Phase shifts and model potentials

Phase Shifts and Model Potentials

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1. Introduction

We wish to present a brief discussion of the problem, of fundamental interest in nuclear physics, concerning the correspondence of a set of measured phase shifts as function of the energy in an elastic scattering, to a possible potential which may serve as model to describe a nuclear collision.

There is of course a large choice of experimental situations and we have to restrict ourselves to very simple cases in order to elucidate without too involved mathematics the type of information which can be derived from such a model potential[1]).

If, for instance, we assume knowledge of the dependence of the scattering amplitude for a certain range of values of energy in a *s*-wave non-relativistic scattering of a spinless particle by an external field, our aim is to find the spatial dependence of the corresponding model potential in the appropriate space region.

It is manifest, from very simple physical considerations, that a knowledge of the scattering amplitude at low energies would enable us to reproduce the behaviour of the potential at large distances only. Conversely a more refined knowledge of the potential at smaller distances should involve information regarding cross sections at high energies.

We wish to anticipate here, as will quantitatively be demonstrated below, the conclusion that the asymptotic behaviour of the potential at large distances is very sensitive to the position and the nature of singularities of the scattering matrix $S(E)$, as function of the energy E[2]). On the other hand, any knowledge of the potential at small distances, if deduced from *s*-eigenwaves only, is insensitive to the analytical properties

[1]) It is a pleasure to acknowledge here the many stimulating discussions on kindred problems, held with Prof. P. SCHERRER during my unforgettable period of studies in his famous Institute. I apologize for not being able to treat here the type of nuclear collisions which have lately been of special interest to Professor SCHERRER. Yet many of the facts appearing in our present discussion are essential to the mastering of more complex situations.

[2]) The connection between the singularities of the S matrix and the behaviour of the potential at large distances, in absence of bound states, has been elucidated by R. JOST [1]. Numbers in brackets refer to References, page 241.

of $S(E)$ at low energies. In fact, it is determined by the analytical behaviour of a GREEN's function near *its spectrum* and for fixed spatial points (WIGNER's $R(E)$ function), for large values of E.

It is a difficult task to extract such information from s-wave experiments, and, therefore, it is preferable to avail oneself to high energy collision data. Since a large number of partial waves are involved in such cases, it becomes necessary to generalize our considerations in order to establish a correspondence between a model potential and the analytical behaviour of the total scattering amplitude as function of the scattering angle at a fixed energy. It is, however, also instructive to consider the problem of the internal region when $R(E)$ is known. This problem will be discussed in section 2.

The mathematical tool necessary to reach the conclusions mentioned above in the most straightforward manner, consists in relating the model potential $U(x)$ to some appropriate kernels $K_e(x, x')$ and $K_i(x, x')$

$$K_e(x,x) = \frac{1}{2}\int_x^\infty U(x')\,dx', \quad K_i(x,x) = \frac{1}{2}\int_0^x U(x')\,dx' \tag{1}$$

K_e and K_i will respectively be called the kernels of MARCHENKO and of GEL'FAND and LEVITAN. The Marchenko kernel K_e gives a very good approximation for large x, when $S(E)$ is known, and is therefore well suited for representation of $U(x)$ in the external region, whereas K_i serves as an approximation for small x, when $R(E)$ is known, and hence can yield the model potential in the internal region.

2. The Model Potential in the Internal Region

We begin with a discussion of the determination of the potential at small distances. Here it is the kernel K_i which plays the important role.

K_i can be defined in several ways, for instance by means of the following integral representation

$$K_i(x, x') = \int_{-\infty}^{+\infty} [d\varrho_0(E') - d\varrho(E')]\,\Phi(E', x)\,\Phi_0(E', x');\ x' \leqslant x \tag{2}$$

the integration taking place along the entire energy spectra belonging to the hermitian hamiltonians H and H_0. Φ and Φ_0 respectively correspond to the eigenwaves of H, and of H_0, which may be termed the hamiltonian of the free motion[3]).

$$H\Phi = E\Phi, \quad H_0\Phi_0 = E\Phi_0, \quad H - H_0 = U(x)$$

[3]) It is not necessary for H_0 to be merely the operator of the kinetical energy: it may correspond to a motion in a given potential whose solutions are known. Some restrictions must be imposed upon the difference $H - H_0 = U(x)$ in order that equation (2) be meaningful. It is sufficient that $\int_0^x U(x')\,dx'$ exists.

