On the asymptotic condition in quantum field theory

On the Asymptotic Condition in Quantum Field Theory*)

von **David Ruelle**
Eidgenössische Technische Hochschule, Zürich

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Abstract. A 'space-like asymptotic condition' is proved which allows Haag's approach to the asymptotic condition to be carried out rigorously in the frame of the WIGHTMAN axioms.

Introduction

Two main approaches exist to the asymptotic condition in 'axiomatic' quantum field theory. One is due to LEHMANN, SYMANZIK, and ZIMMERMANN[16]), and postulates the convergence of field matrix elements to matrix elements of free fields.

The extremely useful reduction formulae which follow yield expressions for the elements of the S-matrix, the analytic properties of which may then be studied, and also systems of equations (τ- or r-equations) expressing essentially the unitarity of the S-matrix[16])[17]). A complete justification of the L.S.Z. formalism involves however a number of new requirements on domains of definition of field operators and continuity of the boundary values of the Green function in p-space. These requirements are not of fundamental physical significance and may well not be independent since e.g. the use of the unitarity condition gives information on the analytic behaviour of the boundary values of the GREEN function[18])[24]). From the purely axiomatic point of view a deeper investigation of the asymptotic condition is necessary and it is probably reasonable to accept provisionally the fact that the L.S.Z. formalism stands at a lower level of rigour than for example that of WIGHTMAN[21]).

The other approach to the asymptotic condition is due to HAAG[9])[10]). HAAG's main idea is that it is possible to construct asymptotic ingoing and outgoing states as strong limits in Hilbert space, if a certain 'space-like asymptotic condition' is verified by the vacuum expectation values of products of field operators. This construction is physically transparent and although the results obtained are less powerful than those of L.S.Z.,

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they give a definition of the asymptotic fields and of the S-matrix of the theory.

In what follows we will show that HAAG's programme may be carried through rigorously in the framework of the Gårding-Wightman axioms if one introduces as a new postulate the completeness of the asymptotic states and spectral conditions connected to this.

The completeness of the asymptotic states is a physically reasonable requirement and is independent of the other axioms as shown by counter-examples (see e.g. ref. [22]) p. 57).

In conclusion, we may introduce in the theory asymptotic states and fields and an observable quantity, the S-matrix.

1. The axioms of a field theory

A field theory according to WIGHTMAN [21]) [22]) is defined by a finite (or at most countable) family of fields $A^{\varkappa}_{\mu}(x)$, which are operator-valued tempered distributions*). This means that to every $\varkappa$ and $\varphi^{\mu} \in \mathcal{S}_4$ there corresponds an operator

$$A^{\varkappa}(\varphi) = \int dx \sum_{\mu} \varphi^{\mu}(x)\; A^{\varkappa}_{\mu}(x) \tag{1}$$

on the Hilbert space $\mathfrak{H}$ of states. These operators, which are not bounded, are assumed to be defined on a common linear manifold D dense in $\mathfrak{H}$ (for a discussion of these points see Appendix).

Furthermore, if $\Phi, \Psi \in D$, $\varphi \to (\Phi, A(\varphi)\, \Psi)$ should be a continuous linear functional on $\mathcal{S}$. The following axioms are then introduced or emphasized

1. The metric in $\mathfrak{H}$ is positive definite.
2. The theory is covariant, i.e. there exists a unitary representation $U(a, A)$ of the covering group of the inhomogeneous proper Lorentz group in $\mathfrak{H}$. An energy-momentum operator P is then defined by

$$U(a, 1) = e^{i P_{\mu} a^{\mu}} . \tag{2}$$

The fields transform according to irreducible representations $S^{\varkappa}_{\mu\nu}(A)$ of the covering group C_2 of the homogeneous proper Lorentz group:

$$U(a, A)\; A^{\varkappa}_{\mu}(x)\; U(a, A)^{-1} = \sum_{\nu} S^{\varkappa}_{\mu\nu}(A^{-1})\; A^{\varkappa}_{\nu}\,(A\, x + a)\, . \tag{3}$$

According to whether $S^{\varkappa}_{\mu\nu}$ is a 'single-' or 'double-valued' representation of the homogeneous proper Lorentz group, $A^{\varkappa}_{\mu}(x)$ is called a Bose field or a Fermi field.

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*) For the theory of distributions see ref. [20]) [8]).

3. There exists a unique state (the vacuum), corresponding to a vector Ω in $\mathfrak{H}$ such that $P\Omega = 0$.
   The vacuum is stable: the spectrum of P belongs*) to $\overline{V}_+$.

4. The theory is local. Let $\sigma_{\varkappa\varkappa'} = -1$ if $A^\varkappa$ and $A^{\varkappa'}$ are both Fermi fields, $\sigma_{\varkappa\varkappa'} = 1$ otherwise, then

$$A^\varkappa_\mu(x)\, A^{\varkappa'}_{\mu'}(x') = \sigma_{\varkappa\varkappa'}\, A^\varkappa_\mu(x')\, A^\varkappa_\mu(x) \quad \text{if} \quad (x'-x)^2 < 0\,. \tag{4}$$

The sign $\sigma_{\varkappa\varkappa'}$ is determined by the theorem of 'connection between spin and statistics'[5]) [2]).

5. $A^\varkappa(\varphi)\, D \subset D$, furthermore D may be taken as the linear manifold of the vectors obtained by applying any polynomial in the operators $A^\varkappa(\varphi)$ to the vacuum.
   That D is then dense in $\mathfrak{H}$ is the axiom of completeness of the theory. (See also Appendix.)

The vacuum expectation values

$$\mathfrak{W}(x_0, x_1, \ldots, x_n) = \langle A^{\varkappa_0}_{\mu_0}(x_0)\, A^{\varkappa_1}_{\mu_1}(x_1) \ldots A^{\varkappa_n}_{\mu_n}(x_n)\rangle_0 \tag{5}$$

are (tensor) tempered distributions as a consequence of Schwartz' kernel theorem (see e.g. ref. [8]) p. 62).

Besides the above axioms, we should exclude the occurrence in $\mathfrak{H}$ of some unphysical irreducible representations corresponding to mass zero of the covering group of the inhomogeneous proper Lorentz group (see e.g. ref. [23])). We will however in what follows use the stronger requirement that there exists a positive lowest mass μ in the theory:

6. Apart from the eigenvalue 0 corresponding to the vacuum, the spectrum of P is contained in $\overline{V}^\mu_+$, $\mu > 0$.

Finally, the introduction of the axiom of completeness of the asymptotic states and the discussion of the related spectral conditions will be possible only later.

We will from now on drop the indices $\varkappa$ and μ of the fields A for notational convenience wherever this does not lead to ambiguity.

