A calculation of the thermal conductivity of superconductors in the intermediate state

A Calculation of the Thermal Conductivity of Superconductors in the Intermediate State

by F. H. J. Cornish and J. L. Olsen.

The Clarendon Laboratory, Oxford, and
Institut für kalorische Apparate und Kältetechnik,
Eidgenössische Technische Hochschule, Zurich.

(24. IV. 1953.)

Abstract. The effect on the intermediate state thermal conductivity of a superconductor of varying rates of energy transfer between lattice and electrons is examined. The variation of thermal conductivity with magnetic field is calculated and shows fair agreement with experimental data.

§ 1. Introduction.

The minimum in the thermal conductivity of superconductors in the intermediate state in a transverse magnetic field observed by Mendelssohn and Olsen[1][2]) has since been shown by various observers: Webber & Spohr[3]); Detwiler & Fairbank[4])[5]); Mendelssohn & Rosenberg[6]); Olsen & Renton[7]); to be a common feature of the low-temperature behaviour of both pure and impure superconductors. Various explanations of the effect have been proposed, several of which involve the assumption of new mechanisms of a fundamental nature [2])[3])[5])[8a]).

It is the object of this paper to show that the observed minimum follows naturally from the fact that the size of the laminar structure of the intermediate state is comparable with the mean free paths of the electrons and of the lattice waves [9])[10]).

§ 2. Simple Considerations.

It has been shown by Meshkowsky & Shalnikov[11]) that the intermediate state of a superconductor is made up of alternate regions of normal and superconducting metal of the order of size suggested by Landau[12]). In the case of a cylinder in a transverse magnetic field these appear to be perpendicular to the axis[8b]).

If the thermal conductivity in the normal state is K_n and that in the superconducting state is K_s, then we may expect that the apparent overall conductivity K of the specimen will be given by

$$1/K = \mu/K_n + (1 - \mu)/K_s \tag{1}$$

where μ ist the fraction of normal material. Now it is known from magnetic experiments [13]) that μ increases approximately linearly with field once a magnetic field of about one half the critical field H_c has been exceeded. It thus appears to follow that $^1/K$ will also vary linearly with field between $^1/K_s$ and $^1/K_n$. This is in fact what has been observed at the higher temperatures [14]) [15]) [9]).

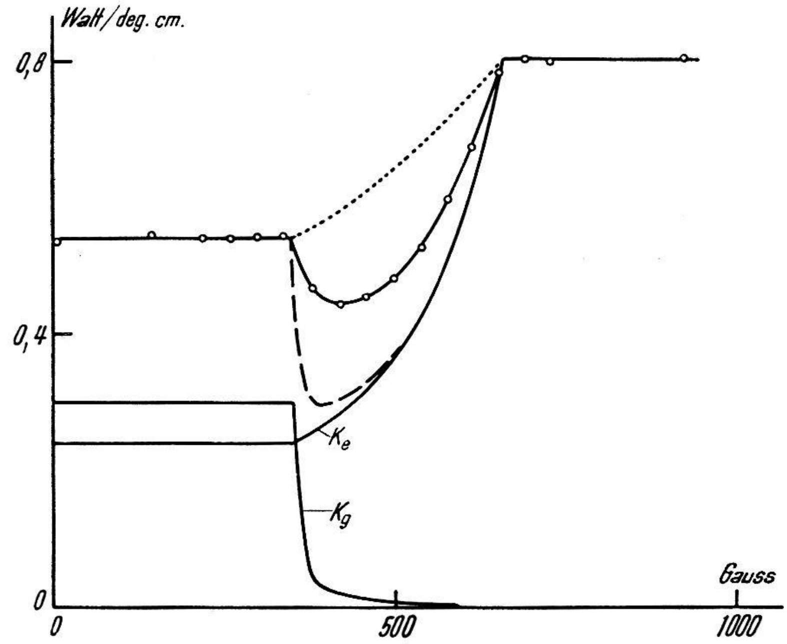

Fig. 1.