Φ and Φ_0 are normalized in such a way that they behave like x for $x \to 0$. $d\varrho$ and $d\varrho_0$ are the differentials of the spectral functions $\varrho(E)$ and $\varrho_0(E)$ which enter into the well-known integral representation of Green's function

$$G(E, x, x') = \int_{-\infty}^{+\infty} d\varrho(E') \frac{\Phi(E', x)\ \Phi(E', x')}{E' - E} \tag{3}$$

The kernel $K_i(x,x')$ is part of the operator which links the Φ's to the Φ_0's.

$$\Phi(E, x) = \Phi_0(E, x) + \int_0^x K_i(x, x')\, \Phi_0(E_j\, x')\, dx' \tag{4}$$

This important equation, known in the literature as the Gel'fand and Levitan integral equation [2], will be derived in section 5 [4]).

It is now clear, as consequence of equations (4) and (2), that

$$K_i^{(0)}(x, x') = \int_{-\infty}^{+\infty} d(\varrho_0 - \varrho)\, \Phi(E', x)\, \Phi_0(E', x') \tag{5}$$

approximates the kernel K_i for small x, and that better approximations can be obtained by iteration of equation (2), using equation (4).

GREEN's function equation (3) may also be written as an integral along a complex path of integration C_i, in the plane of E, which excludes its spectrum as in figure 1:

$$G(E, x, x') = \frac{1}{2\pi i} \int_{C_i} R(E')\, dE' \frac{\Phi(E', x)\ \Phi^*(E', x')}{E' - E}\ ;\ x \leq x' \tag{6}$$

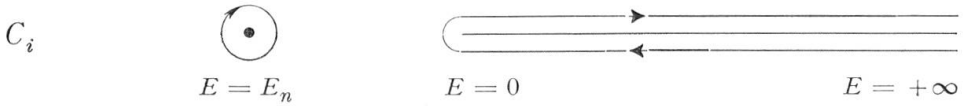

Figure 1

$R(E)$, called a Wigner function, is characterized by simple analytical properties [4].

$R(E)$ is regular for Im $E \neq 0$ and has possible simple poles (bound states) for negative values of E, such as the point $E = E_n$ in Figure 1, with negative residues

$$R(E) \sim -\frac{1}{N\Phi_n} \frac{1}{E - E_n} \quad \text{for } E \text{ near } E_n\,.$$

$N\Phi_n$ is the norm of the bound state eigenfunction: $\Phi(E_n, x)$.

[4]) Our derivation in section 5 is slightly different from the usual approach and it is based essentially on the existence and the analytical properties of the Green functions. It serves also to generalize the Gel'fand and Levitan equation in more complex cases. We refer to our recent paper [3], and take this opportunity to clarify the unfortunate formulation employed therein.

Since Im $R(E)$ has the same sign as Im E and since $R(E^*) = R^*(E)$, it follows from the two equivalent representations, equation (3) and equation (6), of GREEN's function that

$$\frac{1}{\pi} \text{Im } R(E) = \frac{d\varrho}{dE} > 0 \quad \text{for } E \text{ real and positive}.$$

$R(E)$ itself can be expressed as an integral along the energy spectrum

$$R(E) = R(E_0) + \int_{-\infty}^{+\infty} \left(\frac{d\varrho(E')}{E' - E} - \frac{d\varrho(E')}{E' - E_0} \right),$$

where $d\varrho$ is a step function for E' negative with jumps $1/(N\Phi_n)$ at the bound states.

For large E', ϱ and ϱ_0 have the same asymptotic behaviour, so that

$$R_0(E) - R(E) = \int_{-\infty}^{+\infty} \frac{d\varrho_0(E') - d\varrho(E')}{E' - E} . \tag{7}$$

It is obvious that an integral form of the kernel K_i equivalent to equation (2) reads:

$$K_i(x, x') = \frac{1}{2\pi i} \int_{C_i} [R_0(E') - R(E')\, dE'\, \Phi(E', x)\, \Phi_0(E', x') . \tag{8}$$

Therefore the behaviour of $R(E)$ and $R_0(E)$ near their spectra is essential to the evaluation of $U(x)$ in the internal region [see equation (1)].

