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Rigorous control of the non-perturbative corrections to the double expansion in g and $g^2 \ln(g)$ for the ϕ_3^4-trajectory in the hierarchical approximation

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Abstract. We study the renormalization invariant trajectory of the ϕ^4-perturbation of the free field fixed point in the hierarchical approximation. We parametrized it by a running ϕ^4-coupling g with linear step β-function. We rigorously control the non-perturbative corrections to finite order approximants from double perturbation theory in g and $g^2 \ln(g)$. The construction uses a contraction mapping for the extended renormalization group composed of a hierarchical block spin transformation with a flow of g.

1 Introduction

The non-perturbative renormalization of ϕ^4-theory is a central problem of constructive quantum field theory [28, 40, 7]. The state of the art includes phase cell expansions [27, 17, 35, 4, 18], renormalization group techniques [20, 5, 3, 23, 21, 22, 25, 37], and random path representations [6, 19]. However, the non-perturbative renormalization of ϕ^4-theory remains a mathematical enterprise of considerable difficulty. Recent work by Brydges, Dimock, and Hurd [8, 9, 10] aims to simplify the older constructions and to cast renormalization theory into a more conceptual form. The present paper intends to make a modest contribution in the same direction.

A key to the understanding of renormalization theory is Wilson's renormalization group [45, 46]. The traditional starting point is a bare action $S_0(\phi, g_0)$[1]. The goal is to compute a renormalized action as the limit $n \to \infty$ of $S_n(\phi, g_n) = R_L^n(S_0)(\phi, g_0)$ of an iterated renormalization group transformation R_L with scale L. The bare couplings g_0 are tuned in this process so as to obtain a finite limit for g_n. This process can be viewed as a trajectory in the dynamical system (on some space of actions) generated by R_L. If the action is a fixed point $S_\star(\phi)$ of R_L then its renormalization becomes trivial: the bare action and the renormalized action become identical. This fixed point problem has a natural generalization. Consider a curve $S(\phi, g)$ of actions parametrized by a (running) coupling g such that

1. $S(\phi, 0)$ is a fixed point $S_\star(\phi)$ of R_L,
2. $\partial_g S(\phi, g)|_{g=0}$ is an eigenvector $\mathcal{O}(\phi)$ of the linearization of R_L at $S_\star(\phi)$, and
3. $R_L(S)(\phi, g) = S(\phi, \delta_L^{-1}(g))$, where δ_L is a step β-function.

In this case, the bare action and the renormalized action have the same functional dependence of ϕ but correspond to different values of the running coupling g. Renormalization amounts to control the flow generated by the step β-function, a comparatively easy task.

We will study a variant of this renormalization problem, where R_L is a block spin transformation for a three dimensional scalar lattice field theory with hierarchical covariance [32]. It is designed such that the interaction remains local under the renormalization group evolution. The lattice interaction Boltzmann factor factorizes into product of local Boltzmann factors $Z(\phi) = \exp(-V(\phi))$ (one for each lattice site), which are functions of a real variable ϕ. In three dimensions, the hierarchical renormalization group then reduces to the non-linear integral transformation

$$R_L(Z)(\psi) = \left\{ \int \mathrm{d}\mu(\zeta)\, Z\left(L^{-\frac{1}{2}}\psi + \zeta\right) \right\}^{L^3}, \tag{1.1}$$

where $\mathrm{d}\mu(\zeta)$ is the Gaussian measure on $\mathbb{R}$ with mean zero and unit covariance. This transformation and variants of it have been studied by many authors both as a model of constructive renormalization and also because of its properties as a non-linear theory. Rigorous work on hierarchical models (more generally ultralocal renormalization groups) includes:

1. the ϵ-expansion [12]
2. the ϕ_3^4 infrared fixed point [31, 33, 34]

[1] The quantum field is understood to be rescaled to a unit ultraviolet cutoff.

3. the massive perturbation of the ϕ_3^4 fixed point [32]
4. the renormalization group differential equation [16]
5. the ϕ_4^4 infrared problem [37, 1]
6. the ϕ_4^4 ultraviolet problem at negative coupling [24]
7. the renormalized ϕ_D^4-trajectory [44]
8. the $SU(2)$-lattice gauge theory [43]
9. the non-linear σ-model [26, 38]
10. the sine-Gordon model [14, 30]
11. multigrid expansions [37, 39]
12. and random surfaces [11]

Beyond the hierarchical approximation, one has to deal with non-local interactions generated by the renormalization group. Although non-local corrections are rather small in all models brought under control so far, the mathematical apparatus needed to control them is a lot more sophisticated, the main tool being polymer expansions. The virtue of hierarchical models is that they allow to study renormalization effects without this additional burden (or perhaps joy).

In this paper, we continue the work started in [44]. We look for a curve of renormalized interaction Boltzmann factors $Z(\phi, g)$ with the following properties:

1. $Z(\phi, g) = \exp(-g : \phi^4 :) \left(1 + O(g^2)\right)$, i.e., $Z(\phi, g)$ emerges from the trivial fixed point $Z_\star(\phi) = 1$ (the free massless hierarchical field) tangent to a (normal ordered) ϕ^4-vertex, and
2. $R_L(Z)(\psi, \delta_L(g)) = Z(\psi, g)$, i.e., $Z(\phi, g)$ is a fixed point of the extended renormalization group $R_L \times \delta_L^\star$ with a linear step β-function $\delta(g) = L^{-1}g$.

The problem is thus to construct a non-trivial fixed point of the extended renormalization group $S_L = R_L \times \delta_L^\star$. We do this by means of a contraction mapping.[2]

For this purpose, we split $Z = Z_1 + Z_2$, where Z_1 is an approximate fixed point, and where Z_2 is a correction. We iterate the transformation of Z_2 with Z_1 kept fixed. This transformation is shown to contract, provided that Z_1 is in a certain sense a sufficiently good approximation. We compute Z_1 as a polynomial approximant of finite order in a (formal) double perturbation expansion in g and $g^2 \ln(g)$. We then prove that this approximant is indeed sufficiently accurate provided that the order of perturbation theory is at least seven. The result of this construction is the following Theorem.[3]

[2]The linearization of this composed transformation has an interesting marginal eigenvector, $g^2 : \phi^2 :$. It appears in the perturbative calculation (5.11). On the first sight, it seems that we are constructing a one parmater set of curves, all of which are tangent to the $: \phi^4 :$ perturbation. But they are clearly related by a reparametrization $g \to z\,g$ of the coupling parameter.

[3]Z_{QU} and $V^{(r_{max})}$ are explained in the bulk of this paper.