Variation of thermal conductivity of slightly impure lead in a transverse magnetic field at 2·9⁰ K.

o—o—o Experimental results[2])

.... Calculated from equation (1)

– – – – Calculated from equation (4)

The variation of K_e and K_g according to equations (2) and (3) are also shown.

At lower temperatures, however, it is found in some cases (see fig. 1) that when $\frac{1}{2} H_c$ is exceeded there is first a sharp drop in K, and then a slower recovery till the value K_n is reached when $H = H_c$. This is of course quite inconsistent with the above calculation, and in fact with any combination of regions having either the conductivity K_s or K_n.

An interesting result is, however, found if the lattice and the electrons are considered as residing in two quite separate conductors, i. e. if we take their temperatures to be independant. Then we should have for the conductivity of the electrons K_e:—

$$1/K_e = \mu/K_{en} + (1 - \mu)/K_{es} \tag{2}$$

where K_{en} is the electronic conductivity in the normal state, and K_{es} the electronic conductivity in the superconducting state. Similarly we should have for the lattice conductivity K_g

$$1/K_g = \mu/K_{gn} + (1 - \mu)/K_{gs} \tag{3}$$

where K_{gn} and K_{gs} are the lattice conductivities in the normal and superconducting states. The experimental results shown in figure 1 are for a lead specimen for which K_{es}, K_{en}, K_{gs} and K_{gn} are known [2], [9]. The dotted curve shows K as calculated from equation (1). The variations of K_g and K_e according to equations (2) and (3) are also plotted, and their sum

$$K = \frac{1}{\mu/K_{en} + (1-\mu)/K_{es}} + \frac{1}{\mu/K_{gn} + (1-\mu)/K_{gs}} \tag{4}$$

is shown by a broken line. It will be seen that this method of calculation yields a minimum in the conductivity in the intermediate state, and if the situation in an actual superconductor does in any way approach the condition of independent temperature distributions for the electrons and the lattice we may perhaps consider that we have here a possible explanation of the minimum.

It would appear that if the size of the lamina of the intermediate state is small compared to the mean free paths of the electrons or phonons then a calculation of K from equation (4) is justified. If the mean free paths are comparable with the laminar size, on the other hand, then K would lie intermediate between the values given by equations (1) and (4). Finally if the mean free paths are small compared to the laminae we should have K given simply by equation (1).

The measurements of Meshkowsky & Shalnikov indicate that d, the size of the laminar structure, is of the order of 10^{-1} cm, or perhaps smaller. The specimen for which the experimental curve is shown in figure 1 has a diameter, D, 10 times smaller than that of the sphere used by Meshkowsky & Shalnikov. Since d varies as $D^{0.6}$ [11] [12] we may expect that here d will be of the order of 2×10^{-2} cm at most. A mean value for the whole cross section of the cylinder would be considerably smaller. This may be compared with the mean free path in the normal state of the electrons l_{en} = ca. 1×10^{-3} cm, and of the phonons $l_{gn} = 3 \times 10^{-5}$ cm. For the superconducting state it is a little more difficult to make a confident estimate, but the success of the Heisenberg formula[17] for the ratio K_{es}/K_{en} leads to the supposition [8c] that l_{es} lies between l_{en} and $2\, l_{en}$.

Thus, in our case $l_{es} = 1 \times 10^{-3}$ cm is probably a reasonable estimate. For the lattice the greatly increased value of the lattice conductivity gives $l_{gs} = 3 \times 10^{-3}$ cm.

We see that some of the free paths are of an order of magnitude which would give a behaviour tending towards that of equation (4). On the other hand l_{gn} is very small compared to d so that the real case cannot be described by either of the extremes discussed above. It therefore seems worth-while to examine the effect of varying degrees of interaction between the lattice and electrons in more detail.

§ 3. Detailed Calculation.

In order to carry out this calculation we use a simple model to represent the actual situation in the metal.