Consider now a vacuum expectation value like (5), the occurrence of the vacuum as an intermediate state in this expression hides the existence of the positive smallest mass μ of the theory. To remedy this situation, HAAG[9]) has shown that one may define 'truncated' vacuum expectation

*) We denote by $V+$ the open forward cone and let

$$V^\mu_+ = \{p : p \in V_+,\ p^2 > \mu^2\},$$

$\overline{V}_+$ and $\overline{V}^\mu_+$ are the closures of V_+ and V^μ_+.

values where the contributions from the intermediate vacuum state are subtracted in a manner which is symmetric with respect to the permutations of the $n + 1$ field operators.

If ϱ_k is the family of all partitions of the set $\{0, 1, \ldots, n\}$ into $k + 1$ subsets: $X_0, X_1, \ldots, X_k$, and $\mathfrak{W}(x)_{X_j}$ the vacuum expectation value of the product of the fields $A(x_i)$, $i \in X_j$, in natural order, the formula

$$\mathfrak{W}(x) = \sum_{k=0}^{n} \sum_{\varrho k} \sigma \prod_{j=0}^{k} \tilde{\mathfrak{W}}(x)_{X_j} \tag{6}$$

may be used to define $\tilde{\mathfrak{W}}$ recursively on the number of variables. σ is a sign factor originating from axiom 4. Let π be the permutation from $(0, 1, \ldots, n)$ to $(X_0, X_1, \ldots, X_k)$ where the elements of each X_j are written in natural order and let π' be the permutation induced by π on the indices of the Fermi fields, then $\sigma = \pm 1$ according to whether π' is even or odd. We will also write

$$\tilde{\mathfrak{W}}(x_0, x_1, \ldots, x_n) = \langle A(x_0)\, A(x_1) \ldots A(x_n) \rangle_0^T \tag{7}$$

for these truncated vacuum expectation values.

The translational invariance of the theory is expressed in terms of the (possibly truncated) vacuum expectation values by the relation

$$\mathfrak{W}(x_0 + a, x_1 + a, \ldots, x_n + a) = \mathfrak{W}(x_0, x_1, \ldots, x_n) \tag{8}$$

where a is any four-vector.

With the above axioms and the definition of truncation, we are in position to study the behaviour at large space-like separations of certain vacuum expectation values.

2. Space-like asymptotic condition*)

We start with a series of definitions.

Let $\boldsymbol{x}_i$ denote the family $x_{i0}, x_{i1}, \ldots, x_{ir(i)}$ of four-vector variables and

$$\boldsymbol{A}_i(\boldsymbol{x}_i) = A(x_{i0})\, A(x_{i1}) \ldots A(x_{ir(i)}) \,,$$

$$\boldsymbol{A}_i\,(\boldsymbol{x}_i + a_i) = U(a_i, 1)\, \boldsymbol{A}_i(\boldsymbol{x}_i)\, U(a_i, 1)^{-1} .$$

We call $\boldsymbol{A}_i$ a Bose or Fermi operator**) according to whether it contains an even or odd number of Fermi fields.

*) The asymptotic behaviour of the vacuum expectation values has been studied in ref. [6])[1])[13]) but the results obtained there are not adequate for our later purposes.

**) Instead of products of fields, one might take more generally 'cycles', i. e. essentially products of T-products[19]), if these are well-defined.

If $\pi \in \mathfrak{S}_{n+1}$ (symmetric group of degree $n+1$) is the permutation such that $\pi(0, 1, \ldots, n) = (i_0, i_1, \ldots, i_n)$, let $\sigma_\pi = \pm 1$ according to whether π, restricted to the indices of the Fermi operators is even or odd.

We write

$$T^\pi(\boldsymbol{x} + a) = T^\pi(\boldsymbol{x}_0 + a_0, \boldsymbol{x}_1 + a_1, \ldots, \boldsymbol{x}_n + a_n) =$$

$$= \sigma_\pi \langle \boldsymbol{A}_{i_0} (\boldsymbol{x}_{i_0} + a_{i_0}) \boldsymbol{A}_{i_1} (\boldsymbol{x}_{i_1} + a_{i_1}) \ldots \boldsymbol{A}_{i_n} (\boldsymbol{x}_{i_n} + a_{i_n}) \rangle_0 , \tag{1}$$

$$F^\pi_\varphi(a) = \int d\boldsymbol{x}\, \boldsymbol{\varphi}(\boldsymbol{x})\, T^\pi\, (\boldsymbol{x} + a) \tag{2}$$

where $\boldsymbol{\varphi}$ is assumed to belong to the functional space $\mathscr{S}$ (see Appendix).

In general we will take the a_i as purely space-like: $a_i = (0, \vec{a}_i)$ and write in this case $\vec{a}_i$ instead of a_i. Consider a definite configuration of the $\vec{a}_i$. The diameter λ of the configuration is given by

$$\lambda^2 = \max_{i, i'} (\vec{a}_{i'} - \vec{a}_i)^2 .$$

We assume that this maximum is obtained for $i = j$, $i' = j'$ so that $\lambda^2 = (\vec{a}_{j'} - \vec{a}_j)^2$. Consider also the family of all partitions of the set $\{0, 1, \ldots, n\}$ into two subsets X, X' such that $j \in X$, $j' \in X'$. The maximum μ of the distance of the configurations $(\vec{a}_i)_{i \in X}$, $(\vec{a}_{i'})_{i' \in X'}$ is given by

$$\mu^2 = \max_X \left[\min_{i \in X,\, i' \in X'} (\vec{a}_{i'} - \vec{a}_i)^2 \right] .$$

We assume that this maximum is obtained for the partition $X = Y$, $X' = Y'$ and that $\mu^2 = (\vec{a}_{l'} - \vec{a}_l)^2$, $l \in Y$, $l' \in Y'$.

We may now remark that $n\,\mu \geqq \lambda$.

