in the invariants of the elliptic functions P, the energy $\mathscr{E}_0$ shall be replaced by the quantum mechanical expectation value

$$\bar{\mathscr{E}}=\langle\mathscr{H}\rangle=-\epsilon'_{\mathrm{Laser}}+\Delta'l=-\Delta'(L+\tfrac{1}{2}) \qquad (3.38)$$

(the negative sign of $\epsilon'_{\mathrm{Laser}}$ is due to definition (3.6)), and analogously for

$$\bar{\mathscr{E}}^2=\langle\mathscr{H}^2\rangle=N(2L+1)+\Delta'^2(L+\tfrac{1}{2})^2. \qquad (3.39)$$

This means, that in the trajectories of the classical particle described by the Hamiltonian function (3.19), we take the energy equal to the quantum mechanical expectation value [10]. The photon number will thus be a strictly periodic function; we recall, that in Section 2, we have given an estimate of how many periods such an approximation is valid.

We can easily verify that the real period 2ω of the elliptic function $P(\tau)$ coincides in fact with the period T found in Section 2. For this purpose we compute the constants e_1, e_2, e_3 of Weierstrass' $\wp$-function [21], i.e. the roots of the equation

$$4\xi^3 - g_2\xi - g_3 = 0. \tag{3.40}$$

This equation, however, is equivalent to the former cubic resolvent (2.23), where the substitutions $\tilde{\epsilon}' \to -\bar{\mathscr{E}}$, $\tilde{\epsilon}'^2 \to \bar{\mathscr{E}}^2$ have been made. Therefore we have $e_i = y_i$, $i = 1, 2, 3$; and for the real period, being equal to [22]

$$2\omega = 2(e_1 - e_3)^{-1/2} K(m), \quad m = (e_2 - e_3)(e_1 - e_3)^{-1}, \tag{3.41}$$

where $e_1 > e_2 > e_3$, we get the former result.

We now want to discuss briefly the magnitude of the function $P(\tau) = \wp(\tau) - (E/3)$ [23]. In the interval $(0, 2\omega)$, we have the symmetry relation $\wp(\omega - \tau) = \wp(\omega + \tau)$, furthermore Weierstrass' $\wp$-function reaches its minimum $e_1 > 0$ at $\tau = \omega$. For $\tau \to +O$, $\wp(\tau)$ tends to $+\infty$, being monotonously decreasing in the interval $(0, \omega)$. In the case $E > 0$, we get, by exactly the same procedure as in the resonant case [10], the following estimate for $P(\tau)$

$$P(\tau) = O(N^\alpha), \quad \alpha = \frac{\omega - \tau}{\omega}, \quad 0 < \tau \leqslant \omega. \tag{3.42}$$

In the case $E < 0$, Weierstrass' $\wp$-function has the expansion [25]

$$\wp(\tau) = -\frac{|E|}{3} + 2|E|\left(1 - \cos\left(t\sqrt{\Delta^2 - \frac{4}{c^2}}\right)\right)^{-1} + O(1), \tag{3.43}$$

i.e.

$$P(\tau) = O(N), \quad 0 < \tau \leqslant \omega. \tag{3.42'}$$

Let us revert to equation (3.29), where we insert equation (3.37). We first compute D_0 (equation (3.33)) by implicit differentiation of equation (3.26). A calculation analogous to Ref. [10], Appendix II, shows, that with the notation of equation (3.42)

$$D_0 = \frac{1}{2iL} P(1 + O(N^{-\alpha})), \quad 0 \leqslant \alpha \leqslant 1. \tag{3.44}$$

The terms $O(N^{-\alpha})$ originate from the differentiation of the invariants, in the resonant case, these contributions are only $O(N^{-2\alpha})$. In order to evaluate the integral (3.29), we have to consider the integral

$$\lim_{q_1 \to 0} \frac{1}{q_1^L} \int_0^\infty dq_2 \int_0^\infty dq_3\, \mathscr{E}_0 K(q_1, q_2, -\tau) K(q_2, q_3, \tau) y_0(q_3) \sqrt{(2L)!}. \tag{3.45}$$

Recalling the definition (3.15), we have

$$\mathscr{E} K(q_1, q_2, -\tau) = -i \frac{\partial}{\partial\tau} K(q_1, q_2, -\tau) + \frac{i}{2} \frac{1}{D} \frac{\partial D}{\partial\tau} K(q_1, q_2, -\tau). \tag{3.46}$$