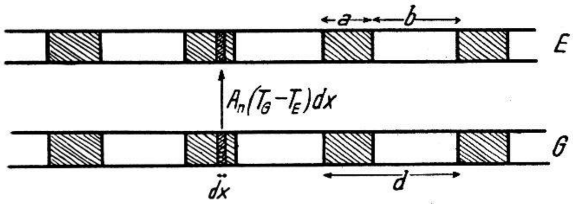

Fig. 2.
Model of superconductor in intermediate state.

We consider, as before, two conductors or systems, one E representing the electrons, and the other, G, representing the lattice. These are both made up of alternate normal and superconducting segments of length a and b respectively (see fig. 2). We shall write

$$\mu = a/(a + b) \text{ and } d = a + b.$$

The conductivities are K_{en}, K_{es}, K_{gn} and K_{gs}, as before. In addition we assume:—

1) The electrons and the lattice form interacting systems in both the normal and superconducting states, the interaction between the two being proportional to their temperature difference.

2) The temperature function in any lamina is, apart from an additive constant, the same as that in any other lamina in the same state. This applies to both the electron and lattice systems.

3) For both electron system, E, and the lattice system, G, the temperature and the heat flow is continuous at the laminar boundaries.

On account of (1) equations of the following form hold for each system in each lamina:—

$$\left.\begin{aligned} K_e \ddot{T}_e &= - A\,(T_g - T_e) \\ K_g \ddot{T}_g &= - A\,(T_e - T_g) \end{aligned}\right\} \qquad (5)$$

where a dot denotes differentiation with respect to a measure along the rod, and A is the interaction coefficient. K_e, K_g and A are assumed to be constant in any given lamina and of the same value in laminae in the same state. Equations like (5) can be solved using (2) and (3), and the following formula is then obtained for the overall conductivity:—

$$\frac{1}{K} = \frac{\mu}{K_n} + \frac{1-\mu}{K_s} + \frac{2}{d} \cdot \frac{(K_{en}/K_n - K_{es}/K_s)^2}{\frac{A_n}{b_n}\coth\frac{1}{2}b_n\mu d + \frac{A_s}{b_s}\coth\frac{1}{2}b_s(1-\mu)d} \tag{6}$$

where

$$b_n^2 = A_n(1/K_{en} + 1/K_{gn}) \text{ and } b_s^2 = A_s(1/K_{es} + 1|K_{gs}).$$

As is to be expected from the method of setting up the equations, this formula yields curves for K lying between those derived from equations (1) and (4). Indeed (6) reduces to (4) when $A \to 0$ and to (1) when $A \to \infty$. Note also that (6) differs from (1) only in an additive always positive term which vanishes when $\mu = 0$ or $\mu = 1$. To test the reasonableness of (6) properly, however, we must be able to indicate possible values for A_n and A_s. An estimate of these constants for a particular case is made after the next section.

§ 4. An alternative form for 1/K.

Clearly from equation (5) it is dimensionally correct to write

$$A_n = K_{in}/l_{in}^2 \qquad A_s = K_{is}/l_{is}^2 \tag{7}$$

where K_{in} and K_{is} represent some kind of interaction conductivity, and l_{in} and l_{is} represent some kind of interaction mean free path. Now we may write (6) in the equivalent form:—

$$\frac{1}{K} = \frac{\mu}{K_n} + \frac{1-\mu}{K_s} + \frac{X_{ns}}{Y_n \coth Z_n\mu + Y_s \coth Z_s(1-\mu)} \tag{8}$$

where

$$X_{ns} = (K_{en}/K_n - K_{es}/K_s)^2$$

$$Y_n = \frac{1}{2}\gamma_n \frac{d}{l_{in}}\sqrt{K_{in}} \qquad Z_n = \frac{1}{2}\frac{1}{\gamma_n}\frac{d}{l_{in}}\sqrt{K_{in}}$$

$$Y_s = \frac{1}{2}\gamma_s \frac{d}{l_{is}}\sqrt{K_{is}} \qquad Z_s = \frac{1}{2}\frac{1}{\gamma_s}\frac{d}{l_{is}}\sqrt{K_{is}}$$

and where

$$\gamma_n = \sqrt{\frac{K_{en}K_{gn}}{K_n}} \qquad \gamma_s = \sqrt{\frac{K_{es}K_{gs}}{K_s}}.$$

§ 5. Numerical evaluation of K for the lead specimen of figure 1.