The truncated vacuum expectation values obtained from (1) by subtracting in a symmetric way the terms for which the vacuum appears as intermediate state between the $\boldsymbol{A}_i$ will be called $\tilde{T}^\pi$ and we write

$$\tilde{F}^\pi_\varphi(a) = \int d\boldsymbol{x}\, \boldsymbol{\varphi}(\boldsymbol{x})\, \tilde{T}^\pi\, (\boldsymbol{x} + a) . \tag{2'}$$

Finally, if $Y = \{i_0, i_1, \ldots, i_k\}$, $Y' = \{i'_0, i'_1, \ldots, i'_{k'}\}$, $k + k' = n - 1$, where the indices in each sequence are written in natural order, we define the permutations I and J such that

$$I(0, 1, \ldots, n) = (0, 1, \ldots, n) ,$$

$$J(0, 1, \ldots, n) = (i_0, i_1, \ldots, i_k, i'_0, i'_1 \ldots, i'_{k'}) .$$

Lemma: *For any positive integer N*

$$\lim_{\lambda \to \infty} \lambda^N [\tilde{F}^I_\varphi(\vec{a}) - \tilde{F}^J_\varphi(\vec{a})] = 0 \tag{3}$$

when the configuration of the $\vec{a}_i$ *remains such that the above defined* j, j', Y, Y', l, l' *stay the same.*

Note first that $\tilde{T}^I(\boldsymbol{x}) - \tilde{T}^J(\boldsymbol{x})$ vanishes when all $x_{i\alpha}$, $i \in Y$ are space-like to all $x_{i'\alpha'}$, $i' \in Y'$, because of locality. $\boldsymbol{\varphi}(\boldsymbol{x})$ therefore does not contribute to the integral

$$\tilde{F}^I_\varphi(\vec{a}) - \tilde{F}^J_\varphi(\vec{a}) = \int d\boldsymbol{x}\, \boldsymbol{\varphi}(\boldsymbol{x})\, [\tilde{T}^I\, (\boldsymbol{x} + \vec{a}) - \tilde{T}^J\, (\boldsymbol{x} + \vec{a})] \tag{4}$$

when $[(x_{i\alpha} - x_{i'\alpha'}) + (a_i - a_{i'})]^2 < 0$ for all $i \in Y$, $i' \in Y'$. Introducing a positive distance by

$$\| x_{i\alpha} - x_{i'\alpha'} \|^2 = (x^0_{i\alpha} - x^0_{i'\alpha'})^2 + (\vec{x}_{i\alpha} - \vec{x}_{i'\alpha'})^2 \,,$$

we see that this is satisfied if

$$\| x_{i\alpha} - x_{i'\alpha'} \|^2 < \frac{\mu^2}{2} \,,$$

$$\text{or} \quad \| x_{i\alpha} - x_{i'\alpha'} \|^2 < \frac{\lambda^2}{2\, n^2} \quad \text{because} \quad n\, \mu \geqq \lambda \,,$$

$$\text{or} \quad \| \boldsymbol{x} \|^2 \equiv \sum_{i=0}^{n} \sum_{\alpha=0}^{r(i)} \| x_{i\alpha} \|^2 < \frac{\lambda^2}{8\, n^2} \,. \tag{5}$$

Inequality (5) defines in $\boldsymbol{x}$-space the inside of a sphere, the radius of which is proportional to λ.

On the other hand, the transformation $\tilde{T}^\pi(\boldsymbol{x}) \to \tilde{T}^\pi\, (\boldsymbol{x} + a)$ is a translation in $\boldsymbol{x}$ for which we may disregard a common additive term α in all a_i because of the translational invariance of the theory. The vector of the translation $\boldsymbol{x} \to \boldsymbol{x} + \vec{a}$ has then a length $\| \vec{a} \|$ smaller than $\lambda \sqrt{L}$ in $\boldsymbol{x}$-space if $L = n + \sum_{i=0}^{n} r(i)$.

Consider now a family of non-negative functions $\mathfrak{h}_\nu(\boldsymbol{x}) \in \mathcal{D}$, ν taking any positive integral value, such that the $\mathfrak{h}_\nu(\boldsymbol{x})$ and their derivatives are bounded uniformly in ν, $\mathfrak{h}_\nu(\boldsymbol{x}) \equiv \mathfrak{h}_\nu(\| \boldsymbol{x} \|) = 0$ if $\| \boldsymbol{x} \| > \nu + 1$ or $\| \boldsymbol{x} \| < \nu - 1$, and $\sum_\nu \mathfrak{h}_\nu(\boldsymbol{x}) = 1$. We may then write

$$\tilde{F}^I_\varphi(\vec{a}) - \tilde{F}^J_\varphi(\vec{a}) = \sum_{\nu > \frac{\lambda}{2\, n \sqrt{2}} - 1} [\tilde{F}^I_{\varphi_\nu}(\vec{a}) - \tilde{F}^J_{\varphi_\nu}(\vec{a})] \tag{6}$$

where $\boldsymbol{\varphi}_\nu(\boldsymbol{x}) = \mathfrak{h}_\nu(\boldsymbol{x})\, \boldsymbol{\varphi}(\boldsymbol{x})$.

Since $\tilde{T}^I(\boldsymbol{x}) - \tilde{T}^J(\boldsymbol{x})$ is a tempered distribution, it may be written as

$$\tilde{T}^I(\boldsymbol{x}) - \tilde{T}^J(\boldsymbol{x}) = D\, g(\boldsymbol{x})$$

where D is a derivative monomial and $g(\boldsymbol{x})$ is a continuous function with at most polynomial increase. Thus

$$\tilde{F}^I_{\varphi_\nu}(\vec{a}) - \tilde{F}^J_{\varphi_\nu}(\vec{a}) = \int d\boldsymbol{x}\, \varphi_\nu(\boldsymbol{x})\, D g\,(\boldsymbol{x} + \vec{a}) = \pm \int d\boldsymbol{x}\, [D\, \varphi_\nu(\boldsymbol{x})]\, g\,(\boldsymbol{x} + \vec{a})\,. \quad (7)$$

The numbers $\max_{\boldsymbol{x}} |D\, \varphi_\nu(\boldsymbol{x})|$ are decreasing faster than any power of ν^{-1}. On the other hand we may write $|g(\boldsymbol{x})| < C\,(1 + \|x\|^2)^{k/2}$ where C is a positive constant. Thus

$$|\tilde{F}^I_{\varphi_\nu}(\vec{a}) - \tilde{F}^J_{\varphi_\nu}(\vec{a})| < S\,(\nu + 1) \max_{\boldsymbol{x}} |D\, \varphi_\nu(\boldsymbol{x})|\, C \max_{\|\boldsymbol{x}\| < \nu+1} (1 + \|\boldsymbol{x} + \vec{a}\|^2)^{k/2}$$

$$< [S\,(\nu + 1) \max_{\boldsymbol{x}} |D\, \varphi_\nu(\boldsymbol{x})|\, C\,(1 + 2\,(\nu + 1)^2)^{k/2}]\,(1 + 2\,L\,\lambda^2)^{k/2}$$

where $S(\nu + 1)$ is the volume of the sphere with radius $\nu + 1$ and we have used the inequality $1 + \|\boldsymbol{x} + \vec{a}\|^2 < (1 + 2\|\boldsymbol{x}\|^2)\,(1 + 2\,\|\vec{a}\|^2)$.