Inserting this representation in the integral (3.45), we get from the first term: $\bar{\mathscr{E}} = -\Delta'(L+\frac{1}{2})$, equation (3.38). By means of equations (3.36) and (3.44), we find for the leading term of the contribution of the second term: $-i\sqrt{E+P}$, the other terms being $O(N^{\alpha})$ smaller. The integral (3.45) is thus equal to

$$-\Delta'(L+\tfrac{1}{2}) - i\sqrt{E+P} + O(N^{1/2-\alpha}), \quad 0<\alpha\leqslant 1. \tag{3.45'}$$

By the same procedure, we find

$$\lim_{q_1\to 0}\frac{1}{q_1^L}\int_0^\infty dq_2\int_0^\infty dq_3\mathscr{E}_0^2 K(q_1,q_2,-\tau)\,K(q_2,q_3,\tau)\,y_0(q_3)\,\sqrt{(2L)!}$$
$$= -2(E+P)\,(L+1) + 4i\Delta'(L+\tfrac{1}{2})\,\sqrt{E+P} + \Delta'^2(L+\tfrac{1}{2})(L+1)$$
$$+\,O(N^{1-\alpha}), \quad 0<\alpha\leqslant 1. \tag{3.47}$$

In a similar way, we see that the contribution of the last term in equation (3.37) is $O(N^{-2\alpha})$ for $0<\alpha\leqslant 1$. Furthermore, the order of magnitude of the 'false term' (3.18) is found to be $O(N^{-1/2-2\alpha})$, $0<\alpha\leqslant 1$, which is small compared to the potential. This result checks the consistency of the procedure.

Summing up, in equation (3.27) only the terms independent of $\bar{\mathscr{E}}_0$ and $\sim\bar{\mathscr{E}}_0$ contribute to the leading term of the average photon number, the other terms being at least $O(N^{\alpha})$ smaller. We thus have

$$\langle n(t)\rangle = -1 + (2L+1)\,\frac{P+E+\dfrac{\Delta'^2}{4}}{P} = -1 + (2L+1)\,\frac{P+N}{P}, \tag{3.48}$$

where $P = O(N^{\alpha})$, $\alpha > 0$; i.e. the average photon number is an even elliptic function of order 2. We point out the remarkable fact, that the detuning constant Δ appears only in the elliptic function $P(\tau)$. Hence, the result is almost the same as in the resonant case.

The result (3.48) represents the leading term of a general even elliptic function of order 2

$$-1 + (2L+1)\,\frac{P+N+\alpha}{P+\beta}, \tag{3.49}$$

where $\alpha, \beta = O(1)$. In order to determine the quantities α, β, we expand the function (3.49) in a power series, using for the elliptic function $P(\tau)$ the expansion [24]

$$P(\tau) = \frac{1}{\tau^2} - \frac{E}{3} + \tfrac{1}{20}\,g_2\,\tau^2 + \tfrac{1}{28}\,g_3\,\tau^4 + \dots. \tag{3.50}$$

We now compare this result with the exact time evolution for short times given in the Appendix. If we successively identify the two leading orders of the coefficients, we find that the corresponding terms $\sim\tau^0$, $\sim\tau^2$, $\sim\tau^4$ are correct, if we take $\alpha = \beta = L+1$. Comparing the terms $\sim\tau^6$ and $\sim\tau^8$, we see that for $\bar{\mathscr{E}}$ and $\bar{\mathscr{E}}^2$ we have to take

$$\bar{\mathscr{E}} = -\Delta'(L+1), \tag{3.51}$$

$$\bar{\mathscr{E}}^2 = N(2L+1) + \Delta'^2(L+1)^2, \tag{3.52}$$

which means that a slight modification $L \to L + \frac{1}{2}$ of our 'trial' values (3.38) and (3.39) is necessary. Then

$$\langle n(t)\rangle = -1 + (2L+1)\,\frac{P+N+L+1}{P+L+1} \tag{3.53}$$

is the correct elliptic function of order 2 for the average photon number. We have shown this result for $P \sim N^{\alpha}$, $\alpha > 0$, i.e. except for $\tau = \omega$ in the case $E > 0$. By continuity in τ, we may now include $\tau = \omega$ as well.