Theorem 1.1 *Let $F_2(g)$ be a continuous positive function of the form*

$$F_2(g) \;=\; g^\sigma \;\exp\bigl(c_\star g^3 + c\,g\bigr) \tag{1.2}$$

with positive constants σ, $c_\star$, and c. Let $Z_{QU}(\phi, g)$ be the quadratic fixed point

$$Z_{QU}(\phi, g) \;=\; \exp\left(a_{QU}(g) - \frac{b_{QU}(g)}{2}\,\phi^2\right) \tag{1.3}$$

of S_L. Let $\mathcal{B}_2$ be the Banach space of functions $Z_2 : \mathbb{R} \times [0, g_{\max}] \to \mathbb{R}$ with respect to the norm

$$\|Z_2\|_{F_2} \;=\; \sup_{(\phi,g)\in\mathbb{R}\times[0,g_{\max}]} \left|\frac{Z_2(\phi, g)}{Z_{QU}(\phi, g)\,F_2(g)}\right|. \tag{1.4}$$

Let $Z_1(\phi, g) \;=\; \exp\left(-V^{(r_{\max})}(\phi, g)\right)$, where $V^{(r_{\max})}(\phi, g)$ is the polynomial approximant of order $r_{\max}$ (in g) of the perturbative solution to the fixed point equation as a double expansion in g and $g^2 \ln(g)$.

For $r_{\max} = 7$, there exist positive constants $g_{\max}$, σ, $c_\star$, c, and C_2 such that the transformation $S_L(Z_1 + Z_2) - Z_1$ is a contraction mapping on the ball

$$\bigl\{Z_2 \in \mathcal{B}_2 \bigm| \|Z_2\|_{F_2} \le C_2\bigr\}. \tag{1.5}$$

It follows that there exists a unique fixed point in this ball. Furthermore, the iteration of the contraction mapping gives a convergent representation for this fixed point.

The renormalized ϕ^4_D-trajectory was constructed in [44] for all dimensions $2 < D < 4$ with the exception of a discrete set of special dimensions, where resonances of power counting factors occur [41]. Unfortunately, the case $D = 3$ is such a resonant case, and was therefore excluded in [44]. The problem is that our renormalization problem does not have a formal power series solution in g in three dimensions. However, it does have a solution as a formal double perturbation expansion in g and $g^2 \ln(g)$. The main content of this paper is to deal with these logarithmic corrections. Polynomial approximants from this double perturbation theory turn out to suffice for the contraction mapping. The backbone of our approach is the contraction mapping. This part is identical in resonant and non-resonant dimensions. To keep this paper selfcontained we have included a section on the contraction mapping with fresh proofs and improved bounds as compared to [44]. In particular, we present

1. a better scheme independent proof of the contraction property,
2. an example of bounds, which are true for all couplings and not only small couplings,
3. a better and more explicit treatment of the tree approximation,
4. and last not least a full stability analysis of the g-$g^2 \ln(g)$-approximants.

Unlike [44], analyticity in ϕ is not used here. This paper is organized as follows. Section two contains a brief review of the hierarchical renormalization group. Section three is devoted to the contraction mapping method. In Section four, we prove a stability bound and an error bound for the first order approximant. It serves as a template for the higher order approximants, which are analyzed in Section five. We conclude with a few remarks and outlooks.

It is well known that the unstable manifold of the trivial fixed point is two dimensional, when the renormalization group is restricted to even vertices and a field independent normalization constant is discarded. In this paper, we select one direction in this two dimensional manifold by imposing the above tangent condition and by imposing the remainder to satisfy a norm condition, to be stated below. The perturbative meaning of this particular curve is that we are constructing the ultraviolet limit of the theory with a minimal number of counterterms. This can be thought of as a local criterion on the flow in the vicinity of the trivial fixed point. A global criterion would be to select the curve by demanding that the theory be massless (or critical). That is, that the curve would end in the non-trivial infrared fixed point rather than in the quadratic high temperature fixed point. The global behavior of our curve requires control of the strong coupling region. The present analysis covers only partially the strong coupling region and cannot tell whether our curve is massive or massless. (Numerical evidence suggests that our curve is massless.) It would be very interesting to translate the global criterion into a local criterion at the trivial fixed point. A first step in this direction would be to analyze the role of reparametrizations and the nature of the non-$\mathcal{C}^{\infty}$-ness implied by the logarithmic corrections. A second step would be to do an analogous construction with a two dimensional β-function. In fact, this second step is rather straight forward. The first step is outside the scope of the present paper. The third step would be to do a construction which is uniform in the coupling parameter. It is conceivable that this can be done in a coupling dependent high temperature picture very much analogous to the construction of the infrared fixed point by Koch and Wittwer [31].

2 Hierarchical renormalization group

The hierarchical renormalization group is the theory of a non-linear integral transformation R acting on a certain space of functions $Z : \mathbb{R} \to \mathbb{R}$. We speak of Z as a hierarchical model in Statistical Physics if Z is positive, continuous, and in a certain sense measurable.

2.1 Renormalization group transformation

Hierarchical renormalization groups come in different versions. We mention the transformations of Dyson [15], Wilson [45], Baker [2],and Gallavotti [20]. The latter version has been investigated as a model for the constructive renormalization of massless scalar field theories by Gawedzki and Kupiainen [23, 24], by Pordt [37, 39], and by Koch and Wittwer [31, 33]. Following their path, we define R as follows.

Definition 2.1 *Let L denote a positive integer, $L \in \{2,3,4,\dots\}$, and let $D = 3$. Let $\alpha = L^D$, $\beta = L^{1-\frac{D}{2}}$, and $\gamma = 1$. Let R be the non-linear integral transformation given by*

$$R(Z)(\psi) = \left\{ \int \mathrm{d}\mu_{\gamma}(\zeta)\, Z(\beta\psi + \zeta) \right\}^{\alpha}, \tag{2.1}$$

where $\mathrm{d}\mu_{\gamma}(\zeta)$ denotes the Gaussian measure[4] *on $\mathbb{R}$ with mean zero and covariance γ.*

[4] The Fourier transform of the Gaussian measure is $\int \mathrm{d}\mu_{\gamma}(\zeta)\, \mathrm{e}^{\mathrm{i}\zeta\psi} = \mathrm{e}^{-\gamma\psi^2/2}$.

L is the block scale. Eventually, we will choose L to be large. D is the dimension of the (Euclidean) space-time. We will restrict our attention to the three dimensional transformation. In the sequel, D and γ are constant, $D = 3$ and $\gamma = 1$. The transformation (2.1) calls to be supplemented by a domain. The following statement defines our model space.

Definition 2.2 *Let $\mathcal{M}$ be the space of continuous even functions $Z : \mathbb{R} \to \mathbb{R}$ with finite norm $\|Z\|_\infty = \sup_{\phi \in \mathbb{R}} |Z(\phi)|$.*

A part of our analysis will be to restrict R to suitable invariant subspaces of $\mathcal{M}$, for instance subspaces of functions with a rapid decrease at infinity. Positive functions form a subset $\mathcal{M}^+$ of $\mathcal{M}$. But $\mathcal{M}^+$ is not a linear subspace of $\mathcal{M}$. Because we intend to use Banach space theory, we take the latter as our starting point.