Obviously the numbers $c_\nu = \max_{\boldsymbol{x}} |D\, \varphi_\nu(\boldsymbol{x})|\, [C\, S\,(\nu + 1)\,(1 + 2\,(\nu + 1)^2)^{k/2}]$ decrease faster than any power of ν^{-1}. Therefore, in the right-hand side of the inequality

$$|\tilde{F}^I_\varphi(\vec{a}) - \tilde{F}^J_\varphi(\vec{a})| < \Bigl(\sum_{\nu > \frac{\lambda}{2\,n\,\sqrt{2}} - 1} c_\nu\Bigr)\,(1 + 2\,L\,\lambda^2)^{k/2}$$

the first factor decreases faster than any power of λ^{-1}. Obviously then

$$\lim_{\lambda \to \infty} \lambda^N\, [\tilde{F}^I_\varphi(\vec{a}) - \tilde{F}^J_\varphi(\vec{a})] = 0$$

for any N, which was what we had set out to prove.

We introduce now in $\boldsymbol{x}$-space the new variables

$$x = x_{i_0 0}\,, \qquad \xi = x_{i_0' 0} - x_{i_0 0}\,, \qquad \xi_i = x_{i0} - x_{i_0 0}\ (i \neq i_0)\,,$$

$$\xi_{i'} = x_{i'0} - x_{i_0' 0}\ (i' \neq i_0')\,,$$

$$\xi_{i\alpha} = x_{i\alpha} - x_{i0}\ (\alpha \neq 0)\,, \qquad \xi_{i'\alpha'} = x_{i'\alpha'} - x_{i'0}\ (\alpha' \neq 0)$$

where $i, i_0 \in Y$; $i', i_0' \in Y'$, and we denote by $\boldsymbol{\xi}$ the family of all ξ_i, $\xi_{i'}$, $\xi_{i\alpha}$, $\xi_{i'\alpha'}$. Then

$$\tilde{T}^\pi = \tilde{T}^\pi(\xi, \boldsymbol{\xi})\,, \qquad \varphi = \varphi(x, \xi, \boldsymbol{\xi})\,.$$

Fourier transforms are defined by

$$\mathfrak{F}\, \tilde{T}^\pi(P, \boldsymbol{P}) = (2\,\pi)^{-2L} \int d\xi\, d\boldsymbol{\xi}\, e^{-i(P\xi + \boldsymbol{P}\boldsymbol{\xi})}\, \tilde{T}^\pi(\xi, \boldsymbol{\xi})\,,$$

$$\check{\varphi}(p, P, \boldsymbol{P}) = (2\pi)^{-2(L+1)} \int dx\, d\xi\, d\boldsymbol{\xi}\, e^{i(px + P\xi + \boldsymbol{P}\boldsymbol{\xi})}\, \varphi(x, \xi, \boldsymbol{\xi})$$

and we have

$$\tilde{F}^{\pi}_{\varphi}(a) = (2\pi)^2 \int dP\, d\boldsymbol{P}\, \check{\varphi}(0, P, \boldsymbol{P})\, \mathfrak{F}\, \tilde{T}^{\pi}(P, \boldsymbol{P}) \times$$

$$\times \exp i \left[P\,(a_{i_0'} - a_{i_0}) + \sum_{i=i_1}^{i_k} P_i\,(a_i - a_{i_0}) + \sum_{i'=i_1'}^{i'_{k'}} P_{i'}\,(a_{i'} - a_{i_0'}) \right]. \qquad (8)$$

This equation shows in particular that $F^{\pi}_{\varphi}(a)$, $\tilde{F}^{\pi}_{\varphi}(a)$ belong to the functional space O_M of the infinitely continuously differentiable functions with slow increase.

Let now $K \in \mathfrak{S}_{n+1}$ be the permutation such that $K\,(0, 1, \ldots, n) = (i_0', i_1', \ldots, i_{k'}', i_0, i_1, \ldots, i_k)$ then $\mathfrak{F}\, \tilde{T}^J(P, \boldsymbol{P})$ vanishes unless $P \in \overline{V}^{\mu}_{+}$ and $\mathfrak{F}\, \tilde{T}^K(P, \boldsymbol{P})$ vanishes unless $P \in \overline{V}^{\mu}_{-}$. This results from axiom 6 and from the truncation of vacuum expectation values.

If we define

$$\check{\psi}(p, P, \boldsymbol{P}) = h(P)\, \check{\varphi}(p, P, \boldsymbol{P}) \in \mathcal{S}$$

where $h \in O_M$ is equal to 1 in $\overline{V}^{\mu}_{+}$ and vanishes out of V_{+}, we have obviously

$$\tilde{F}^J_{\psi}(a) = \tilde{F}^J_{\varphi}(a)\,, \qquad \tilde{F}^K_{\psi}(a) = 0\,. \qquad (9)$$

Now, in exactly the same way as we proved (6) and under the same conditions we obtain

$$\lim_{\lambda \to \infty} \lambda^N\, [\tilde{F}^J_{\psi}(\vec{a}) - \tilde{F}^K_{\psi}(\vec{a})] = 0 \qquad (10)$$

which implies

$$\lim_{\lambda \to \infty} \lambda^N\, \tilde{F}_{\varphi}(\vec{a}) = 0 \qquad (11)$$

where we have written $\tilde{F}_{\varphi}(a)$ instead of $\tilde{F}^I_{\varphi}(a)$.

Equation (11) obviously holds for all the possible choices of j, j', Y, Y', l, l' introduced at the beginning of this section, and since there is only a finite number of such choices, it holds when $\lambda^2 = \max_{i,i'} (a_i - a_{i'})^2$ goes to infinity without any further restriction on the configuration of the a_i.

According to (7), taking any partial derivative D with respect to the a_i of $\tilde{F}_{\varphi}(a)$ simply amounts to modifying φ, so that

$$\lim_{\lambda \to \infty} \lambda^N\, D\, \tilde{F}(\vec{a}) = 0\,. \qquad (12)$$

Therefore, considering $\tilde{F}_{\varphi}(a)$ as a function of the differences between the a_i, we have:

Theorem 1: *$\tilde{F}_\varphi(\vec{a})$ as well as $D_0 \tilde{F}_\varphi(\vec{a})$, where D_0 is any derivative monomial with respect to the a_i^0, are functions in $\mathcal{S}_{3n}$.*

Let us now introduce the regularized local fields

$$\int dx\, \varphi(x-a)\, A(x) = U(a,1) \int dx\, \varphi(x)\, A(x)\, U(a,1)^{-1}$$

and more general operators

$$B_i(a_i) = U(a_i, 1)\, \boldsymbol{A}_i(\boldsymbol{\varphi}_i)\, U(a_i, 1)^{-1} \quad (\boldsymbol{\varphi} \in \mathcal{S}) \tag{13}$$

of a type considered by HAAG.

We call x_i the former variables a_i, x denoting the family $x_0, x_1, \ldots, x_n$ of four-vector variables, and write

$$F(x) = \langle B_0(x_0)\, B_1(x_1) \ldots B_n(x_n) \rangle_0 .$$

This means essentially that we have taken $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 \otimes \boldsymbol{\varphi}_1 \otimes \cdots \otimes \boldsymbol{\varphi}_n$.