By the same procedure, the expectation value of the quantity $n^2(\tau)$ can be computed. The leading term is found to be an even elliptic function of order 4

$$\langle n^2(t)\rangle = 1 + (2L+1)\,\frac{P+N}{P^2}\,(P(2L-1) + 2N(L+1)). \tag{3.54}$$

Comparing this result with the expansion given in the Appendix, we see that

$$\langle n^2(t)\rangle = 1 - 3(2L+1)\,\frac{P+N+L+1}{P+L+1} + (2L+1)\,(2L+2)\,\frac{(P+N+L+1)\,(P+N+L+2)}{(P+L+1)(P+L+2)} \tag{3.55}$$

gives the two leading orders correctly up to τ^8, if we again make the substitutions (3.51) and (3.52). From equations (3.53) and (3.55) we finally obtain for the mean square variance of the photon number

$$\langle \sigma^2(t)\rangle = \langle n^2(t)\rangle - \langle n(t)\rangle^2 = (2L+1)\,N\,\frac{P-L}{(P+L+1)^2}\,\frac{P+N+L+1}{P+L+2}. \tag{3.56}$$

Here again the constant Δ appears in the function P only.

4. Extension to Other Initial States and Discussion

In the two preceding sections, we considered the initial state with all molecules excited and few additional photons ($L = O(1)$). We now investigate the time evolution in the *superradiant* case, i.e. the number $2L$ of photons being present at $t = 0$ is $O(N)$. The procedure is the same as in the case $L = O(1)$. Thus, for the leading order of the average photon number we get

$$\langle n(t)\rangle = -1 + 2L\,\frac{P+N+L}{P+L}. \tag{4.1}$$

Since this result agrees with equation (3.53) up to corrections $O(1)$ and equation (3.53) is correctly matched to the exact time evolution for short times, the former result can be immediately taken over:

$$\langle n(t)\rangle = -1 + (2L+1)\,\frac{P+N+L+1}{P+L+1}. \tag{4.2}$$

In the invariants g_2 and g_3, we have to insert again $\bar{\mathscr{E}}$ and $\bar{\mathscr{E}}^2$ given by equations (3.51) and (3.52). For the quantity $\langle n^2(t)\rangle$ and the mean square variance we also get the former results (3.55) and (3.56) respectively.

In the superradiant case $L = O(N)$, a general discussion of equation (4.1) is rather cumbersome; we therefore confine ourselves to a representative example. Choosing

$$2L = N\left(1 - \frac{\Delta^2 c^2}{4}\right) + 2l_1, \quad l_1 = O(1), \quad \frac{\Delta^2 c^2}{4} < 1, \tag{4.3}$$

we find from equations (3.40) and (3.41) for the leading term of the real period

$$T = 2\omega = 2\,\frac{K(m = \frac{1}{2})}{\sqrt{2N}\,\sqrt[4]{1 - \frac{\Delta^2 c^2}{4}}} = 2\,\frac{1{,}854}{\sqrt{2N}\,\sqrt[4]{1 - \frac{\Delta^2 c^2}{4}}}. \tag{4.4}$$

In order to get an estimate for the time interval, over which we expect a periodic behaviour, we compare the periods T_1, T_2 for $\epsilon_1 = \epsilon_{\text{Laser}}$ and $\epsilon_2 = \epsilon_{\text{Laser}} + \sqrt{\langle\delta H^2\rangle} = \epsilon_{\text{Laser}} + \sqrt{2LN}$. It results, that after about n_0 periods, where

$$n_0 = \frac{1}{2}\left|\frac{T_1}{T_1 - T_2}\right| = 2\sqrt{N}\,\sqrt{\frac{4}{\Delta^2 c^2} - 1} = O(N^{1/2}), \quad \Delta \neq 0, \tag{4.5}$$

the time evolution has become aperiodic. Whereas in the resonant case $\Delta = 0$ [10], we have $n_0 = O(N)$, equation (4.5) shows, that for $\Delta \neq 0$ we have periodicity only over $n = O(N^{1/2})$, $n \ll n_0$, periods; in this sense the resonant case is unstable.

In Section 2, we found (equation (2.37)) $n_0 = 2L\log N$, if there are only few additional photons at $t = 0$; this means that the periodicity is disturbed much earlier than in the superradiant case. Furthermore, comparing equations (2.36) and (4.4) for the period T, we see that in the case $L = O(1)$ T is depending much stronger on $\frac{1}{4}\Delta^2 c^2$ than for $L = O(N)$. The relative change of the period as a function of $|\Delta| c$ is shown in Figure 2.