Lemma 2.1 *The transformation R acts on the space $\mathcal{M}$. It satisfies the bound $\|R(Z)\|_\infty \leq \|Z\|_\infty^\alpha$.*

2.2 Trivial fixed point and tangent map

R has two trivial non-zero fixed points in $\mathcal{M}^+$, the ultraviolet fixed point $Z_{UV}(\phi) = 1$, and the (Gaussian) high temperature fixed point

$$Z_{HT}(\phi) = A_{HT}\, \mathrm{e}^{-\frac{b_{HT}}{2}\phi^2}, \tag{2.2}$$

where

$$A_{HT} = (\alpha\beta^2)^{\frac{\alpha}{2(\alpha-1)}} = L^{\frac{1}{1-L^{-D}}}, \qquad b_{HT} = \frac{\alpha\beta^2 - 1}{\gamma} = L^2 - 1. \tag{2.3}$$

In this paper, we will study the unstable manifold of the ultraviolet fixed point. Basic knowledge about it is gained from the linearized renormalization group. Recall the following well known facts.

Lemma 2.2 *Let $\mathcal{O} \in \mathbb{R}[\phi^2]$ be an even polynomial in ϕ with coefficients in $\mathbb{R}$ of the form $\mathcal{O}(\phi) = \phi^{2n}$ + lower powers of ϕ^2. Let $Z : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$Z(\phi, g) = \mathrm{e}^{-g\,\mathcal{O}(\phi)}. \tag{2.4}$$

(I) For all $g \in \mathbb{R}^+$, $Z(\cdot, g) \in \mathcal{M}^+$, and $Z(\phi, 0) = Z_{UV}(\phi)$. (II) For all $g \in \mathbb{R}^+$, $R(Z)(\cdot, g)$ is continuously differentiable in g, and $\lim_{g\to 0^+} \partial_g R(Z)(\cdot, g)$ defines a linear operator

$$D_{Z_{UV}}(R)(\mathcal{O})(\psi) = \alpha \int \mathrm{d}\mu_\gamma(\zeta)\, \mathcal{O}(\beta\psi + \zeta). \tag{2.5}$$

on $\mathbb{R}[\phi^2]$.

The linear operator $D_{Z_{UV}}(R)$ is the tangent map of R at Z_{UV}. This tangent map is diagonalizable.

Lemma 2.3 *Let H_{2n} be the Hermite polynomial of order $2n$ [29, 8.95]. Let $P_{2n} \in \mathbb{R}[\phi^2]$ be given by*

$$P_{2n}(\phi, v) = \left(\frac{v}{2}\right)^n \mathrm{H}_{2n}\left(\frac{\phi}{\sqrt{2v}}\right), \qquad v = \frac{\gamma}{1-\beta^2}. \tag{2.6}$$

We have that $D_{Z_{UV}}(R)(P_{2n}) = \lambda_{2n} P_{2n}$, where $\lambda_{2n} = \alpha\beta^{2n} = L^{\sigma_{2n}}$ with scaling dimension $\sigma_{2n} = D + n(2-D) = 3 - n$.

In three dimensions, P_0, P_2, and P_4 are relevant, P_6 is marginal, and all others are irrelevant. From the set of eigenfunctions, we select the relevant non-quadratic member P_4. Its explicit form is

$$P_4(\phi, v) = \phi^4 - 6 v \phi^2 + 3 v^2, \qquad \sigma_4 = 4 - D = 1. \tag{2.7}$$

Another way of saying that P_4 is an eigenfunction of the tangent map with eigenvalue λ_4 is the following. Let $V(\phi, g) = g P_4(\phi, v)$. Then we have that

$$D_{Z_{UV}}(R)(V)(\psi, \delta g) = V(\psi, g), \qquad \delta^{-1} = \alpha\beta^4 = L^{\sigma_4}. \tag{2.8}$$

In other words, $V(\phi, g)$ is a fixed point of $D_{Z_{UV}}(R) \times \delta^\star$, the tangent map extended by a flow of g. The linear function $g \to \delta g$ will serve as our step β-function.

As in [44], the goal of this paper is to construct a non-linear analogue of (2.8), which is a restricted homeomorphism from the tangent space at Z_{UV} to an invariant cone in $\mathcal{M}$, originating at Z_{UV}.

2.3 Extended renormalization group

¿From now on, we turn our attention from points in $\mathcal{M}$ to parametrized curves in $\mathcal{M}$, originating at Z_{UV}.

Definition 2.3 *Let $g_{\max} \in \mathbb{R}^+$ and $\mathbb{G} = [0, g_{\max}]$. Let $\mathcal{N}$ be the space of continuous functions $Z : \mathbb{R} \times \mathbb{G} \to \mathbb{R}$ such that $Z(\phi, g) = Z(-\phi, g)$ for all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$, $Z(\phi, 0) = Z_{UV}(\phi)$, and $\|Z\|_\infty = \sup_{(\phi,g)\in\mathbb{R}\times\mathbb{G}} |Z(\phi, g)|$ is finite.*

For technical reasons, we introduced a maximal coupling $g_{\max}$. By default, this maximal coupling is an arbitrary large number in the following. Restrictions on $g_{\max}$ will be explicitely stated.

Definition 2.4 *Let $\sigma_4 = 4 - D$ and $\delta = L^{-\sigma_4}$. Let S be the non-linear transformation given by*

$$S(Z)(\psi, g) = (R \times \delta^\star)(Z)(\psi, g) = R(Z)(\psi, \delta g). \tag{2.9}$$

Lemma 2.4 *The transformation S acts on the space $\mathcal{N}$. It satisfies the bound $\|S(Z)\|_\infty \leq \|Z\|_\infty^\alpha$.*

This bound is an immediate consequence of $\delta\mathbb{G} \subset \mathbb{G}$ (since $\delta g < g$) and

$$\sup_{(\phi,g)\in\mathbb{R}\times\mathbb{G}} |S(Z)(\phi, g)| \leq \left\{ \sup_{(\phi,g)\in\mathbb{R}\times\delta\mathbb{G}} |Z(\phi, g)| \right\}^\alpha \tag{2.10}$$

Functions in $\mathcal{N}$ are bounded but not necessarily decaying at large fields. To bound non-perturbative contributions, we will need an exponential decay at large fields. The following observation suggests how to select a subspace $\mathcal{N}_{QU} \subset \mathcal{N}$, which suits this purpose.

Lemma 2.5 *Let $Z_{QU} : \mathbb{R} \times \mathbb{G} \to \mathbb{R}^+$ be the positive continuous function given by*

$$Z_{QU}(\phi, g) = \exp\left\{ a_{QU}(g) - \frac{b_{QU}(g)}{2} \phi^2 \right\}, \tag{2.11}$$

where

$$a_{QU}(g) = \frac{\alpha-1}{2\alpha}\sum_{n=1}^{\infty}\alpha^{-n}\log\left\{\frac{1+(\delta^{-n}g)^{\rho}}{1+g^{\rho}}\right\} \tag{2.12}$$

and

$$b_{QU}(g) = \frac{\alpha\beta^2-1}{\alpha\gamma}\frac{g^{\rho}}{1+g^{\rho}}, \tag{2.13}$$

where $\rho = \frac{2}{4-D} = 2$. *The function* Z_{QU} *is an element of* $\mathcal{N}$. *It is a fixed point of* S.