We say that B_i is a Bose or a Fermi field according to whether $\boldsymbol{A}_i$ is a Bose oi a Fermi operator. The definition of the truncated vacuum expectation values $\tilde{F}(x)$ is then obvious and theorem 1 insures that $\tilde{F}(\vec{x})$, $D_0 \tilde{F}(\vec{x})$ where D_0 is any time derivative monomial, belong to $\mathcal{S}_{3n}$ as functions of the differences between the $\vec{x}_i$.

The physical meaning of the theorem appears if we take for instance $\Phi(x) = B(x)\, \Omega$ with $\|\Phi\| = 1$, then

$$\lim_{x^0 = 0,\, \|\vec{x}\| \to \infty} (\Phi(x), B'(0)\, \Phi(x)) = \langle B'(0) \rangle_0$$

i.e. the state $\Phi(x)$ is asymptotically localizable in the sense of KNIGHT[14]).

3. Asymptotic behaviour of the solutions of the Klein-Gordon equation

Let $f(x)$, $x = (x^0, \vec{x})$, be a positive-frequency solution of the Klein-Gordon equation:

$$(\square - m^2)\, f(x) = 0, \qquad m > 0, \qquad f \in \mathcal{S}'. \tag{1}$$

If we write

$$f(x) = (2\pi)^{-2} \int dp\, e^{-ipx}\, \check{f}(p) \tag{2}$$

we have

$$\check{f}(p) = \theta(p^0)\, \delta(p^2 - m^2)\, \tilde{f}(\vec{p}). \tag{3}$$

We assume that $\tilde{f}(\vec{p})$ is infinitely continuously differentiable with compact support*):

$$\tilde{f}(\vec{p}) \in \mathcal{D}_3 . \tag{4}$$

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*) This assumption might be weakened but it is easy to see that the lemma below does not hold for arbitrary normalizable solutions of (1).

Let now u be a vector such that $(u^0)^2 + \vec{u}^2 = 1$ (u is on the euclidean unit sphere) and let

$$f_u(t) \equiv f(t\,u) = (2\,\pi)^{-2} \int dp\; e^{-i(p\,u)\,t}\; \check{f}(p)\,. \qquad (5)$$

We may then write

$$f_u(t) = (2\,\pi)^{-1/2} \int ds\; e^{-i\,s\,t}\, F_u(s)\,,$$

$$F_u(s) = (2\,\pi)^{-3/2} \int dp\; \delta\,(s - p\,u)\; \check{f}(p)\,. \qquad (6)$$

From (3), (4), (6) one sees easily that if the line $t\,u$ intersects with the forward sheet of the hyperboloid $p^2 = m^2$ at a point outside of the support of $\check{f}(p)$, $F_u(s)$ is an infinitely differentiable function of s with compact support.

This is always the case if u is sufficiently close to the light-cone or space-like.

In order to treat the case where $t\,u$ intersects with the forward sheet of the hyperboloid $p^2 = m^2$ inside of the support of $\check{f}(p)$ we put u along the time axis by a Lorentz transformation. Then

$$\begin{aligned} F_u(p^0) &= (2\,\pi)^{-3/2} \int d\vec{p}\; \theta(p^0)\; \delta\,(p^2 - m^2)\; \tilde{f}\,(\vec{p}) = \\ &= \frac{1}{2}\,(2\,\pi)^{-3/2}\, \theta(p^0) \int |\vec{p}\,|\; d(\vec{p}^{\,2})\; \delta\,(\vec{p}^{\,2} - ((p^0)^2 - m^2)) \int d\Omega\; \tilde{f}\,(\vec{p}) = \\ &= \theta\,(p^0 - m)\, \sqrt{(p^0)^2 - m^2}\; g\big((p^0)^2 - m^2\big) \end{aligned} \qquad (7)$$

where

$$g(\vec{p}^{\,2}) = \frac{1}{2}\,(2\,\pi)^{-3/2} \int d\Omega\; \tilde{f}\,(\vec{p})$$

g is infinitely continuously differentiable on the closed positive semi-axis.

We may also write

$$F_u(p^0) = \sqrt{p^0 - m}\; \tilde{g}\,(p^0 - m)\,,$$

where

$$\tilde{g}\,(p^0 - m) = \theta\,(p^0 - m)\, \sqrt{p^0 + m}\; g\,\big((p^0)^2 - m^2\big)\,.$$

Then

$$\begin{aligned} f_u(t) &= (2\,\pi)^{-1/2} \int ds\; e^{-i\,s\,t}\, \sqrt{s + m}\; \tilde{g}\,(s - m) = \\ &= (2\,\pi)^{-1/2}\, e^{i\,m\,t} \int ds\; e^{-i\,s\,t}\, \sqrt{s}\; \tilde{g}(s)\,. \end{aligned} \qquad (8)$$

We may write

$$\sqrt{s}\; \tilde{g}(s) = \sqrt{s}\; \tilde{g}(0)\, e^{-s} + \sqrt{s}\,\big(\tilde{g}(s) - \tilde{g}(0)\, e^{-s}\big)$$

where the Fourier transform of the first term decreases like $|t|^{-3/2}$ in view of the formula

$$(2\,\pi)^{-1/2} \int_0^\infty ds\; e^{-i s t}\, s^{-1/2} = \frac{1+i}{2}\, t^{-1/2}\,. \tag{9}$$

The second derivative of the second term is absolutely integrable, so that the Fourier transform of this term is bounded by $a(u)\,|t|^{-2}$.

Allowing again u to move, we have shown that

$$|f_u(t)| < A(u)\,|t|^{-3/2} \tag{10}$$

where $A(u)$ is continuous.

Furthermore, when u is outside of a certain cone, contained in the future light cone and determined by the support of $\check{f}(p)$, $f_u(t)$ decreases faster in $|t|$ than any inverse polynomial, uniformly in u on the compacts of the unit euclidean sphere. From this we may now conclude

Lemma:

1. $\max\limits_{\vec{x}} |f(t, \vec{x})|$ *decreases like* $t^{-3/2}$ *when* $t \to +\infty$,

2. $\int d\vec{x}\, |f(t, \vec{x})|$ *does not increase faster than* $t^{3/2}$ *when* $t \to +\infty$.

The second statement comes from the fact that the region in which $f(t, \vec{x})$ does not decrease faster than any inverse polynomial has a volume which increases like t^3.

These results obviously extend to negative-frequency solutions of (1).

4. HAAG's strong convergence asymptotic condition and the construction of asymptotic fields

Using the space-like asymptotic condition (theorem 1) and the asymptotic behaviour of the solutions of the Klein-Gordon equation (lemma), HAAG has shown that one may construct asymptotic states as strong limits in Hilbert space.