Let us now discuss the time evolution of the average photon number given by equation (4.2). We first assume $|\Delta| c < 2$. In Ref. [10], graphic representations are given for the resonant case. In our case $\Delta \neq 0$, the general shape of the pulsations is the same, however, as we have just seen, their period is a function of Δ. Furthermore, the height of the pulses also depends strongly on Δ. In fact, assuming $L \ll N$, we get from equation (2.30)

$$\begin{aligned} P(\omega) &= e_1 - \frac{E}{3} \\ &= \frac{(L+1)\,\Delta^2 c^2}{\left(1 - \frac{\Delta^2 c^2}{4}\right)4} + \left[\frac{L^2}{1 - \frac{\Delta^2 c^2}{4}} + \frac{2L + 1 + \Delta^2 c^2 (L+1)^2}{4\left(1 - \frac{\Delta^2 c^2}{4}\right)^2}\right]^{1/2}. \end{aligned} \tag{4.6}$$

If $L \gg 1$, equation (4.6) takes the form

$$P(\omega) = L\left(1 + \frac{\Delta^2 c^2}{4}\right)\left(1 - \frac{\Delta^2 c^2}{4}\right)^{-1}, \tag{4.7}$$

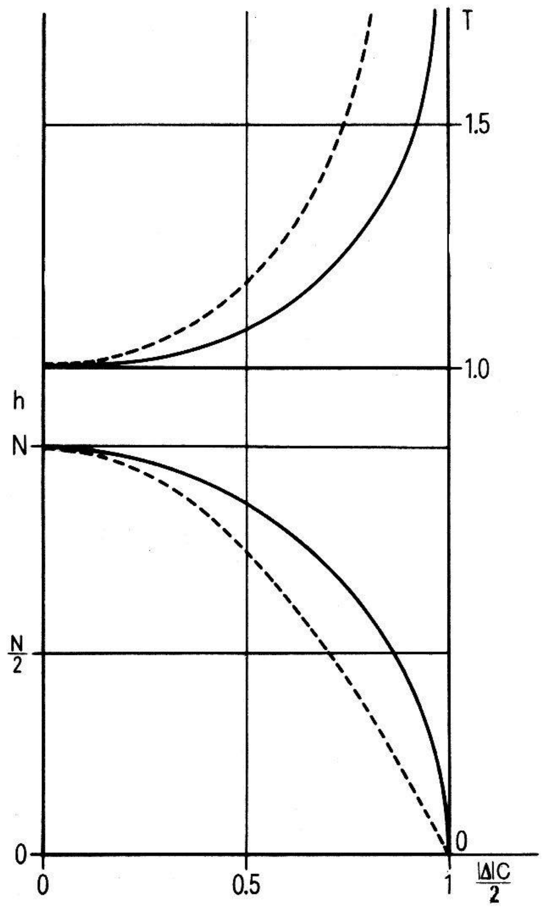

Figure 2
Maximal height h and period T of the pulsations for the fully excited initial state, ---- $L = O(1)$, —— superradiant.

and for the maximal height of the pulse we thus obtain

$$\langle n(\omega)\rangle = N\left(1 - \frac{\Delta^2 c^2}{4}\right). \tag{4.8}$$

In Figure 2 we have depicted the quantity $\langle n(\omega)\rangle - \langle n(0)\rangle \overset{\text{def}}{=} h$. We remark, that this quantity tends to $O(1)$ for $|\Delta| c \to 2$, i.e. the laser action vanishes. The analogous procedure in the superradiant case yields

$$P(\omega) = N\left[\left(1 - \frac{\Delta^2 c^2}{4}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{\Delta^2 c^2}{4}\right)\right], \tag{4.9}$$

and therefore

$$\langle n(\omega)\rangle - \langle n(0)\rangle = N\left(1 - \frac{\Delta^2 c^2}{4}\right)^{1/2}, \tag{4.10}$$

see Figure 2.

As in the resonant case, the mean square variance of the photon number is entirely different for $L = O(1)$ and $L = O(N)$ respectively. If $L = O(1)$, $\langle\sigma^2(t)\rangle$ is proportional to N^2, whereas in the superradiant case, it is $O(N)$. The general shape of the curves for $\langle\sigma^2(t)\rangle$ is the same as for $\Delta = 0$ [10]. However, the property that $\sigma(\omega) = O(1)$ for

$L = E/3$, i.e. that at $\tau = \omega$ the photon number is almost sharp, is true only for $\Delta = 0$; in fact, from equations (3.56) and (4.9), we get

$$\langle \sigma^2(\omega) \rangle = N \frac{\Delta^2 c^2}{4} + O(1). \tag{4.11}$$