The assignment $g \mapsto Z_{QU}(\cdot, g)$ is a continuous parametrized curve in $\mathcal{M}^+$. There is no reason, not to set $\mathbb{G} = \mathbb{R}^+$ at this point (or $g_{\max} = \infty$). Then Z_{QU} becomes a curve of Gauss-functions, which connects the two trivial fixed points of R, the ultraviolet fixed point $Z_{QU}(\phi, 0) = Z_{UV}(\phi)$ to the high temperature fixed point $Z_{QU}(\phi, \infty) = Z_{HT}(\phi)$.

Definition 2.5 *Let* $\mathcal{N}_{QU}$ *be the subspace of* $\mathcal{N}$, *consisting of functions with finite norm*

$$\|Z\|_{QU} = \sup_{(\phi,g)\in\mathbb{R}\times\mathbb{G}}\left|\frac{Z(\phi,g)}{Z_{QU}(\phi,g)}\right|, \tag{2.14}$$

completed to a Banach space.

Functions in $\mathcal{N}_{QU}$ are in particular continuous in g. Their most useful property is the inbuilt bound

$$|Z(\phi,g)| \leq \|Z\|_{QU}\, Z_{QU}(\phi,g). \tag{2.15}$$

At fixed g, it compares the decay of Z at large ϕ with the decay of the fixed point Z_{QU}. (2.15) serves as the basic large field bound in this paper.

Lemma 2.6 *The non-linear transformation* S *acts on* $\mathcal{N}_{QU}$. *(It leaves invariant* $\mathcal{N}_{QU} \subset \mathcal{N}$.*) Let* $Z \in \mathcal{N}_{QU}$. *Then* $\|S(Z)\|_{QU} \leq \|Z\|_{QU}^{\alpha}$.

We can now state our goal more precisely as to find a non-trivial fixed point of S in $\mathcal{N}_{QU}$ (besides the g-independent trivial fixed points of R, and besides Z_{QU}). To this aim, we need to introduce yet another subspace of $\mathcal{N}_{QU}$.

Definition 2.6 *Let* $g_{\max} < \infty$. *Let* $F : \mathbb{G} \to \mathbb{R}^+$ *be a continuous positive function with finite norm* $\|F\|_{\infty} = \sup_{g\in\mathbb{G}}|F(g)|$. *Let* $\mathcal{N}_F$ *be the subspace of* $\mathcal{N}_{QU}$ *consisting of functions with finite norm*

$$\|Z\|_{F} = \sup_{(\phi,g)\in\mathbb{R}\times\mathbb{G}}\left|\frac{Z(\phi,g)}{Z_{QU}(\phi,g)\,F(g)}\right|. \tag{2.16}$$

(2.16) refines (2.15). Its motive is to gain control over the g-dependence of Z.[5] It will be crucial to carefully choose the function F. To give a flavour of, what kind of function F should be, notice the following fact. $Z \in \mathcal{N}_F$ implies that

$$|Z(\phi,g)| \leq \|Z\|_F\, Z_{QU}(\phi,g)\, F(g). \tag{2.17}$$

[5] The information that Z is an element of $\mathcal{N}_{QU}$ implies no more information about its g-dependence than that, for all $\phi \in \mathbb{R}$, $Z(\phi,g)$ is a bounded function of g.

and thus

$$|S(Z)(\phi,g)| \;\leq\; \|Z\|_F^\alpha \; Z_{QU}(\phi,g)\; F(\delta\, g)^\alpha. \tag{2.18}$$

If F is such that the condition $F(\delta\, g)^\alpha \leq F(g)$ holds for all $g \in \mathbb{G}$, then $\mathcal{N}_F$ is invariant under S. Furthermore, the ball $\|Z\|_F \leq 1$ is then mapped to itself under S. Perturbation theory suggests functions of the form $F(g) = C\, g^\sigma \exp(c\, g^\tau)$. Depending on the actual values of C, σ, and τ, the above condition may pose a restriction on $g_{\max}$.

3 Contraction mapping

In this section, we present our method to construct a non-trivial fixed point of S. As in [44], we split $Z = Z_1 + Z_2$ into two parts Z_1 and Z_2, where Z_1 is an approximate fixed point, and where Z_2 is an error term. Z_1 will be kept fixed. We intend to estimate the non-linear mapping of the Z_2 under certain assumptions on Z_1.

Lemma 3.1 *Let Z_1 and Z_2 be two elements of $\mathcal{N}_{QU}$. Then*

$$S(Z_1 + Z_2) - Z_1 \;=\; T_1(Z_1) + T_2(Z_1, Z_2), \tag{3.1}$$

where

$$T_1(Z_1) \;=\; S(Z_1) - Z_1 \tag{3.2}$$

and

$$T_2(Z_1, Z_2) \;=\; S(Z_1 + Z_2) - S(Z_1). \tag{3.3}$$

Since S acts on $\mathcal{N}_{QU}$, the mappings T_1 and T_2 are both well defined. The norm of $T_1(Z_1)$ measures the quality of the approximate fixed point Z_1. Equipped with a bound on $T_1(Z_1)$, the issue is to find bounds on Z_2 which iterate under (3.1).

3.1 Invariant ball $\mathcal{B}$

The dependence of Z_1 and Z_2 on g will be controlled with two different functions F_1 and F_2 to be specified below.

Lemma 3.2 *Let $F_i : \mathbb{G} \to \mathbb{R}^+$, $i = 1, 2$, be two continuous positive functions. Abbreviate $\mathcal{N}_i = \mathcal{N}_{F_i}$ and $\|Z_i\| = \|Z_i\|_{F_i}$. Assume two functions Z_i with $Z_i \in \mathcal{N}_i$. Thus for all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$,*

$$|Z_i(\phi,g)| \;\leq\; \|Z_i\|\; Z_{QU}(\phi,g)\; F_i(g). \tag{3.4}$$

Then $T_2(Z_1, Z_2)$ is an element of $\mathcal{N}_{QU}$. For all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$ we have that

$$|T_2(Z_1,Z_2)(\phi,g)| \;\leq\; \left\{ \Big(\|Z_1\|\, F_1(\delta g) + \|Z_2\|\, F_2(\delta g) \Big)^\alpha - \Big(\|Z_1\|\, F_1(\delta g) \Big)^\alpha \right\} Z_{QU}(\phi,g). \tag{3.5}$$

Proof $T_2(Z_1, Z_2)$ has the integral representation

$$
\begin{aligned}
T_2(Z_1,Z_2)(\psi,g) &= \int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t}\, S(Z_1+t\,Z_2)(\psi,g) \\
&= \int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t} \left\{ \int \mathrm{d}\mu_\gamma(\zeta)\ (Z_1+t\,Z_2)(\beta\psi+\zeta,\delta g) \right\}^{\alpha} \\
&= \int_0^1 \mathrm{d}t\, \alpha \left\{ \int \mathrm{d}\mu_\gamma(\zeta)\ (Z_1+t\,Z_2)(\beta\psi+\zeta,\delta g) \right\}^{\alpha-1} \\
&\quad \times \int \mathrm{d}\mu_\gamma(\zeta)\ Z_2(\beta\psi+\zeta,\delta g)
\end{aligned} \tag{3.6}
$$