3. The Model Potential in the External Region

The discussion of the behaviour of the model potential at large distances can be carried out in complete analogy with that given in section 2 for the internal region.

One has to introduce the Marchenko kernel

$$K_e(x, x') = \frac{1}{2\pi} \int_{C_e} dp' [S(p') - S_0(p')]\, f^{\text{out}}(p', x)\, f_0^{\text{out}}(p', x') . \tag{9}$$

In this instance the scattering matrices $S(p)$ and $S_0(p)$ (where $E = p^2$) corresponding to the hamiltonians H and H_0, play the dominant role.

f^{out} is the outgoing wave normalized to have the behaviour of e^{ipx} as $x \to \infty$.

The path of integration C_e is along all the real values of the momentum p (with a small positive imaginary part) and in addition avoids those singularities of S and S_0 which correspond to bound states (see Figure 2).

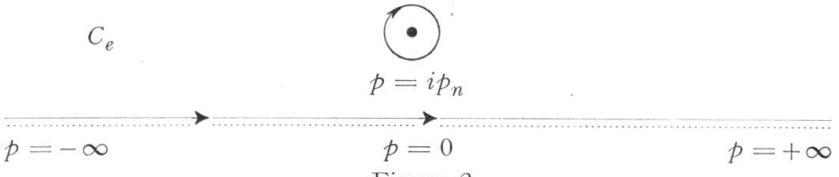

Figure 2

In the complex p half-plane Π (which corresponds to the complex E plane considered in section 2 with a cut along the positive real axis) the eigenwaves f^{out} and f_0^{out} are analytical functions of p[5]).

Therefore K_e and $U(x)$ in the external region [equation (1)] are essentially determined by the behaviour of $S(p)$ and $S_0(p)$ near their singularities, in the Π half-plane, which are not bound states.

$K_e(x, x')$ enters in the integral equation

$$f^{\text{out}}(p, x) = f_0^{\text{out}}(p, x) + \int_x^\infty K_e(x, x')\, f^{\text{out}}(p, x')\, dx' \tag{10}$$

as it will be proved in section 5, together with the form given by equation (9) of the Marchenko kernel K_e.

Now, a first approximation of $K_e(x, x')$ valid for large x is

$$K_e^{(0)}(x, x') = \frac{1}{2\pi} \int_{C_e} dp' [S(p') - S_0(p')]\, f_0^{out}(p'x)\, f_0^{out}(p'x'), \tag{11}$$

as follows from the equations (9) and (10). Better approximations for large x are obtained by the iterative procedure already mentioned for the case of the internal region. As we shall discuss in the next section, one can in certain special cases even evaluate $K_e(x, x')$ exactly.

4. Simple Examples

H_0 is chosen to be the operator of the kinetical energy with the required eigenfunctions

$$\begin{cases} f_0^{\text{out}}(p, x) = e^{ipx} \\ S_0(p) = 1 \end{cases} \qquad \begin{cases} \Phi_0(E, x) = \dfrac{\sin \sqrt{E}\, x}{\sqrt{E}} \\ R_0(E) = i\sqrt{E} \end{cases} \qquad 0 < \arg E < 2\pi$$

We consider now an S matrix with two distinct simple poles at the points $p = i\alpha$ and $p = i\beta$ of the half-plane $\Pi(\text{Im } p > 0)$. Of these two poles, one only, for instance $p = i\alpha$, is a bound state.

$$S(p) = \frac{p + i\alpha}{p - i\alpha} \frac{p + i\beta}{p - i\beta} \quad \text{and} \quad \beta > \alpha\ . \tag{12}$$

For a first approximation of the model potential at large distances, we have to evaluate the kernel given by equation (11)

$$K_e^{(0)}(x, x') = \frac{1}{2\pi} \int_{C_c} \left(\frac{p + i\alpha}{p - i\alpha} \frac{p + i\beta}{p - i\beta} - 1 \right) e^{ip(x+x')}\, dp = -2\beta \frac{\beta + \alpha}{\beta - \alpha} e^{-\beta(x+x')} \tag{13}$$

from this

$$U^{(0)}(x) = -2 \frac{d}{dx} K_e^{(0)}(x, x) = 8\beta^2 \frac{\alpha + \beta}{\alpha - \beta} e^{-2\beta x}. \tag{14}$$

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[5]) A sufficient condition for the existence of K_e is that $\int_0^\infty x |U(x)|\, dx < \infty$.

