Comment on a paper of Amrein, Martin, and Misra

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Comment on a Paper of Amrein, Martin, and Misra

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Abstract. We point out a gap in the proof of a proposition in a paper of Amrein, Martin and Misra and give a proof of their result.

The paper by Amrein, Martin and Misra [1] is an interesting and important contribution to the literature of time-dependent scattering theory. In this note we point out a gap in the proof of their Proposition 1 and furnish a proof of this result.

The scattering theory of AMM is based upon three conditions which they call (A1), (A2), and (A3). Using their notation we define the unitary groups $V_t = \exp(-iHt)$ and $U_t = \exp(-iH_0t)$ $(-\infty < t < \infty)$ which respectively describe the total evolution and free evolution of the scattering system, and the von Neumann algebra $\mathscr{A}_0$ consisting of all bounded linear operators on the Hilbert space $\mathscr{H} = L^2(R^3)$ that commute with all spectral projections of the positive self-adjoint operator H_0. We will be concerned with the following two conditions of AMM:

(A1) There exists a projection operator P on $\mathscr{H}$ such that

a) $[P, V_t] = 0, t \in (-\infty, \infty)$,

b) for every operator $A \in \mathscr{A}_0$ there exist two operators $A_\pm$ such that

$$\operatorname*{s-lim}_{t\to\pm\infty} V_t^* A V_t P = A_\pm = \mu_\pm(A), \tag{1}$$

c) $[P, A_\pm] = 0$.

(A3) For every vector $f \in \mathscr{H}$ there exists a vector $g \in P\mathscr{H}$ such that for all $A \in \mathscr{A}_0$

$$\lim_{t\to-\infty} (g, V_t^* A V_t g) = (f, Af).$$

In their Proposition 1 AMM want to prove that the images $\mu_\pm(\mathscr{A}_0)$ of the mappings defined by (1) are von Neumann algebras by assuming only their first condition (A1). After establishing that $\mu_\pm$ are *-homomorphisms they quote a theorem of Feldman and Fell [2] thereby arriving at the conclusion that $\mu_\pm$ are ultraweakly continuous on $\mathscr{A}_0$. However, the theorem of Feldman and Fell does not apply in this case, because, as one easily proves, $\mathscr{A}_0$ is not properly infinite. This result was also discovered by AMM subsequent to the publication of their paper [3].

It is possible to prove that $\mu_-(\mathscr{A}_0)$ is a von Neumann algebra by making use of (A3) as well as (A1). W. O. Amrein has informed the author that such a proof has been discussed in detail by Mourre [4]. However, in order to use this method to prove that $\mu_+(\mathscr{A}_0)$ is a von Neumann algebra it is necessary to require the validity of (A3) also in the limit $t\to+\infty$. Call this new condition $(A3)_+$. Alternatively, $(A3)_+$ can be derived by assuming the validity of (A1), (A2) and (A3) and that the scattering system is time reversal invariant [1]. With the additional condition $(A3)_+$ the scattering operator S is necessarily unitary when it exists, as is noted in [1]. Thus, this method of proof seems to be too restrictive (at least to the author).

In view of the above remarks it would appear to be useful to have a proof of the proposition in question assuming only the validity of (A1). We will give such a proof.

Proposition. Assume the validity of (A1). Then $\mu_\pm$ are ultraweakly continuous on $\mathscr{A}_0$.

Once this much has been proved, the results stated by AMM as their Proposition 1 can be obtained by, for example, following the procedure in the latter part of their proof. In the proof given below we will use freely the result of AMM that $\mu_\pm$ are *-homomorphisms on $\mathscr{A}_0$ without proving it again. We note that Lavine [5] proved that $\mu_\pm$ are *-isomorphisms on a certain operator algebra under more restictive assumptions. In the present situation $\mu_\pm$ are not necessarily isomorphisms as in the cases discussed by Mourre [4] and Lavine [5].

Proof. We write $\mathscr{A}_0$ as the direct sum of finite and properly infinite von Neumann algebras $(\mathscr{A}_0)_G$ and $(\mathscr{A}_0)_{I-G}$ ([6], Proposition 8, p. 98), and consider the finite summand $(\mathscr{A}_0)_G$ by assuming that $\mathscr{A}_0$ is finite.

Let A denote an arbitrary non-zero element of $\mathscr{A}_0'$. Then A is a bounded normal operator and consequently has a polar decomposition

$$A = U|A|, \quad |A| \in \mathscr{A}_0', \tag{2}$$

where U is unitary and $|A|$ is bounded, positive, and self-adjoint. It follows that $A \in \operatorname{Ker}\mu_\pm$ if and only if $|A| \in \operatorname{Ker}\mu_\pm$.

Since $|A|$ is bounded and self-adjoint, and the spectrum of H_0 is the non-negative real axis [1], we can write,

$$(\Psi, |A|\Psi) = \int_0^\infty f(\lambda)\, d(\Psi, E_0(\lambda)\,\Psi), \quad \Psi \in \mathscr{H}, \quad f \in \mathscr{B}, \tag{3}$$

where E_0 denotes the resolution of the identity of H_0 and $\mathscr{B}$ the class of all real-valued bounded Borel measurable functions on $[0, \infty)$. It follows that

$$F(\delta) = E_0(f^{-1}(\delta)) \tag{4}$$

for every Borel set δ of the spectrum of $|A|$, where F denotes the resolution of the identity of $|A|$ ([7], Corollary X.2.10).