Let $f_i(x)$ be positive frequency solutions of the Klein-Gordon equation

$$\check{f}_i(p) = \theta(p^0)\,\delta(p^2 - m_i^2)\,\tilde{f}_i(\vec{p})\,, \qquad \tilde{f}_i \in \mathcal{D}_3$$

and let

$$B_i^f(x_i^0) = 2\,\pi\, i \int d\vec{x}_i \left[f_i(x_i)^* \frac{\partial}{\partial x_i^0} B_i(x_i) - \frac{\partial}{\partial x_i^0} f_i(x_i)^* B_i(x_i)\right]. \tag{1}$$

Then, HAAG's result is the following

Theorem 2: *We assume that the state $B_i(x_i)^* \Omega$ belongs to a discrete irreducible representation Γ_i with mass m_i of the covering group of the inhomogeneous proper Lorentz group, and that $B_i(x_i)\,\Omega = 0$. Let $\Phi(t)$ be a vector obtained by applying to the vacuum a product of n operators $B_i^f(t)^*$ or $B_i^f(t)$. Then*

$$\lim_{t \to +\infty} \Phi(t) = \Phi_{\text{out}} \quad \text{and} \quad \lim_{t \to -\infty} \Phi(t) = \Phi_{\text{in}}$$

exist in the norm and define asymptotic states.

We will also use the symbol Φ_{ex} where ex may be replaced consistently by in or out. For a discussion of the physical reasons which justify the interpretation of the Φ_{ex} as asymptotic states, we refer the reader to HAAG's papers[9])[10])[4]).

The necessity of finding suitable fields B_i brings limitations to the construction of these asymptotic states.

We know that there may exist in $\mathfrak{H}$, apart from the vacuum, discrete irreducible representations of the covering group of the inhomogeneous proper Lorentz group corresponding to positive masses and different spin values[23]). These representations generate a subspace $\mathfrak{H}_1$ of $\mathfrak{H}$.

Our formulation, stated below, of the axiom of completeness of the asymptotic states implies that there exist fields B_i^* such that, applied to the vacuum, they yield vectors belonging to representations of a family which already generates $\mathfrak{H}_1$.

In a physically reasonable theory this should follow from spectral properties connected with the stability condition and selection rules.

In order to prove theorem 2, we consider the expression

$$\left\langle \frac{d}{dt}\,\Phi(t) \,\middle|\, \frac{d}{dt}\,\Phi(t) \right\rangle ,$$

perform the derivations and expand into a sum of products of truncated vacuum expectation values. These are of the form

$$I(t) = \int d\vec{x}_0\, d\vec{x}_1 \ldots d\vec{x}_k\; f_0'(\vec{x}_0, t)\; f_1'(\vec{x}_1, t) \ldots f_k'(\vec{x}_k, t)\; \tilde{F}'(\vec{x}_1 - \vec{x}_0, \ldots, \vec{x}_k - \vec{x}_{k-1})$$

where $\tilde{F}' \in \mathscr{S}_{3k}$. It follows then from the lemma of section 3 that $I(t)$ behaves like $|t|^{-3/2\,(k-1)}$ at infinity. Because of our assumptions, factors with $k = 0$ do not appear. On the other hand, terms containing only factors with $k = 1$ vanish because of the identity $d/dt\, B_i^f(t)^*\, \Omega = 0$. Finally we see that $\| d/dt\, \Phi(t) \|$ decreases at infinity like $|t|^{-3/2}$ so that this expression is integrable and $\Phi(t)$ has strong limits as $t \to \pm\infty$.

We supplement this argument of HAAG by a series of remarks.

1. Φ_{ex} is independent of the particular Lorentz frame used to define it.

To see this take an infinitesimally different Lorentz frame, $\Phi(t)$ becomes $\Phi(t) + d\Phi(t)$ but an argument similar to the above one shows that $\lim_{t\to\pm\infty} \| d\Phi(t)\| = 0$.

2. The scalar product of two ingoing or of two outgoing asymptotic state vectors Φ_{ex} is a sum of products of factors of the form $\langle B'^{f'}(t)\; B^f(t)^*\rangle_0$ which are independent of t. These products are preceeded by a $\pm$ sign as required by the definition of truncated vacuum expectation values when Fermi fields are present.

We introduce now the following new assumption in the theory:

Axiom of completeness of the asymptotic states

The finite linear combinations of ingoing state vectors Φ_{in} defined by (2) form a dense subspace D_{in} in Hilbeıt space.

Because of the TCP theorem[12]), a similar statement holds for the outgoing states.

3. Linear operators B^f_{out}, $(B^f_{out})^*$, B^f_{in}, $(B^f_{in})^*$ are defined on D_{out}, D_{in} by

$$B^f_{ex}\Phi_{ex} = \lim_{t\to\pm\infty} B^f(t)\;\Phi(t),\; (B^f_{ex})^*\,\Phi_{ex} = \lim_{t\to\pm\infty} \big(B^f(t)\big)^*\,\Phi(t)\,. \tag{2}$$

All we have to check is that if $\Psi(t) = \sum_{i=0}^{l} \Phi_i(t)$ and $\Phi_{ex} = 0$ then $\lim_{t\to\pm\infty} B^f(t)\;\Psi(t) = 0$, but since the vectors $\Psi(t)$, $B^f(t)^*\,B^f(t)\;\Psi(t)$ both have a limit, that of $\Psi(t)$ being zero, we have indeed $\lim_{t\to\pm\infty} \big(\Psi(\mathrm{t}),\; B^f(t)^*\;B^f(t)\;\Psi(t)\big) = 0$.

Obviously $(B^f_{ex})^*$ is the hermitean conjugate of B^f_{ex}. Let

$$\begin{aligned} f_{\alpha_1\ldots\alpha_{2s}}(x) &= (2\pi)^{-2}\int dp\; e^{-ipx}\;\theta(p^0)\;\delta\,(p^2 - m^2)\;\tilde f_{\alpha_1\ldots\alpha_{2s}}(\vec p)\,, \\ g_{\alpha_1\ldots\alpha_{2s}}(x) &= (2\pi)^{-2}\int dp\; e^{-ipx}\;\theta(p^0)\;\delta\,(p^2 - m^2)\;\tilde g_{\alpha_1\ldots\alpha_{2s}}(\vec p)\,. \end{aligned} \tag{3}$$

$\tilde f_{\alpha_1\ldots\alpha_{2s}}$, $\tilde g_{\alpha_1\ldots\alpha_{2s}} \in \mathscr{S}^*$, be positive frequency solutions of the Klein-Gordon equation, symmetric with respect to the undotted spinor indices $\alpha_1, \ldots, \alpha_{2s}$.