Let us consider briefly a strongly detuned laser, $|\Delta| c > 2$, with $L = O(1)$. Weierstrass' $\wp$-function may then be written

$$\wp(\tau) = -\frac{|E|}{3} + 2|E| \left(1 - \cos\left(t\left(\Delta^2 - \frac{4}{c^2}\right)^{1/2}\right)\right)^{-1} + O(1)$$

(equation (3.43)), and for the leading term of the average photon number we obtain

$$\langle n(t) \rangle = 2L + \frac{2L+1}{2} \left(\frac{\Delta^2 c^2}{4} - 1\right)^{-1} \left(1 - \cos\left(t\left(\Delta^2 - \frac{4}{c^2}\right)^{1/2}\right)\right) + O(N^{-1}). \tag{4.12}$$

This means that the height of the pulses is only $O(1)$. Summing up, we see that for $L = O(1)$, laser action is possible only if $|\Delta| c < 2$.

Finally we want to discuss another initial state of interest: the *fully depleted state*, where at $t = 0$ all N molecules are in the lower level and R photons present. This case is important in non-linear optics.

In the matrix representation, this initial state is described by

$$\mathbf{e}(t = 0) = (1, 0, \ldots, 0), \tag{4.13}$$

the corresponding function $y_0(z)$ has the form

$$y_0(z) = (R!)^{-1/2} z^{R-L} \exp\left(-\tfrac{1}{4}z^2 + \frac{\Delta'}{2} z\right). \tag{4.14}$$

The method developed in Section 3 can easily be adapted to this problem – we confine ourselves to a brief outline of the main differences in the procedure. Up to equation (3.27) the approach is the same, however the photon number is now determined by the highest power of the expression $e^{i\mathcal{H}\tau} \mathcal{A} e^{-i\mathcal{H}\tau} y_0$. The average photon number is thus given by

$$\langle n(t) \rangle = \lim_{q_1 \to \infty} q_1^{-(R-L)} \exp\left(\tfrac{1}{4}q_1^2 - \frac{\Delta'}{2} q_1\right) \int_0^\infty dq_2 \int_0^\infty dq_3$$
$$\times K(q_1, q_2, -\tau) \left[\tfrac{1}{2}q_2^2 - \frac{\Delta'}{2} q_2 + L + q_2 \frac{\partial}{\partial q_2}\right] K(q_2, q_3, \tau)\, y_0(q_3). \tag{4.15}$$

Having performed the differentiation, we introduce, instead of the variable q_2,

$$\mathscr{E}_\infty = \lim_{q_1 \to \infty} \mathscr{E}(q_1, q_2, \tau). \tag{4.16}$$

From equation (3.34), the substitution formula is found to be

$$q_2 = \frac{1}{2} \frac{iP' + \Delta' P - \mathscr{E}_\infty}{P + E + \dfrac{\Delta'^2}{4}}. \tag{4.17}$$

The resulting integrals are evaluated by the method developed in Section 3. For the leading term of the average photon number one gets

$$\langle n(t)\rangle = R\,\frac{P+\frac{1}{2}(R-N)}{P+\frac{1}{2}(R+N)}. \tag{4.18}$$

We now match this even elliptic function of order 2 to the exact time evolution for short times (see Appendix). One finds, that the two leading orders of all coefficients up to τ^8 are correct, if in the invariants of the function $P(\tau)$ we take

$$\bar{\mathscr{E}} = -\frac{\Delta'}{2}(R+N), \tag{4.19}$$

$$\overline{\mathscr{E}^2} = RN + \tfrac{1}{4}\Delta'^2(R+N)^2, \tag{4.20}$$

respectively. We note that these values do not correspond exactly to the expectation values

$$\langle\mathscr{H}\rangle = -\frac{\Delta'}{2}(N+L+\tfrac{1}{2}) = -\frac{\Delta'}{2}(R+N+\tfrac{1}{2}), \tag{4.21}$$

$$\langle\mathscr{H}^2\rangle = RN + \tfrac{1}{4}\Delta'^2(R+N+\tfrac{1}{2})^2; \tag{4.22}$$

in fact, the modification $L \to L-\frac{1}{2}$ is necessary in the terms containing Δ'. The quantity $\langle n^2(t)\rangle$ can be computed by the same method, and for the mean square variance we finally get

$$\langle\sigma^2(t)\rangle = RN\,\frac{P^2-\frac{1}{4}(R-N)^2}{(P+\frac{1}{2}(R+N))^2\,(P+\frac{1}{2}(R+N)-1)}. \tag{4.23}$$

We remark, that in the fully depleted case also the detuning constant Δ appears only in the elliptic function $P(\tau)$. Apart from this, the results (4.18) and (4.23) are therefore the same as in the resonant case.