The following is by no means a rigorous development. Its purpose is merely to show that by reasonable arguments values for the parameters may be assigned which in (8) yield a thermal conductivity curve in satisfactory agreement with experiment.

In the case of the lead specimen for which experimental data are given in figure 1 we have to a good approximation

$$\left.\begin{array}{llll} K_n &= 0\cdot 80 & K_s &= 0\cdot 54 \\ K_{en} &= 0\cdot 80 & K_{es} &= 0\cdot 24 \\ K_{gn} &= 0\cdot 0033 & K_{gs} &= 0\cdot 30 \text{ watt/deg cm.} \end{array}\right\} \quad (9)$$

This corresponds to:—

$$\left.\begin{array}{llll} l_{en} &= 1\times 10^{-3}\text{ cm} & l_{es} &= 1\times 10^{-3}\text{ cm} \\ l_{gn} &= 3\times 10^{-5}\text{ cm} & l_{gs} &= 3\times 10^{-3}\text{ cm.} \end{array}\right\} \quad (10)$$

We shall take $d = 10^{-2}$ cm.

The real problem of our calculation, of course, lies in a choice of A_n and A_s. It seems reasonable, however, to suppose that $K_{in}/l_{in}{}^2$ will be of the order of $K_{en}/l_{en}{}^2$ or $K_{gn}/l_{gn}{}^2$, and similarly that $K_{is}/l_{is}{}^2$ will be of the order of $K_{es}/l_{es}{}^2$ or K_{gs}/l_{gs}. It is difficult to decide whether the greater or the smaller of these possible values are the appropriate ones to use in the calculation, so we shall evaluate K for both of these possibilities. Now

$$\left.\begin{array}{ll} \dfrac{K_{en}}{l_{en}^2} = \dfrac{0\cdot 8}{(10^{-3})^2} = 8\times 10^5 & \dfrac{K_{gn}}{l_{gn}^2} = \dfrac{0\cdot 00\,33}{(3\times 10^{-5})^2} = 3\times 10^6 \\ \dfrac{K_{es}}{l_{es}^2} = \dfrac{0\cdot 2}{(10^{-3})^2} = 2\times 10^5 & \dfrac{K_{gs}}{l_{gs}} = \dfrac{0\cdot 30}{(3\times 10^{-3})^2} = 3\times 10^4. \end{array}\right\} \quad (11)$$

Consequently we obtain the following equation for K when the larger values for A_n and A_s are chosen:—

$$\frac{1}{K} = \frac{\mu}{0\cdot 80} + \frac{1-\mu}{0\cdot 54} + \frac{0\cdot 31}{0\cdot 5\coth 150\,\mu + 0\cdot 8\coth 6\,(1-\mu)} \quad (12)$$

and when the smaller values for A_n and A_s are chosen we find:—

$$\frac{1}{K} = \frac{\mu}{0\cdot 80} + \frac{1-\mu}{0\cdot 54} + \frac{0\cdot 31}{0\cdot 26\coth 78\,\mu + 0\cdot 32\coth 2\cdot 4\,(1-\mu)}\,. \quad (13)$$

In figure 3 is shown the variation of K with magnetic field given by (12) and (13) assuming that μ increases linearly from zero to unity as the field increases from $\frac{1}{2}H_c$ to H_c.