Let

$$I = \begin{pmatrix}1 & \\ & 1\end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} & -i\\ i & \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & \\ & -1\end{pmatrix} \tag{4}$$

and

$$[\partial^{\dot\alpha\beta}] = I\,\partial_0 + \sum_{i=1}^{3}\sigma_i^T\,\partial_i\,, \qquad [p^{\dot\alpha\beta}] = I\,p^0 - \sum_{i=1}^{3}\sigma_i^T\,p_i \tag{5}$$

where σ_i^T is the transposed of σ_i.

If the following expression

$$(f, g) = 2\pi i \int \vec{dx}\, f^*_{\dot\alpha_1 \ldots \dot\alpha_{2s}}(x)\, \overleftrightarrow{\partial}_0 \left(-\frac{1}{i m}\, \partial^{\dot\alpha_1 \beta_1}\right) \ldots \left(-\frac{1}{i m}\, \partial^{\dot\alpha_{2s} \beta_{2s}}\right) \times \times g_{\beta_1 \ldots \beta_{2s}}(x) \tag{6}$$

makes sense, it defines a Lorentz invariant scalar product of f and g.

We may then write

$$(f, g) = 2\pi i \int \vec{dx} \int dp_1\, \theta(p_1^0)\, \delta(p_1^2 - m^2) \int dp_2\, \theta(p_2^0)\, d\,(p_2^2 - m^2) \times$$
$$\times (-i)\,(p_1^0 + p_2^0)\, e^{i p_1 x}\, e^{-i p_2 x}\, \tilde f^*_{\dot\alpha_1 \ldots \dot\alpha_{2s}}(\vec{p}_1)\, \frac{p^{\dot\alpha_1 \beta_1}}{m} \cdots \frac{p^{\dot\alpha_{2s} \beta_{2s}}}{m}\, \tilde g_{\beta_1 \ldots \beta_{2s}}(\vec{p}_2) =$$
$$= \int dp\, \theta(p^0)\, \delta(p^2 - m^2)\, \tilde f^*_{\dot\alpha_1 \ldots \dot\alpha_{2s}}(\vec{p})\, \frac{p^{\dot\alpha_1 \beta_1}}{m} \cdots \frac{p^{\dot\alpha_{2s} \beta_{2s}}}{m}\, \tilde g_{\beta_1 \ldots \beta_{2s}}(\vec{p})\,. \tag{7}$$

With the help of a function $\varphi \in \mathcal{S}$, we have defined an operator $B(x)$, and we assume that the vector $B(x)^* \Omega$ belongs to the representation Γ with mass m and spin s.

Every vector Φ in Γ may be represented uniquely by a family $h_{\alpha_1 \ldots \alpha_{2s}}$ of positive-frequency solutions of the Klein-Gordon equation normalizable in the sense of equation (6)+). We introduce only undotted spinor indices in order to avoid the introduction of subsidiary conditions on the free fields[7]).

Because of remark 1 and the completeness of asymptotic states, it is obvious that B^f_{ex} is completely determined by $(B^f)^* \Omega$ and therefore by the corresponding $h_{\alpha_1 \ldots \alpha_{2s}}$. It is furthermore an anti-linear functional of $h_{\alpha_1 \ldots \alpha_{2s}}$ and we may write $B^f_{\text{ex}} = B_{\text{ex}}\,(\tilde h_{\alpha_1 \ldots \alpha_{2s}})$. We will now show that $B_{\text{ex}}(\tilde h_{\alpha_1 \ldots \alpha_{2s}})$ is defined for all $\tilde h_{\alpha_1 \ldots \alpha_{2s}} \in \mathcal{D}$.

First, one checks easily that if $g_{\alpha_1 \ldots \alpha_{2s}}$ is associated with the vector $(2\pi)^2\, B(0)^* \Omega$, then

$$B^f_{\text{ex}} = B_{\text{ex}}\,(\tilde h_{\alpha_1 \ldots \alpha_{2s}}) = B_{\text{ex}}\,(\tilde f\, \tilde g_{\alpha_1 \ldots \alpha_{2s}})\,. \tag{8}$$

Let now

$$B'(0) = \int_{C_2} d\mu(A)\, \varphi(A)\, U(A, 0)\, B(0)\, U(A, 0)^{-1}$$

where $\varphi \in \mathcal{D}$ and has its support in a sufficiently small neighbourhood of unity in C_2 (covering group of the homogeneous proper Lorentz group). This transformation $B \to B'$ corresponds to a simple change $\varphi \to \varphi'$,

+) $h_{\alpha_1 \ldots \alpha_{2s}}$ is completely determined by $(\Phi, U(a, A)\, \Phi)$ i. e. in our case by the WIGHTMAN functions.

but has the effect that $\tilde{g}_{\alpha_1 \ldots \alpha_{2s}}$ becomes $\tilde{g}'_{\alpha_1 \ldots \alpha_{2s}}$ which is infinitely continuously differentiable. Therefore $\tilde{h}'_{\alpha_1 \ldots \alpha_{2s}} = \tilde{f}\, \tilde{g}'_{\alpha_1 \ldots \alpha_{2s}} \in \mathcal{D}$.

The tensor formed by the $2s+1$ numbers $\tilde{g}'_{\alpha_1 \ldots \alpha_{2s}}(0)$ may be assumed to be different from zero, and because of the irreducibility of the corresponding representation of the covering group of the three dimensional rotation group, one may by applying elements of this group obtain $2s+1$ linearly independent tensors $\tilde{g}'^{m}_{\alpha_1 \ldots \alpha_{2s}}(0)$, $1 \leqq m \leqq 2s+1$. The corresponding functions $\tilde{g}'^{m}_{\alpha_1 \ldots \alpha_s}(p)$ are then linearly independent in a neighbourhood of the origin, and it is thus possible to find a linear combination

$$\tilde{h}'_{\alpha_1 \ldots \alpha_{2s}}(\vec{p}) = \sum_{m=1}^{2s+1} \tilde{f}^m(\vec{p})\, \tilde{g}'^{m}_{\alpha_1 \ldots \alpha_{2s}}(\vec{p}) \,, \qquad \tilde{f}^m \in \mathcal{D} \tag{9}$$

which has only one non-vanishing component, this component being different from zero at the origin.

By multiplying $\tilde{h}'_{\alpha_1 \ldots \alpha_{2s}}(p)$ with functions in $\mathcal{D}$, acting with C_2 and taking linear combinations it is then easy to obtain any $\tilde{h}_{\alpha_1 \ldots \alpha_s} \in \mathcal{D}$. We may now define asymptotic fields corresponding to the representation Γ for negative as well as positive-frequency solutions of the Klein-Gordon equation by

$$A^{\Gamma}_{\text{ex}}(h) = B_{\text{ex}}(\tilde{h}) \,, \qquad A^{\Gamma}_{\text{ex}}(h^*) = (-)^{2s+1} \left(B^*_{\text{ex}}(\tilde{h})\right)^* \tag{10}$$

where h is a positive frequency solution of the Klein-Gordon equation. Since the vacuum expectation values of products of A_{ex} are those of free fields, the A_{ex} are free fields.