The discussion of these results runs in analogy to the case of the fully excited initial state. It turns out however that it is not possible to describe the period and the maximal height of the pulses by similarly simple equations as in the case treated initially. We thus confine ourselves to depict the numerical values for $L = O(1)$ and $2L = N(1-\Delta^2c^2/4)$, see Figure 3.

We remark that for $L = O(1)$, the period increases strongly as $|\Delta|c$ tends towards 0; this is due to the fact that for $\Delta = 0$ the parameter m of the complete elliptic integral $K(m)$ determining the period, equation (3.41), is $m = 1-\eta$, $\eta = O(N^{-1})$, [10]. Thus, in the resonant case, the period is $O((1/N)\log N)$, whereas for $|\Delta|c = O(1)$, it is only $O(N^{-1})$, the parameter m being equal to $m = 1-\eta$, $\eta = O(1)$. This difference in the order of magnitude of the quantity η also accounts for a drastic change in the time evolution. In the resonant case, the aperiodicity already begins after a few oscillations [10]; in the non-resonant case however, an analogous computation shows that $O(N^{1/2})$ oscillations are periodic.

Finally, we consider the superradiant initial state $2L = N(1-(\Delta^2c^2/4))$. Figure 3 shows that the period now depends weakly on the quantity $|\Delta|c$ in contrast to the case discussed above. Furthermore, the periodic approximation is valid for $O(N^{1/2})$ periods, if $|\Delta|c \neq 0$; whereas in the resonant case, the aperiodicity starts after $O(N)$ periods only. Thus we see that in the fully depleted case, as well as in the fully excited case, an

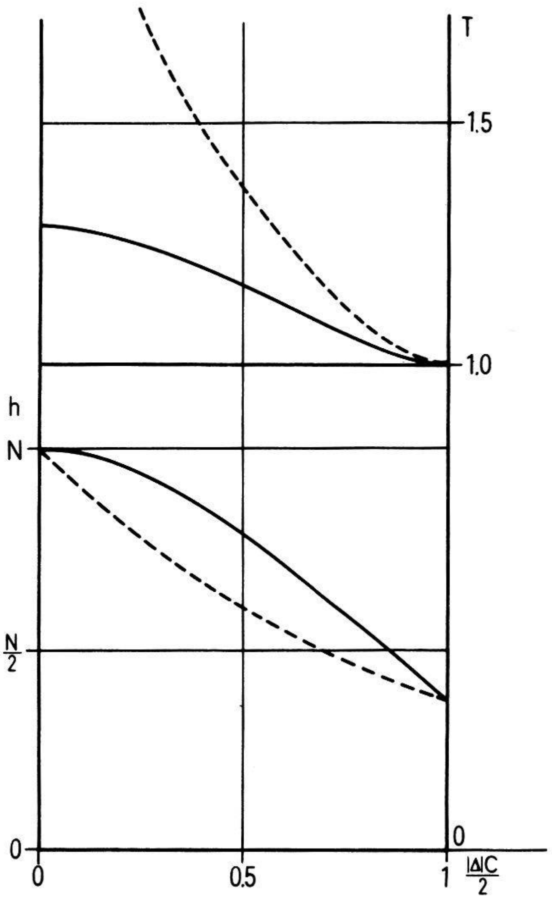

Figure 3
Maximal height h and period T of the pulsations for the fully depleted initial state, ---- $L = O(1)$, —— superradiant.

increase in the number of photons present initially augments the range, where the physical quantities show a periodic behaviour; besides, near resonance, the period as well as the height of the pulsations become less sensitive to a change of the quantity Δc.

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I wish to express my gratitude to Prof. Dr. G. Scharf, under whose direction the present work has been done, for his advice and many enlightening discussions.