Using the bound (3.4), it follows that

$$
\begin{aligned}
&|T_2(Z_1,Z_2)(\psi,g)| \\
&\leq \int_0^1 \mathrm{d}t\, \alpha \Big(\|Z_1\|\, F_1(\delta g) + t\, \|Z_2\|\ F_2(\delta g) \Big)^{\alpha-1} \\
&\quad \times \|Z_2\|\ F_2(\delta g) \left\{ \int \mathrm{d}\mu_\gamma(\zeta)\ Z_{QU}(\beta\psi+\zeta,\delta g) \right\}^{\alpha} \\
&= \int_0^1 \mathrm{d}t\, \frac{\partial}{\partial t} \Big(\|Z_1\|\, F_1(\delta g) + t\, \|Z_2\|\, F_2(\delta g) \Big)^{\alpha}\ Z_{QU}(\psi,g) \\
&= \left\{ \Big(\|Z_1\|\, F_1(\delta g) + \|Z_2\|\, F_2(\delta g) \Big)^{\alpha} - \Big(\|Z_1\|\, F_1(\delta g) \Big)^{\alpha} \right\} \\
&\quad \times Z_{QU}(\psi,g). \quad \square
\end{aligned} \tag{3.7}
$$

Differing from [23], no split into small and large fields is required here. Small and large fields are taken care of simultaneously by the ϕ-dependence of Z_{QU}. For the right hand side of (3.5), we notice the elementary estimate

$$
\begin{aligned}
&\Big(\|Z_1\|\, F_1(\delta g) + \|Z_2\|\, F_2(\delta g) \Big)^{\alpha} - \Big(\|Z_1\|\, F_1(\delta g) \Big)^{\alpha} \leq \\
&\quad \alpha \Big(\|Z_1\|\, F_1(\delta g) + \|Z_2\|\, F_2(\delta g) \Big)^{\alpha-1}\ \|Z_2\|\ F_2(\delta g).
\end{aligned} \tag{3.8}
$$

Lemma 3.3 *Let Z_i be as in Lemma 3.2. Let Z_1 be such that in addition $\|T_1(Z_1)\| = \|T_1(Z_1)\|_{F_2}$ is finite. Then we have that for all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$,*

$$|T_1(Z_1)(\phi,g)| \leq \|T_1(Z_1)\|\ Z_{QU}(\phi,g)\ F_2(g). \tag{3.9}$$

Let n be an integer, $n \in \{1, 2, 3, \dots\}$. Assume that F_1 and F_2 conspire such that

$$\alpha \Big(\|Z_1\|\, F_1(\delta g) + (n+1)\ \|T_1(Z_1)\|\, F_2(\delta g) \Big)^{\alpha-1}\ F_2(\delta g) \leq \frac{n}{n+1}\, F_2(g). \tag{3.10}$$

Let $\mathcal{B}$ be the ball in $\mathcal{N}_2$ given by

$$\|Z_2\| \leq (n+1)\ \|T_1(Z_1)\|. \tag{3.11}$$

Then it follows that $\mathcal{B}$ is invariant under the transformation (3.1). For all $Z_2 \in \mathbb{B}$ and for all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$,

$$|S(Z_1+Z_2)(\phi,g) - Z_1(\phi,g)| \leq (n+1)\ \|T_1(Z_1)\|\ Z_{QU}(\phi,g)\ F_2(\phi,g). \tag{3.12}$$

Proof ¿From (3.5), (3.8), (3.10), and (3.11), it follows that

$$\begin{aligned}\|S(Z_1+Z_2)-Z_1\| &\le \|T_1(Z_1)\| + \|T_2(Z_1,Z_2)\| \\ &\le \|T_1(Z_1)\| + \frac{n}{n+1}\,\|Z_2\| \\ &\le \|T_1(Z_1)\| + \frac{n}{n+1}\,(n+1)\,\|T_1(Z_1)\| \\ &= (n+1)\,\|T_1(Z_1)\|. \quad \square \end{aligned} \tag{3.13}$$

We remark that the additional assumptions made in Lemma 3.3 do not clash with those made in Lemma 3.2.

Lemma 3.3 suggests the following strategy. We look for functions Z_1, F_1, and F_2 such that $\|Z_1\|_{F_1}$ and $\|T_1(Z_1)\|_{F_2}$ are both finite. Furthermore, F_1 and F_2 have to satisfy (3.10). Then we have an invariant ball of error terms Z_2 in the norm associated with F_2.

3.2 Contraction mapping

Our next task is to establish the contraction property for the transformation (3.1) on the ball $\mathcal{B}$.

Lemma 3.4 *Let Z_1, Z_2, and Z_2' be as in Lemma 3.2. Then we have for all $(\phi, g) \in \mathbb{R} \times \mathcal{G}$ the bound*

$$\begin{aligned}&\left|S(Z_1+Z_2')(\phi,g) - S(Z_1+Z_2)(\phi,g)\right| \le \\ &\alpha\left(\|Z_1\|\,F_1(\delta g) + \sup_{t\in[0,1]}\left\|Z_2 + t\,(Z_2'-Z_2)\right\|\,F_2(\delta g)\right)^{\alpha-1} F_2(\delta g) \\ &\qquad \times Z_{QU}(\psi,g)\,\|Z_2'-Z_2\|\end{aligned} \tag{3.14}$$

Proof ¿From the integral representation

$$\begin{aligned}&S(Z_1+Z_2')(\psi,g) - S(Z_1+Z_2)(\psi,g) \\ &= \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\,S\big(Z_1+Z_2+t\,(Z_2'-Z_2)\big)(\psi,g) \\ &= \int_0^1 \mathrm{d}t\,\frac{\partial}{\partial t}\left\{\int \mathrm{d}\mu_\gamma(\zeta)\,\big(Z_1+Z_2+t\,(Z_2'-Z_2)\big)(\beta\psi+\zeta,\delta g)\right\}^{\alpha} \\ &= \int_0^1 \mathrm{d}t\,\alpha\left\{\int \mathrm{d}\mu_\gamma(\zeta)\,\big(Z_1+Z_2+t\,(Z_2'-Z_2)\big)(\beta\psi+\zeta,\delta g)\right\}^{\alpha-1} \\ &\quad\times\int \mathrm{d}\mu_\gamma(\zeta)\,\big(Z_2'-Z_2\big)(\beta\psi+\zeta,\delta g)\end{aligned} \tag{3.15}$$

it follows that

$$\begin{aligned}
&\left|S(Z_1 + Z_2')(\psi,g) - S(Z_1 + Z_2)(\psi,g)\right| \\
&\leq \int_0^1 \mathrm{d}t\,\alpha \left(\|Z_1\|\, F_1(\delta g) + \left\|Z_2 + t\,(Z_2' - Z_2)\right\|\, F_2(\delta g) \right)^{\alpha-1} \\
&\quad \times\ F_2(\delta g) \left\{ \int \mathrm{d}\mu_\gamma(\zeta)\ Z_{QU}(\beta\psi + \zeta, \delta g) \right\}^{\alpha} \|Z_2' - Z_2\| \\
&\leq \alpha \left(\|Z_1\|\, F_1(\delta g) + \sup_{t\in[0,1]} \left\|Z_2 + t\,(Z_2' - Z_2)\right\|\, F_2(\delta g) \right)^{\alpha-1} F_2(\delta g) \\
&\quad \times\ Z_{QU}(\psi,g)\ \|Z_2' - Z_2\|. \quad \square
\end{aligned} \tag{3.16}$$

The contraction property follows from this estimate in cooperation with Lemma 3.3.