Now suppose $A \in \operatorname{Ker}\mu_\pm$. From (1) and (2) one finds

$$\lim_{t\to\pm\infty} \|V_t^*|A|V_t g\| = 0, \quad \text{all } g \in P\mathscr{H}. \tag{5}$$

Since $V_t^* F V_t$ is the resolution of the identity of $V_t^*|A|V_t$ we find, using (4), (5) and ([7], Corollary X.7.3), that

$$\lim_{t\to\pm\infty} \|V_t^* E_0(f^{-1}(\delta))\, V_t g\| = 0 \tag{6}$$

for all $g \in P\mathscr{H}$ and all Borel sets δ of the spectrum of $|A|$.

Take f to be the characteristic function of a Borel set Δ of $[0, \infty)$, $f = \chi_\Delta$. Then $f \in \mathscr{B}$ and we find from (3), $|A| = E_0(\Delta)$, which is non-zero by assumption. The set $\delta = \{0, 1\}$ is a Borel set of the spectrum of this operator so that

$$E_0(f^{-1}(\delta)) = E_0([0, \infty)) = I = \text{identity operator}$$

and we obtain a contradiction with (6). Consequently,

$$E_0(\Delta) \notin \operatorname{Ker} \mu_\pm \tag{7}$$

for all Borel sets Δ of $[0, \infty)$ such that $E_0(\Delta) \neq 0$.

Let f_n denote a simple function,

$$f_n = \sum_{i=1}^{n} \alpha_i \chi_{\Delta_i}, \quad n < \infty, \tag{8}$$

where the α_i are non-zero real numbers and $\{\Delta_i\}$ is a sequence of disjoint Borel sets of $[0, \infty)$. Then $f_n \in \mathscr{B}$ and the corresponding operator is obtained from (3),

$$B_n = \sum_{i=1}^{n} \alpha_i E_0(\Delta_i).$$

Such operators will be called simple. Since $\mu_\pm$ are linear we find

$$\mu_\pm(B_n) = \sum_{i=1}^{n} \alpha_i \mu_\pm(E_0(\Delta_i)). \tag{9}$$

From the disjointness of the sequence $\{\Delta_i\}$ and the standard properties of a spectral measure one finds that $\{E_0(\Delta_i)\}$ is a sequence of pairwise orthogonal projections. The homomorphisms $\mu_\pm$ preserve this property so that $\{\mu_\pm(E_0(\Delta_i))\}$ are also sequences of this type. If $B_n \neq 0$ then there is at least one Borel set Δ_k occurring in the sequence $\{\Delta_i\}$ of (8) such that $E_0(\Delta_k) \neq 0$ so that (7) obtains. Take $\Psi_\pm \neq 0$ in the range of $\mu_\pm(E_0(\Delta_k)) \neq 0$ so that (9) yields

$$\mu_\pm(B_n)\Psi_\pm = \alpha_k \Psi_\pm \neq 0,$$

and thus $\mu_\pm(B_n) \neq 0$ or

$$B_n \notin \operatorname{Ker} \mu_\pm \tag{10}$$

for all non-zero simple operators.

It follows from (3) that for each positive operator $|A| \neq 0$ there exists a Borel set Δ_0 of $[0, \infty)$ such that f assumes only positive values on Δ_0 and $E_0(\Delta_0) \neq 0$. We then find that, since $f\chi_{\Delta_0}$ belongs to $\mathscr{B}$ and is non-negative, there exists an increasing sequence of non-negative simple functions $\{f_n\}$ which converges pointwise to $f\chi_{\Delta_0}$ [8]. Because $f\chi_{\Delta_0}$ majorizes f_n for each n we find that $f_n \in \mathscr{B}$ for all n and that the simple operators B_n corresponding to f_n by (3) are positive and bounded. Moreover, each f_n vanishes outside the Borel set Δ_0 and for a sufficiently large value of n (n_0 say) $f_{n_0} > 0$ on Δ_0. It follows that $B_{n_0} \neq 0$ so that we obtain from (10),

$$\mu_\pm(B_{n_0}) > 0 \tag{11}$$

because $\mu_\pm$ preserve positivity. Since $f\chi_{\Delta_0} - f_{n_0} \in \mathscr{B}$ is non-negative we find that the operator $|A|E_0(\Delta_0) - B_{n_0}$ to which this function corresponds by (3) is positive. We now again use the fact that $\mu_\pm$ preserve positivity coupled with (11) to show that

$$|A|E_0(\Delta_0) \notin \operatorname{Ker}\mu_\pm \qquad \text{for } |A| \neq 0,$$

from which $|A| \notin \operatorname{Ker}\mu_\pm$ immediately follows.

Thus, we have shown that $\operatorname{Ker}\mu_\pm \cap \mathscr{A}_0' = \{0\}$. Then, since $\mathscr{A}_0$ has been assumed finite, one finds $\operatorname{Ker}\mu_\pm = \{0\}$ ([6], Corollaire 1 of Proposition 2, p. 256). Hence, $\mu_\pm$ are injective on the finite summand $(\mathscr{A}_0)_G$. It follows that the restrictions of $\mu_\pm$ to $(\mathscr{A}_0)_G$ are direct summand representations in the sense of Fell [9] and are consequently ultraweakly continuous because $(\mathscr{A}_0)_G$ is of type I [2, 9].

We can now follow AMM and invoke the theorem of Feldman and Fell [2] to deduce that the restrictions of $\mu_\pm$ to the properly infinite summand $(\mathscr{A}_0)_{I-G}$ are ultraweakly continuous. Finally, we use the linearity of $\mu_\pm$, the ultraweak continuity of their restrictions to the two summands, and the characterization of ultraweakly continuous homomorphisms in [2] to prove that $\mu_\pm$ are ultraweakly continuous on $\mathscr{A}_0$.

W. O. Amrein has recently informed the author that the result of this note has also been proved by V. Georgescu (unpublished) by a different method.

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