APPENDIX

The exact time evolution for short times

The average photon number at the time t is equal to

$$\langle n(t)\rangle = \sum_{x=0}^{N} (R-x)\,|e_x(t)|^2, \tag{A.1}$$

where the state vector $\mathbf{e}(t)$ is given by

$$\mathbf{e}(t) = e^{-iHt}\,\mathbf{e}(0). \tag{A.2}$$

As initial state $\mathbf{e}(0)$ we first take the case with all molecules in the upper level, i.e. $\mathbf{e}(0) = (0,0,\ldots,1)$. In order to get a power series expansion in t of $\langle n(t)\rangle$ we expand the exponential function in equation (A.2). After an elaborate computation one arrives at

$$\langle n(t)\rangle = 2L + t^2 \frac{1}{c^2}(2L+1) + t^4 \frac{1}{c^4}(2L+1)\frac{1}{3N}\left(N - 2L - 2 - \frac{\Delta^2 c^2}{4}N\right)$$
$$+ t^6 \frac{1}{c^6}(2L+1)\frac{1}{N^2}\left[\tfrac{1}{360}\Delta^4 c^4 N^2 + \tfrac{1}{45}\Delta^2 c^2 N(-N+2L+2)\right.$$
$$\left.+ \tfrac{1}{45}(2N^2 - 26NL - 26N + 8L^2 + 34L + 26)\right] + t^8 \frac{1}{c^8}\frac{1}{N^3}$$
$$\times (2L+1)\left[-\tfrac{1}{4}\tfrac{1}{5040}\Delta^6 c^6 N^3 + \tfrac{1}{1680}\Delta^4 c^4 N^2(N - 2L - 2)\right.$$
$$+ \tfrac{1}{420}\Delta^2 c^2 N(-N^2 - 4L^2 + 22LN - 26L + 22N - 22)$$
$$+ \tfrac{1}{1260}(4N^3 - 240N^2 L + 480NL^2 - 32L^3 - 240N^2$$
$$\left.+ 1446LN - 528L^2 + 966N - 1230L - 734)\right] + \ldots.$$

By the same procedure, we compute the average square photon number given by

$$\langle n^2(t)\rangle = \sum_{x=0}^{N} (R-x)^2 |e_x(t)|^2. \tag{A.3}$$

The result of the series expansion reads

$$\langle n^2(t)\rangle = 4L^2 + t^2 \frac{1}{c^2}(2L+1)(4L+1) + t^4 \frac{1}{c^4}(2L+1)$$
$$\times \frac{1}{3N}\left[-(4L+1)\tfrac{1}{4}\Delta^2 c^2 N + 10NL - 8L^2 + 7N - 16L - 8\right]$$
$$+ t^6 \frac{1}{c^6}(2L+1)\frac{1}{N^2}\left[(4L+1)\tfrac{1}{360}\Delta^4 c^4 N^2 + \tfrac{1}{45}\Delta^2 c^2 N\right.$$
$$\times (-19LN + 8L^2 + 25L - 16N + 17) + \tfrac{1}{45}(32L^3 - 224L^2 N$$
$$\left.+ 68LN^2 + 62N^2 + 264L^2 - 520NL - 296N + 468L + 236)\right]$$
$$+ t^8 \frac{1}{c^8}(2L+1)\frac{1}{N^3}\frac{1}{1260}\left[-(4L+1)\tfrac{1}{16}\Delta^6 c^6 N^3 + \tfrac{1}{4}\Delta^4 c^4 N^2(138NL\right.$$
$$- 24L^2 + 129N - 156L - 132) + \Delta^2 c^2 N(-48L^3 + 768NL^2 -$$
$$- 264N^2 L - 828L^2 - 255N^2 + 1968NL - 1728L + 1200N - 948)$$
$$- 128L^4 + 3936L^3 N - 4992L^2 N^2 + 520LN^3 - 4160L^3 +$$
$$- 23400L^2 N - 12288LN^2 + 508N^3 - 18552L^2 + 39204LN$$
$$\left.- 7296N^2 - 27476L + 19740N - 12956\right] + \ldots.$$

For the fully depleted initial state, i.e. for $\mathbf{e}(0) = (1,0,\ldots,0)$, we have

$$\langle n(t)\rangle = R - t^2\frac{1}{c^2}R + t^4\frac{1}{c^4}\frac{R}{3N}(R+N-1+\tfrac{1}{4}\Delta^2c^2N) - t^6\frac{1}{c^6}\frac{R}{N^2}$$
$$\times[\tfrac{1}{360}\Delta^4c^4N^2 + \tfrac{1}{45}\Delta^2c^2N(R+N-1) + \tfrac{1}{45}(2R^2+2N^2+13RN$$
$$-13R-13N+11)] - t^8\frac{1}{c^8}\frac{R}{N^3}[-\tfrac{1}{5040}\tfrac{1}{4}\Delta^6c^6N^3 - \tfrac{1}{1680}$$
$$\times\Delta^4c^4N^2(R+N-1) - \tfrac{1}{1260}\Delta^2c^2N(3R^2+3N^2+33RN-33R$$
$$-33N+30) - (120R^2N+120RN^2+4R^3+4N^3-120R^2-120N^2$$
$$-483RN+363R+363N-247)] + \ldots.$$