Lemma 3.5 *Let Z_1, Z_2, and Z_2' be as in Lemma 3.3. For all $Z_2 \in \mathcal{B}$ and $Z_2' \in \mathcal{B}$, we have that*

$$\|S(Z_1 + Z_2') - S(Z_1 + Z_2)\| \ \leq\ \frac{n}{n+1}\, \|Z_2' - Z_2\|. \tag{3.17}$$

Proof Since $\mathcal{B}$ is convex, it follows that for all $t \in [0,1]$,

$$\left\|Z_2 + t\,(Z_2' - Z_2)\right\|\, F_2(\delta\, g) \ \leq\ (n+1)\ \|T_1(Z_1)\|. \tag{3.18}$$

We can therefore use (3.11) to conclude the asserted bound from (3.14), namely

$$\begin{aligned}
&\left|S(Z_1 + Z_2')(\phi,g) - S(Z_1 + Z_2)(\phi,g)\right| \\
&\leq \frac{n}{n+1}\, Z_{QU}(\phi,g)\, F_2(g)\, \|Z_2' - Z_2\|. \quad \square
\end{aligned} \tag{3.19}$$

Corollary 3.1 *Let $\mathcal{B}$ be as in Lemma 3.3. $\mathcal{B}$ is invariant under the transformation (3.1). The transformation (3.1) is a contraction mapping. The transformation (3.1) has a unique fixed point in $\mathcal{B}$.*

The iteration of the contraction mapping is a convergent scheme to compute the fixed point. Let $Z_{2,0} = 0$ and define a sequence of (non-perturbative) approximants by

$$Z_{2,n+1} \ =\ S(Z_1 + Z_{2,n}) - Z_1. \tag{3.20}$$

As $n \to \infty$, this sequence converges to the desired fixed point. We cannot say much about this limit at this level of generality. However, if Z_1 is strictly positive than it follows immediately that the fixed point is non-negative.

3.3 Estimates for all couplings

Our next subject is to construct examples of the two functions F_1 and F_2 with the property (3.10). These examples are tailor made for perturbation theory.

Suppose that we are told an approximate fixed point Z_1 from an independent calculation, whose large field decay is (at least) Gaussian. Then we proceed as follows.

1. We determine F_1 such that

$$\sup_{\phi\in\mathbb{R}}\left|\frac{Z_1(\phi,g)}{Z_{QU}(\phi,g)}\right| \leq F_1(g). \tag{3.21}$$

Then $\|Z_1\|_{F_1} \leq 1$. Since it contains information about the large field behavior of Z_1, we call this bound the *stability bound* on Z_1.

2. We determine F_2 from an estimate on $T_1(Z_1)$ such that

$$\sup_{\phi\in\mathbb{R}}\left|\frac{T_1(Z_1)(\phi,g)}{Z_{QU}(\phi,g)}\right| \leq C_1\, F_2(g) \tag{3.22}$$

for some finite constant C_1. Then $\|T_1(Z_1)\|_{F_2} \leq C_1$. We call this second bound the *error bound* on Z_1.

3. We check that F_1 and F_2 satisfy (3.10).

If F_1 and F_2 are functions of the type of this section, then the validity of (3.10) is guaranteed by the below estimates.

Remember that Z_1 plays the role of an approximate fixed point. This means that $T_1(Z_1)$ should be small compared with Z_1. A construction for all couplings should in particular cover the case of small couplings. Inspired by perturbation theory, we choose

$$F_2(g) = g^{\sigma}\, F_1(g). \tag{3.23}$$

It says that the error term goes to zero as the coupling goes to zero. The speed of this process is given by the exponent σ.

Lemma 3.6 *Let $F_1 : \mathbb{G} \to \mathbb{R}^+$ be a positive continuous function. Let $\|Z_1\|_{F_1} \leq 1$. Let σ be a positve real number, with $\sigma > \sigma_\star$, where*[6]

$$\sigma_\star = \frac{D}{4-D} = 3. \tag{3.24}$$

Let $F_2(g) = g^{\sigma} F_1(g)$. Let $\|T_1(Z_1)\|_{F_2} \leq C_1$ for some positive constant C_1. Let $n \in \mathbb{N}$. Put $C_2 = (n+1)C_1$. Assume that F_1 has the following property. Let there exists a positive constant C_F such that for all $g \in \mathbb{G}$,

$$e^{(\alpha-1)\,C_2\,(\delta g)^{\sigma}}\, F_1(\delta g)^{\alpha} \leq C_F\, F_1(g). \tag{3.25}$$

Then we have the following estimate. There exists a positive real number $L_{\min}$ such that for L larger than this number, we have the bound

$$\alpha\left(\|Z_1\|\, F_1(\delta\, g) + (n+1)\,\|T_1(Z_1)\|\, F_2(\delta\, g)\right)^{\alpha-1} F_2(\delta\, g) \leq \frac{n}{n+1}\, F_2(g). \tag{3.26}$$

[6] The exponent $\sigma_\star$ is such that $\alpha\delta^{\sigma_\star} = 1$. For $\sigma > \sigma_\star$ the block volume α is beaten by the power of the step β-function.

Proof ¿From the assumptions (3.22), (3.23), and (3.25), it follows that

$$\begin{aligned}
&\alpha \left(\|Z_1\| \, F_1(\delta g) + (n+1) \, \|T_1(Z_1)\| \, F_2(\delta g) \right)^{\alpha - 1} \; F_2(\delta g) \\
&\leq \; \alpha \left(1 + C_2 \, (\delta g)^\sigma \right)^{\alpha-1} \; (\delta g)^\sigma \; F_1(\delta g)^\alpha \\
&\leq \; \alpha \, \mathrm{e}^{(\alpha-1) \, C_2 \, (\delta g)^\sigma} \; (\delta g)^\sigma \; F_1(\delta g)^\alpha \\
&\leq \; \alpha \, (\delta g)^\sigma \; C_F \; F_1(g) \\
&= \; \alpha \delta^\sigma \; C_F \; F_2(g).
\end{aligned} \tag{3.27}$$

For $\sigma > \sigma_\star$, the L-dependent factor $\alpha\delta^\sigma \; = \; L^{D+(D-4)\,\sigma}$ can be made arbitrary small by taking L to be large. □

We learn that (3.10) (identical with (3.26)) holds provided that F_1 satisfies (3.25). An example of a function F_1 which satisfies (3.25) without a restriction on $g_{\max}$ is the following.