The simple form, equation (12), we have chosen for the S matrix permits an exact determination of the model potential. In fact, from equations (10) and (11) one deduces an integral equation for the kernel K_e itself

$$K_e(x, x') = K_e^{(0)}(x, x') + \int_x^\infty K_e(x, x'')\, K_e^{(0)}(x'', x')\, dx''. \tag{15}$$

It is now clear that if $K_e(x, x')$ degenerates into the product of two functions of x and x', as in our case [see equation (13)], a solution

$$K_e(x, x') = K_e^{(0)}(x, x') \Big/ \left(1 - \int_x^\infty K_e^{(0)}(x, x')\, dx'\right) \tag{16}$$

exists. Substituting equation (13) in equation (16), we get

$$K_e(x, x') = 2\beta\,(\alpha + \beta)\, e^{-2\beta\,(x+x')} \Big/ [(\alpha - \beta) - (\alpha + \beta)\, e^{-2\beta x}].$$

The corresponding potential is

$$U(x) = -2\,\frac{d}{dx}\,\frac{2\,\beta\,(\alpha+\beta)\, e^{-2\beta x}}{(\alpha-\beta)-(\alpha+\beta)\, e^{-2\beta x}}\,, \tag{17}$$

of which equation (14) is a first asymptotic approximation, valid for large x.

The extension of these considerations to the case of S matrices with any finite number of simple poles is straightforward and leads always to an exact determination of the potential[6]).

The value of $U(x)$, equation (17), for $x = 0$ is

$$U(0) = 2\,(\alpha^2 - \beta^2). \tag{18}$$

This value can be found using the first approximation of the Gel'fand and Levitan kernel, equation (8). One has

$$K_e^{(0)}(x,x) = \frac{1}{N\Phi^{(0)}(\alpha, x)}\,\Phi^{(0)}(\alpha, x) + \frac{\alpha^2-\beta^2}{\pi}\int_0^\infty \frac{p\,dE}{E+\alpha^2}\,\frac{\sin^2 px}{E}\,.$$

The first term can be neglected, being of order x^2, and the second term yields

$$\frac{\alpha^2-\beta^2}{\pi}\cdot\frac{\pi}{2\,\alpha}\cdot(1 - e^{-2\,\alpha x}) \simeq (\alpha^2 - \beta^2)\, x\,.$$

From this and equation (1)

$$U^{(0)}(0) = 2\,\frac{dK^{(0)}}{dx}\,(x, x)\Big|_{x=0} = 2\,(\alpha^2 - \beta^2)$$

which coincides with the value equation (18) derived from the exact form of the potential.

For a hermitian hamiltonian, the singularities of the corresponding S matrix must lie symmetrically with respect to the imaginary axis in the

[6]) Such potentials are known in the literature as Bargmann potentials.

half-plane Π, or in particular cases on it. To S matrices with poles outside the imaginary axis correspond oscillating potentials in the external region. An infinite number of poles or even branch points may occur in correspondence with potentials having but relatively simple spatial behaviour. The S matrix of an exponential potential, for example, has an infinite number of simple poles on the imaginary p axis. Among these poles the one with the smallest imaginary part is identical to the unique pole of the potential considered above, provided the potentials have the same slope at infinity.

Assuming an S matrix regular everywhere with the exception of a cut from $p = i\,m/2$ to $p = i\,\infty$ on the imaginary p-axis, the kernel

$$K_e^{(0)}(x, x') = \frac{1}{2\pi}\int_{C_e} (S(p) - 1)\; e^{ip(x+x')}\, dp$$

becomes

$$= \frac{1}{2\pi}\int_{\frac{im}{2}+\varepsilon}^{i\infty+\varepsilon} 2\,i\ \mathrm{Im}\ S(p)\; e^{ip(x+x')}\, dp,$$

where use has been made of the property $S(-p^*) = S^*(p)$. Choosing $\mathrm{Im}\ S(i\,p) = 2\,\pi\,C$, one obtains

$$K_e^{(0)}(x,x) = -C\ \frac{e^{-2px}}{x}\Bigg|_{p=m/2}^{p=\infty} \text{ for large } x \quad \simeq \quad C\,\frac{e^{-mx}}{x}$$

which corresponds to a Yukawa potential in the external region.