Clearly both the calculated curves are much too steep at the end points. The most satisfactory of the two curves is that using the smaller of the possible values for A_n and A_s. Further reduction in A would of course improve the agreement still more, and it would appear to be of great interest if a more reliable calculation of these quantities could be made. The calculated curve would also

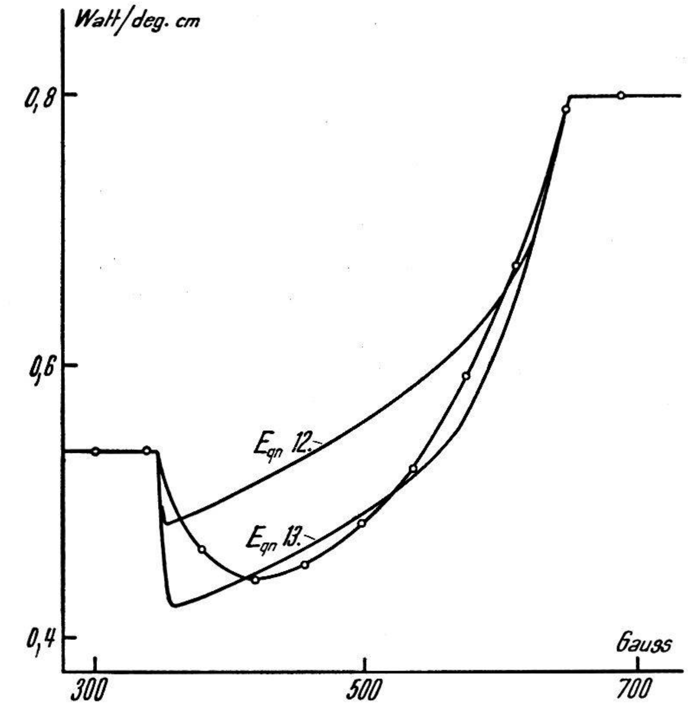

Fig. 3.
Calculated variation of the thermal conductivity of lead in the intermediate state according to (12) and (13) compared with experimental results o—o—o.

be improved if a smaller value for d were chosen, and this would probably be justifiable. In view of the extreme crudeness of our model, however, little would appear to be gained in attempting to choose the parameters at our disposal to give complete agreement with experiment.

§ 6. Summary.

The foregoing calculations show the extent to which a rather crude model can describe the observed effects in the thermal conductivity of the intermediate state. No fundamentally new processes have been invoked, but agreement which seems numerically fairly satisfactory has been obtained. These calculations lend plausibility to the suggestion that the intermediate state minimum is simply caused by the incomplete separation of the material into normal and superconducting regions.

References.

[1]) K. Mendelssohn and J. L. Olsen, Proc. Phys. Soc. A. **63**, 2 (1950).
[2]) K. Mendelssohn and J. L. Olsen, Phys. Rev. **80**, 859 (1950).
[3]) R. T. Webber and D. A. Spohr, Phys. Rev. **84**, 384 (1951).
[4]) D. P. Detwiler and H. A. Fairbank, Phys. Rev. **86**, 574 (1952).
[5]) D. P. Detwiler and H. A. Fairbank, Phys. Rev. **88**, 1049 (1952).
[6]) K. Mendelssohn and H. M. Rosenberg (1953) to be published.
[7]) J. L. Olsen and C. A. Renton, Phil. Mag. **43**, 946 (1952).
[8]) D. Shoenberg, Superconductivity, Cambridge 1952; a: p. 86, b: p. 112, c: p. 204.
[9]) J. L. Olsen, Proc. Phys. Soc. A, **65**, 518 (1952).
[10]) J. L. Olsen and H. M. Rosenberg, Adv. Phys. **2**, 28 (1953).
[11]) A. G. Meshkowsky and A. I. Shalnikov, J. Phys. U.S.S.R. **11**, 1 (1947).
[12]) L. D. Landau, J. Phys. U.S.S.R. **7**, 99 (1943).
[13]) M. Désirant and D. Shoenberg, Proc. Roy. Soc. A, **194**, 63 (1948).
[14]) K. Mendelssohn and R. B. Pontius, Phil. Mag. **24**, 777 (1937).
[15]) W. J. de Haas and H. Rademakers, Phys. **7**, 992 (1940).
[16]) C. G. Kuper, Phil. Mag. **42**, 961 (1951).
[17]) W. Heisenberg, Z. Naturforsch. **3**a, 65 (1948).