Lemma 3.7 *Let $c_\star$ and c be positive constants. Let σ be a positive constant such that $\sigma > \sigma_\star$. Let F_1 be the function*

$$F_1(g) \; = \; \exp\left(c_\star \, g^{\sigma_\star} + c \, g^\sigma\right). \tag{3.28}$$

For all positive constants C_2 such that

$$C_2 \; \leq \; \frac{1 - \alpha\delta^\sigma}{\alpha\delta^\sigma} \; c, \tag{3.29}$$

this function F_1 satisfies the bound

$$\mathrm{e}^{(\alpha-1) \, C_2 \, (\delta g)^\sigma} \; F_1(\delta \, g)^\alpha \; \leq \; F_1(g). \tag{3.30}$$

Proof For this particular function F_1, we have that

$$\begin{aligned}
&\exp\left\{(\alpha-1) \, C_2 \, (\delta g)^\sigma\right\} \; \exp\left\{c_\star \, g^{\sigma_\star} + \alpha \, c \, (\delta \, g)^\sigma\right\} \\
&\leq \; \exp\left\{c_\star \, g^{\sigma_\star} + \alpha\delta^\sigma (C_2 + c) \, g^\sigma\right\} \\
&\leq \; \exp\left\{c_\star \, g^{\sigma_\star} + c \, g^\sigma\right\}. \quad \square
\end{aligned} \tag{3.31}$$

Lemma 3.7 contains a restriction on C_2. To avoid a conflict with (3.10), the constant c has to be chosen to be sufficiently large.

The pair F_i given by (3.28) and (3.23) (with $\sigma > \sigma_\star$) satisfies all properties needed for the contraction mapping. Remarkably, it involves no restriction on the size of $g_{\max}$. With this pair, we could construct the ϕ^4-theory at arbitrary large couplings. The problem is to compute an approximate fixed point Z_1, which satisfies the stability bound (3.21) with this function F_1, and which satisfies the error bound with a sufficiently large exponent σ. Unfortunately, we have not succeeded to prove these bounds for approximants from perturbation theory.

3.4 Estimates for small couplings

Therefore, we supplement the bounds for all couplings by suitable bounds for small couplings. There are two sources of constraints on the value of $g_{\max}$. One source is that we may not be able to prove the stability bound on Z_1 for arbitrary large couplings. Also we may not be able to prove the error bound on $T_1(Z_1)$ for arbitrary large couplings. In this section, we will not speak about this source of problems. These constraints cannot be addressed before we actually compute Z_1. The second source is that the function F_1, which we find from (3.21) (say by defining F_1 by equality), might not meet the requirement (3.25) for arbitrary large couplings. If both effects come together, we have to put $g_{\max}$ equal to the minimum from these constraints.

Lemma 3.8 *Let $c_\star$ and c be a positive constants. Let τ be a positive constant such that $\tau < \sigma_\star$. Let F_1 be the function*

$$F_1(g) \;=\; \exp\left(c_\star\, g^{\sigma_\star} + c\, g^{\tau}\right). \tag{3.32}$$

Let C_2 and C_F be positive constants, with $C_F > 1$. For all values of L, there exists a maximal coupling $g(L)$ such that, for all g less than this maximal coupling, we have that

$$\mathrm{e}^{(\alpha-1)\,C_2\,(\delta g)^{\sigma}}\; F_1(\delta g)^{\alpha} \;\leq\; C_F\, F_1(g). \tag{3.33}$$

Proof The estimate is very similar to the one for all couplings. The identity

$$\mathrm{e}^{(\alpha-1)\,C_2\,(\delta g)^{\sigma}}\; F_1(\delta g)^{\alpha} \;=\; \mathrm{e}^{c\,(\alpha\delta^{\tau}-1)\,g^{\tau}}\; F_1(g) \tag{3.34}$$

yields the condition

$$\mathrm{e}^{(\alpha-1)\,C_2\,(\delta g)^{\sigma}+c\,(\alpha\delta^{\tau}-1)\,g^{\tau}} \;\leq\; C_F \tag{3.35}$$

on g. Given exponents σ and τ, together with constants C_2 and c, and a scale L, (3.35) determines $g(L)$. □

To get an idea how $g(L)$ behaves as a function of L, we neglect the the first term in (3.35). Then

$$g(L) \;\leq\; \left(\frac{\ln\left(C_F\right)}{c\,(\alpha\delta^{\tau}-1)}\right)^{\frac{1}{\tau}} \tag{3.36}$$

shows that $g(L)$ shrinks as a certain power of L. A typical value of τ in our perturbation theory is one. Since $\sigma_\star = 3$ in three dimensions, we are in the small coupling case. The good news is that the maximal coupling is not necessarily ridiculously small.

It is likely that there exist other functions F_i which meet the requirements of the above contraction mapping. The particular ones considered here are made for approximants Z_1 from perturbation theory. The interesting question remains whether this contraction mapping works with other approximants, for instance approximants from numerical work. This would be more in the spirit of the fixed point construction of [31].

4 Linear approximation

The simplest approximation is a pure ϕ^4-vertex. In this section, we will prove a stability bound and an error bound for this linear approximation defined by

$$Z(\phi,g) \;=\; \mathrm{e}^{-V(\phi,g)}, \qquad V(\phi,g) \;=\; g\,P_4(\phi,v). \tag{4.1}$$

(To simplify the notation, we write Z instead of Z_1.) The linear approximation will not suffice to obtain a contraction mapping in three dimensions. But it is a part of, and also an instructive example for, the finer bounds to be presented below.

4.1 Stability bound

The classical stability bound for (4.1) relies on analyticity in the field variable a strip around the real axis [23]. We proceed differently therefrom [37, 44].

Lemma 4.1 *The polynomial $P_4(\phi,v)$ is bounded from below by*

$$P_4(\phi,v) \;\geq\; \frac{1}{2}\,\phi^4 - 15\,v^2. \tag{4.2}$$

Proof

$$\begin{aligned} P_4(\phi,v) &= \phi^4 - 6\,v\,\phi^2 + 3\,v^2 \\ &= \left(\epsilon\,\phi^2 - \frac{3\,v}{\epsilon}\right)^2 - \left(\frac{3\,v}{\epsilon}\right)^2 + (1-\epsilon^2)\,\phi^4 + 3\,v^2 \\ &\geq (1-\epsilon^2)\,\phi^4 + 3\left(1 - \frac{3}{\epsilon^2}\right)v^2. \end{aligned} \tag{4.3}$$

For $\epsilon^2 = \frac{1}{2}$, the assertion follows. □

Lemma 4.2 *For all $(\phi,g) \in \mathbb{R}\times\mathbb{G}$, we have that (4.1) is bounded from above by*

$$Z(\phi,g) \;\leq\; Z_{QU}(\phi,g)\,\mathrm{e}^{a_1(g)} \tag{4.4}$$

with

$$a_1(g) \;\leq\; 15\,v^2\,g + \frac{1}{8}\,g^{2\rho-1} - a_{QU}(g). \tag{4.5}$$

Proof For any c, we have the elementary bound

$$\phi^4 \;=\; (\phi^2 - \frac{c}{2})^2 + c\,\phi^2 - \frac{c^2}{4} \;\geq\; c\,\phi^2 - \frac{c^2}{4}. \tag{4.6}$$

Put $c \;=\; g^{\rho-1}$ to obtain

$$\frac{g}{2}\phi^4 \;\geq\; \frac{g^\rho}{2}\phi^2 - \frac{g^{2\rho-1}}{8}. \tag{4.7}$$

The values of L, D, and γ are such that (2.13) is bounded from above by

$$b_{QU}(g) \;=\; \frac{L^2-1}{L^D\,\gamma}\,\frac{g^\rho}{1+g^\rho} \;<\; \frac{L^{2-D}}{\gamma}\,g^\rho \;<\; g^\rho. \quad \square \tag{4.8}$$

The bound (4.5) suggests a function F_1 of the form (3.28).