The approximation of the wave function itself, can be reached at the same time as that of the model potential. For s-waves and $f_0^{\mathrm{out}} = e^{ipx}$

$$K_e(x,x') = \frac{1}{2\pi i}\int_{C_e} (S(p') - 1)\, f^{\mathrm{out}}(p'x)\, e^{ip'x'}\, dp'$$

and equation (10) becomes

$$f^{\mathrm{out}}(px) = e^{ipx} - \frac{1}{2\pi i}\int_{C_e} \frac{S(p')-1}{p+p'}\, f^{\mathrm{out}}(p'x)\, e^{i(p+p')x}\, dp'\,. \tag{19}$$

This can be solved immediately if S has isolated poles only. We shall illustrate it, taking for S the same form, equation (12), as considered before.

Putting $$e^{-ipx}\, f^{\mathrm{out}}\,(p, x) = \varphi(p, x)$$

equation (19) reads

$$\varphi(p, x) = 1 + 2\,i\beta\ \frac{\alpha+\beta}{\alpha-\beta}\ \frac{1}{p+i\,\beta}\ e^{-2\beta x}\,\varphi(i\,\beta, x)\,. \tag{20}$$

Therefore

$$\varphi\,(i\,\beta, x) = 1\ \Big/\ \left[1 - \frac{\alpha+\beta}{\alpha-\beta}\ e^{-2\beta x}\right]$$

and from equations (20) and (19) one deduces

$$f^{\text{out}}(p, x) = e^{ipx}\left\{1 + 2i\beta\,\frac{\alpha+\beta}{p+i\beta}\,\frac{e^{-2\beta x}}{(\alpha-\beta) - (\alpha+\beta)\,e^{-2\beta x}}\right\}$$

which is the exact form of the outgoing wave function corresponding to the potential given by equation (17).

The extension to S matrices with a finite number of poles is straightforward and will not be reported here.

5. On the Derivation of the Integral Equations

[equations (4) and (10)]

We wish to indicate briefly in this final section how the Gel'fand and Levitan equation [equation (4)] and the corresponding equation (10) for the outgoing waves can be derived. We treat the case of s-waves and consider potentials whose Schrödinger equation possesses a unique Green function.

Let us call $\Phi(E, x)$ and $\Phi_s(E, x)$ the two independent solutions of the wave equation $H\Phi = E\Phi$ which satisfy the boundary conditions

$$\begin{aligned} \Phi(E, 0) &= 0 & \Phi_s(E, 0) &= 1 \\ \Phi'(E, 0) &= 1 & \Phi_s'(E, 0) &= 0 \end{aligned} \tag{21}$$

Φ, Φ_s and their derivatives with respect to x are entire functions[3]) of E. A square integrable solution

$$F(E, x) = \Phi_s(E, x) + R(E)\,\Phi(E, x) \tag{22}$$

is obtained if

$$R(E) = -\lim_{x\to\infty} \frac{\Phi s(E, x)}{\Phi(E, x)}\,.$$

$R(E)$ is well defined for $\operatorname{Im} E \neq 0$ and is a meromorphic function of E. Green's function reads then

$$G(E, x, x') = \begin{cases} \Phi(E, x)\; F(E, x') & x < x' \\ \Phi(E, x')\; F(E, x) & x > x' \end{cases} \tag{23}$$

We shall again introduce the same quantities, defined above for the hamiltonian H_0, in the corresponding case of the wave equation $H_0\Phi_0 = E\Phi_0$.