The above considerations show furthermore that any vector obtained by applying a polynomial in the $A(h)$, $\tilde{h} \in \mathcal{D}$, to the vacuum may be obtained directly by HAAG's limiting procedure. The S-matrix may thus be constructed from our knowledge of the Wightman functions, it is obviously unitary and TCP invariant[12]).

In conclusion I wish to thank both Professor R. JOST and Professor M. FIERZ for interesting discussions and very helpful criticisms.

Appendix

In this appendix, some facts about the Hilbert space $\mathfrak{H}$ of a field theory have been collected for the convenience of the reader.

First, we want to point out that it is equivalent to introduce the fields as 'operator-valued tempered distributions' or as 'operator-valued distributions' and to assume that the vacuum expectation values are tempered distributions as is done by WIGHTMAN[21]).

Let thus D_0 be the linear manifold obtained by applying polynomials in the $A^{\varkappa}(\varphi)$, $\varphi \in \mathcal{D}$, to Ω. Since also $A^{\varkappa}(\varphi)^*$ is considered as field operator and is defined on D_0, which is assumed to be dense in $\mathfrak{H}$, it follows that the operator $A^{\varkappa}(\varphi)$ has a closed extension. The intersection H_0 of the domains of the smallest closed extensions of all $A^{\varkappa}(\varphi)$ may thus be strictly bigger than D_0. That this is the case is seen as follows.

The vector

$$A^{\varkappa_0}(\varphi_0)\, A^{\varkappa_1}(\varphi_1) \ldots A^{\varkappa_n}(\varphi_n)\, \Omega\,, \qquad \varphi_i \in \mathcal{D}$$

is a continuous function of $\varphi_0 \otimes \varphi_1 \otimes \cdots \otimes \varphi_n$ considered as an element of $\mathcal{S}_{4(n+1)}$. Extension by continuity allows then to define vectors

$$\boldsymbol{A}(\boldsymbol{\varphi})\, \Omega = \int dx_0\, dx_1 \ldots dx_n\, \boldsymbol{\varphi}(x_0, x_1, \ldots, x_n)\, A^{\varkappa_0}(x_0)\, A^{\varkappa_1}(x_1) \ldots A^{\varkappa_n}(x_n)\, \Omega$$

$$\boldsymbol{\varphi} \in \mathcal{S}_{4(n+1)}$$

and the linear manifold D_1 spanned by them is contained in H_0.

It is also obvious that one may define operators $\boldsymbol{A}(\boldsymbol{\varphi})$ on D_1 by $\boldsymbol{A}(\boldsymbol{\varphi}_1)\, \boldsymbol{A}(\boldsymbol{\varphi}_2)\, \Omega = \boldsymbol{A}(\boldsymbol{\varphi}_1 \otimes \boldsymbol{\varphi}_2)\, \Omega$ and that if $\Phi \in D_1$, $\boldsymbol{A}(\boldsymbol{\varphi})\, \Phi$ is a continuous function $S \to \mathfrak{H}$. This settles our first point.

We remark now that $\mathfrak{H}$ is necessarily separable. This follows immediately from the density of D_1 in $\mathfrak{H}$, the continuity of $A(\boldsymbol{\varphi})\, \Omega$ and the fact that $\mathcal{S}$ is a separable space*).

We conclude with a remark on the completeness axiom.

We will say that a bounded operator C commutes (weakly) with an operator $A(\varphi)$, $\varphi \in \mathcal{D}$ if

$$\big(C^* \Phi,\, A(\varphi)\, \Psi\big) = \big(A(\varphi)^* \Phi,\, C\, \Psi\big) \tag{1}$$

for all Φ, Ψ in the dense domain of definition of all $A(\varphi)$. This definition has the advantage that no assumption has to be made about the density of D_0 or the range of C.

Then Haag formulates the completeness axiom (irreducibility) as follows[11])[3]) (H): if a bounded operator C commutes with all $A(\varphi)$, then C is a multiple of the identity.

We will now show that (H) is a theorem in the frame of the Wightman axioms including completeness and the existence of a positive smallest mass in the theory. Consider any operator C satisfying (1) for all $A(\varphi)$.

It then also satisfies (1) when $A(\varphi)$ is replaced by $\boldsymbol{A}(\boldsymbol{\varphi})$ and $\Phi, \Psi \in D_1$. We may suppose $C\,\Omega \neq 0$, since if $C\,\Omega = 0$, $C\,\Phi = 0$ for any $\Phi \in D_1$ and therefore $C = 0$.

---

*) Both $\mathcal{D}$ and $\mathcal{S}$ are separable as one may check directly. For $\mathcal{S}$, this follows also (see ref. [15]), p. 373) from the fact that it is a Fréchet space and a Montel space[20]).

We have thus $\|C\,\Omega\| = \varrho > 0$, $\langle C\rangle_0 = \alpha$, $|\alpha| \leqq \varrho$.

Let $L\,(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_r)$ be a linear combination of the $\boldsymbol{A}(\boldsymbol{\varphi})$ such that $\|(C - L(\boldsymbol{\varphi}))\,\Omega\| < \varepsilon$, then $\varrho\,\varepsilon > |(\Omega, c^*\,c\,\Omega) - (\Omega, L(\boldsymbol{\varphi})^*\,C\,\Omega)|$.

But we may, by multiplying the $\boldsymbol{\varphi}_i$ by suitable functions in p-space obtain a new operator $L(\boldsymbol{\psi})$ such that

$$L(\psi)\,\Omega = L(\boldsymbol{\varphi})\,\Omega\,, \qquad \Omega\,L(\psi) = (\Omega, L(\boldsymbol{\varphi})\,\Omega)\,\Omega\,.$$

Then

$$p\,\varepsilon > |p^2 - (\Omega, L(\psi)^*\,C\,\Omega)| = |p^2 - (\Omega, C\,L(\psi)^*\,\Omega)| =$$

$$= |p^2 - \alpha\,(\Omega, L(\psi)^*\,\Omega)|$$

and finally since

$$\lim_{\varepsilon \to 0}\,(\Omega, L(\psi)\,\Omega) = \alpha$$

we have

$$\alpha\,\alpha^* = p^2, \qquad C\,\Omega = \alpha\,\Omega\,.$$

From this it follows immediately that $C\,\Phi = \alpha\,\Phi$ for any Φ in D_1 and C is therefore a multiple of the identity.

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