$$\langle n^2(t)\rangle = R^2 - t^2\frac{1}{c^2}R(2R-1) + t^4\frac{1}{c^4}\frac{R}{N}(\tfrac{1}{12}\Delta^2c^2N(2R-1)$$
$$+\tfrac{1}{3}(2R^2+5RN-6R-4N+4)) - t^6\frac{1}{c^6}\frac{R}{N^2}[\tfrac{1}{360}\Delta^4c^4N^2(2R-1)$$
$$+\tfrac{1}{90}\Delta^2c^2N(4R^2+19RN-21R-17N+17) + \tfrac{1}{45}(4R^3+56R^2N$$
$$+34RN^2-58R^2-174RN-32N^2+140R+118N-86)]$$
$$+t^8\frac{1}{c^8}\frac{R}{N^3}[\tfrac{1}{5040}\tfrac{1}{4}\Delta^6c^6N^3(2R-1) + \tfrac{1}{1680}\Delta^4c^4N^2(2R^2+23RN$$
$$-24R-22N+22) + \tfrac{1}{1260}\Delta^2c^2N(6R^3+192R^2N+132RN^2$$
$$-195R^2-129N^2-666RN+534R+474N-345) + \tfrac{1}{1260}$$
$$\times(8R^4+492R^3N+1248R^2N^2+260RN^3-496R^3-4614R^2N$$
$$-3888RN^2-256N^3+3366R^2+10344RN+2640N^2-6716R$$
$$-6222N+3838)] + \ldots.$$

REFERENCES

[1] R. DICKE, Phys. Rev. *93*, 99 (1954).
[2] P. SMITH, *Mode Selection in Lasers*, Proceedings of the IEEE, Vol. 60/4, p. 422 (1972).
[3] H. HAKEN, *Handbuch der Physik*, Band XXV/2c, p. 28 (Springer 1970).
[4] H. HAKEN, *Handbuch der Physik*, Band XXV/2c, p. 10 (Springer 1970).
[5] M. TAVIS and F. CUMMINGS, Phys. Rev. *188*, 692 (1969).
[6] W. MALLORY, Phys. Rev. *188* 1976 (1969).
[7] D. WALLS and R. BARAKAT, Phys. Rev. *A1*, 446 (1970).
[8] G. SCHARF, Helv. Phys. Acta *43*, 806 (1970).
[9] R. BONIFACIO and G. PREPARATA, Phys. Rev. *A2*, 336 (1970).
[10] G. SCHARF, To appear.
[11] K. HEPP and E. LIEB, Annals of Phys. *76*, 360 (1973).
[12] L. LANDAU and E. LIFSCHITZ, *Quantenmechanik*, p. 65 (Akademie Verlag 1965).
[13] L. LANDAU and E. LIFSCHITZ, *Quantenmechanik*, pp. 171 and 178 (Akademie Verlag 1965).
[14] E. DEHN, *Algebraic Equations*, p. 90 (Dover 1960).
[15] A. ERDELYI, *Higher Transcendental Functions*, Vol. 2, p. 307 (McGraw-Hill 1963).

[16] P. BYRD and M. FRIEDMANN, *Handbook of Elliptic Integrals*, p. 298 (Springer 1954).
[17] W. GRÖBNER and N. HOFREITER, *Integraltafel*, p. 63, formula 9c (Springer 1949).
[18] M. ABRAMOWITZ and I. STEGUN, *Handbook of Math. Functions*, pp. 599, 600 (Dover 1965).
[19] P. CHOQUARD, Helv. Phys. Acta *28*, 138 (1955).
[20] K. WEIERSTRASS, Math. Werke, Vol. V, 4ff (1915).
[21] E. WHITTAKER and G. WATSON, *A Course of Modern Analysis*, pp. 437, 443 (Cambridge 1965).
[22] M. ABRAMOWITZ and I. STEGUN, *Handbook of Math. Functions*, p. 649 (Dover 1965).
[23] M. ABRAMOWITZ and I. STEGUN, *Handbook of Math. Functions*, p. 630 (Dover 1965).
[24] M. ABRAMOWITZ and I. STEGUN, *Handbook of Math. Functions*, p. 635 (Dover 1965).
[25] H. HANCOCK, *Lectures on the Theory of Elliptic Functions*, Vol. I, p. 344 (Dover 1958).