For the derivation of equation (4) we begin by writing

$$\begin{aligned} &\frac{1}{2\pi i}\oint \frac{dE'}{E'-E}\,\{\Phi(E', x)\; F_0(E', x) - \Phi_0(E', x)\; F(E', x)\} \\ &\qquad = \Phi(E, x)\; F_0(E, x) - \Phi_0(E, x)\; F(E, x) \\ &\frac{1}{2\pi i}\oint \frac{dE'}{E'-E}\,\{\Phi(E', x)\; F_0'(E', x) - \Phi_0'(E', x)\; F(E', x) + 1\} \\ &\qquad = 1 + \Phi(E, x)\; F_0'(E, x) - \Phi_0'(E, x)\; F(E, x) \end{aligned} \tag{24}$$

where $0 < \arg E < 2\pi$ and E does not belong to the spectra of H and H_0.

The integration in the complex E plane is carried counterclockwise around the point E. One must now enlarge the path of integration so that it reaches the vicinity of the spectra of H and H_0 while avoiding them. This path will be called C_i and is of the type illustrated in figure 1. For the shift of the integration line, the point $E = \infty$ is harmless, since the eigenwaves of H and H_0 which enter in equation (24) have a common limit for large E. No contribution to the integral will thus result from the arc at infinity. Noting equation (22), one obtains,

$$\Phi(E, x)\, F_0(E, x) - \Phi_0(E, x)\, F(E, x) = \frac{1}{2\pi i} \int_{C_i} \frac{dE'}{E' - E} [R_0(E') - R(E')]\, \Phi(E', x)\, \Phi_0(E', x)$$

$$1 + \Phi(E, x)\, F_0'(E, x) - \Phi_0'(E, x)\, F(E, x) = \frac{1}{2\pi i} \int_{C_i} \frac{dE'}{E' - E} [R_0(E') - R(E')]\, \Phi(E', x)\, \Phi_0'(E', x).$$

Taking into account that

$$\Phi_0'(E, x)\, F_0(E, x) - \Phi_0(E, x)\, F_0'(E, x) = 1 \qquad \text{[see equations (21), (22)]}$$

one deduces furthermore

$$\Phi(E, x) = \Phi_0(E, x) + \frac{1}{2\pi i} \int_{C_i} \frac{dE'}{E' - E}\, \Phi(E', x) \cdot [\Phi_0(E', x)\, \Phi_0'(E, x) - \Phi'(E', x)\, \Phi_0(E, x)]. \tag{25}$$

Hence from

$$\frac{d}{dx}[\Phi_0(E', x)\, \Phi_0'(E, x) - \Phi_0'(E', x)\, \Phi_0(E, x)] = (E' - E)\, \Phi_0(E, x)\, \Phi_0(E', x)$$

it finally follows

$$\begin{cases} \Phi(E, x) = \Phi_0(E, x) + \int_0^x K_i(x, x')\, \Phi_0(E, x')\, dx' \\ K_i(x, x') = \frac{1}{2\pi i} \int_{C_i} dE'\, [R_0(E') - R(E')]\, \Phi(E', x)\, \Phi_0(E', x'), \end{cases}$$

quod erat demonstrandum.

Employing considerations of the same nature, we shall now derive equation (10)[7]. For x large and $0 < \arg E < 2\pi$ we shall write

$$\begin{aligned} \Phi_s(E, x) &\simeq A_s(E)\, e^{-i\sqrt{E}\, x} \\ \Phi(E, x) &\simeq A(E)\, e^{-i\sqrt{E}\, x}. \end{aligned} \tag{26}$$

[7]) This type of equations has been found by V. A. MARCHENKO [5].

Then equation (23) becomes

$$R(E) = - A_s(E) \,/\, A(E)$$

$A(E)$ and $A_s(E)$ are analytical functions of E in the plane Π, with a cut along the positive real E axis. If $f^{\text{out}}(E, x)$ be a solution of the wave equation which for large x and in Π behaves like

$$f^{\text{out}}(E, x) \simeq e^{i\sqrt{E}\,x}, \tag{27}$$

an independent solution is

$$f^{\text{in}}(E, x) = f^{\text{out}}(\omega E, x) \simeq e^{-i\sqrt{E}\,x}, \quad \text{where } \omega = e^{-2in}. \tag{28}$$

Then

$$\Phi(E, x) = A(\omega E)\, f^{out}(E, x) + A(E)\, f^{out}(\omega E, x), \tag{29}$$

since $\Phi(E, x) = \Phi(\omega E, x)$ for every x and since equation (26) must hold for large x.

Similarly

$$\Phi_s(E, x) = A_s(\omega E)\, f^{\text{out}}(E, x) + A_s(E)\, f^{\text{out}}(\omega E, x), \tag{30}$$

furthermore

$$\Phi_s(E, x)\, \Phi'(E, x) - \Phi_s'(E, x)\, \Phi(E, x) = 1$$

$$f^{\text{out}}(\omega E, x)\, f^{\text{out}}(E, x) - f^{\text{out}}(E, x)\, f^{\text{out}}(\omega E, x) = 2i\sqrt{E}$$

[cf. equations (21), (27) and (30)]. From the equations (29) and (30) it follows

$$A_s(E)\, A(\omega E) - A_s(\omega E)\, A(E) = \frac{1}{2i\sqrt{E}}$$

$$f^{\text{out}}(E, x) = 2i\sqrt{E}[A_s(E)\, \Phi(E, x) - A(E)\, \Phi_0(E, x)]$$

which shows the analiticity of $f^{\text{out}}(E, x)$ and $f'^{\,\text{out}}(E, x)$ in the cut plane Π. We finally need the function

$$\psi(E, x) = - \frac{\Phi(E, x)}{2i\sqrt{E}\, A(E)} = \frac{1}{2i\sqrt{E}} [S(E)\, f^{\text{out}}(E, x) - f^{\text{in}}(E, x)] \tag{31}$$

where

$$S(E) = - A(\omega E) \,/\, A(E) \text{ is the } S \text{ matrix.}$$

$\psi(E, x)$ is analytical in the cut plane Π, with exception of the bound states $[A(E) = 0]$ where simple poles appear.

In analogy with equation (24) we write

$$\frac{1}{2\pi i} \oint \frac{dE'}{E' - E} [\psi(E', x)\, f_0^{\text{out}}(E', x) - \psi_0(E', x)\, f^{\text{out}}(E', x)]$$
$$= \psi(E, x)\, f_0^{\text{out}}(E, x) - \psi_0(E, x)\, f^{\text{out}}(E, x)$$

$$\frac{1}{2\pi i} \oint \frac{dE'}{E' - E} [1 + \psi(E', x)\, f_0'^{\,\text{out}}(E', x) - \psi_0'(E', x)\, f^{\text{out}}(E', x)] \tag{32}$$
$$= 1 + \psi(E, x)\, f_0'^{\,\text{out}}(E, x) - \psi_0'(E, x)\, f^{\text{out}}(E, x)$$

for E not belonging to the spectra of H and H_0. One proceeds as before.

The shifting of the integration line to the vicinity of the spectra of H and H_0 is here also permissible, since no contribution comes from the arc at infinity. Furthermore the terms which contain $f^{\text{in}}(E,x)$ are missing [cf. equations (31), (28)]. One obtains:

$$\frac{1}{2\pi i}\int_{C_e}\frac{dE'}{E'-E}\;\frac{1}{2i\sqrt{E'}}\left(S(E')-S_0(E')\right)f^{\text{out}}(E',x)\,f_0^{\text{out}}(E',x)$$
$$=\psi(E,x)\,f_0^{\text{out}}(E,x)-\psi_0(E,x)\,f^{\text{out}}(E,x)$$
$$\frac{1}{2\pi i}\int_{C_e}\frac{dE'}{E'-E}\;\frac{1}{2i\sqrt{E'}}\left(S(E')-S_0(E')\right)f^{\text{out}}(E',x)\,f_0'^{\text{out}}(E',x) \tag{33}$$
$$=1+\psi(E,x)\,f_0'^{\text{out}}(E,x)-\psi_0'(E,x)\,f^{\text{out}}(E,x).$$

The elimination of ψ and the trivial equation

$$f_0^{\text{out}}(E,x)\,f_0'^{\text{out}}(E',x)-f_0'^{\text{out}}(E,x)\,f_0^{\text{out}}(E',x)$$
$$=(E'-E)\int_x^\infty f^{\text{out}}(E,x')\,f^{\text{out}}(E',x')\,dx'$$

leads to the proof of equation (10). At the same time the integral form equation (9) of the Marchenko kernel is thus established.

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