Lemma 4.3 *Let Z be given by (4.1). Let $F_1 : \mathbb{G} \to \mathbb{R}^+$ be the function*

$$F_1(g) \;=\; \exp\left(15\,v^2\,g + \frac{g^{2\rho-1}}{8}\right). \tag{4.9}$$

Then Z is bounded in the norm (2.16). We have that $\|Z\|_{F_1} \;\leq\; 1$.

Proof For all $g \;\geq\; 0$, $a_{QU}(g) \;\geq\; 0$, wherefore

$$\|Z\|_{F_1} \;=\; \sup_{(\phi,g)\in\mathbb{R}\times\mathbb{G}} \left|\frac{Z(\phi,g)}{Z_{QU}(\phi,g)\,F_1(g)}\right| \;\leq\; \sup_{g\in\mathbb{G}} \mathrm{e}^{-a_{QU}(g)} \;\leq\; 1. \quad \square \tag{4.10}$$

In three dimensions, the value of ρ is two and that of $\sigma_\star$ is three. (Recall (2.13) and (3.24).) Coincidentally, $2\rho - 1 \;=\; \sigma_\star$. But the second term in the exponent in (4.9) is only linear in g. Although the stability bound (4.9) holds for any value of g, it restricts our contraction mapping to small couplings.

4.2 Error bound

Equipped with this stability bound, one is led to estimate $T_1(Z)$ in the norm given by (3.23), where σ is a suitable exponent. As we will see, in fact as we know from [44], such a bound indeed holds. But the exponent σ is only one half and therefore smaller than $\sigma_\star$. The linear approximation therefore suffices only for a construction in low dimensions. Let us nevertheless see how the exponent one half comes about. For this purpose, we consider the following interpolation formula.

Definition 4.1 *Let $X : \mathbb{R} \times \mathbb{G} \times [0,1] \to \mathbb{R}$ be defined by*

$$X(\psi,g,t) \;=\; \left\{\int \mathrm{d}\mu_{t\,\gamma}(\zeta)\;\exp\Big(-\int \mathrm{d}\mu_{(1-t)\,\gamma}(\xi)\;V(\beta\psi + \zeta + \xi, \delta g)\Big)\right\}^\alpha \tag{4.11}$$

In the limit of a vanishing covariance, the Gaussian measure $d\mu_\gamma(\zeta)$ becomes a Dirac measure $d\zeta\, \delta(\zeta)$. Therefore, (4.10) interpolates between the exponentiated linerarized renormalization group transformation (2.5)

$$X(\psi, g, 0) \;=\; \exp\Big(-\alpha \int d\mu_\gamma(\xi)\; V(\beta\psi + \xi, \delta g)\Big) \tag{4.12}$$

and the full renormalization group transformation

$$X(\psi, g, 1) \;=\; \left\{\int d\mu_\gamma(\zeta)\; \exp\Big(-V(\beta\psi + \zeta, \delta g)\Big)\right\}^{\alpha}, \tag{4.13}$$

both transformations being extended by the flow of g. The usefulness of this interpolation relies on the following property.

Lemma 4.4 *For $t \in (0,1)$, X is continuously differentiable in t. We have that*

$$\begin{aligned}
&\frac{\partial}{\partial t} X(\psi, g, t) \;=\\
&\alpha\left\{\int d\mu_{t\gamma}(\zeta)\; \exp\Big(-\int d\mu_{(1-t)\gamma}(\xi)\; V(\beta\psi + \zeta + \xi, \delta g)\Big)\right\}^{\alpha-1}\\
&\times\left\{\int d\mu_{t\gamma}(\zeta)\; \exp\Big(-\int d\mu_{(1-t)\gamma}(\xi)\; V(\beta\psi + \zeta + \xi, \delta g)\Big)\right.\\
&\left.\times\frac{\gamma}{2}\Big(\frac{\partial}{\partial\zeta}\int d\mu_{(1-t)\gamma}(\xi)\; V(\beta\psi + \zeta + \xi, \delta g)\Big)^2\right\}.
\end{aligned} \tag{4.14}$$

Proof

$$\begin{aligned}
&\frac{\partial}{\partial t}\int d\mu_{t\gamma}(\zeta)\; \exp\Big(-\int d\mu_{(1-t)\gamma}(\xi)\; V(\phi + \zeta + \xi)\Big)\\
&= \int d\mu_{t\gamma}(\zeta)\; \Big[\frac{\gamma}{2}\frac{\partial^2}{\partial\zeta^2} + \frac{\partial}{\partial t}\Big] \exp\Big(-\int d\mu_{(1-t)\gamma}(\xi)\; V(\phi + \zeta + \xi)\Big)\\
&= \int d\mu_{t\gamma}(\zeta)\; \exp\Big(-\int d\mu_{(1-t)\gamma}(\xi)\; V(\phi + \zeta + \xi)\Big)\\
&\times\left\{\frac{\gamma}{2}\Big(\frac{\partial}{\partial\zeta}\int d\mu_{(1-t)\gamma}(\xi)\; V(\phi + \zeta + \xi)\Big)^2\right.\\
&\left.-\Big[\frac{\gamma}{2}\frac{\partial^2}{\partial\zeta^2} + \frac{\partial}{\partial t}\Big]\int d\mu_{(1-t)\gamma}(\xi)\; V(\phi + \zeta + \xi)\right\}
\end{aligned} \tag{4.15}$$

and

$$\Big[\frac{\gamma}{2}\frac{\partial^2}{\partial\zeta^2} + \frac{\partial}{\partial t}\Big]\int d\mu_{(1-t)\gamma}(\xi)\; V(\phi + \zeta + \xi) \;=\; 0. \quad \square \tag{4.16}$$

Notice that (4.14) is of second order in V due to the cancellation (4.16). Notice furthermore that (4.14) is non-negative. The integral of (4.14) yields a representation for $T_1(Z)$, which can be used to derive an upper bound of the desired form.

Lemma 4.5 *Let Z be given by (4.1). Let X be given by (4.11). Then we have that*

$$T_1(Z)(\psi, g) = X(\psi, g, 1) - X(\psi, g, 0) = \int_0^1 \mathrm{d}t \, \frac{\partial}{\partial t} X(\psi, g, t). \tag{4.17}$$

Proof Because V is an eigenvector of the extended linearized renormalization group (2.8),

$$\alpha \int \mathrm{d}\mu_\gamma(\xi) \, V(\beta\psi + \zeta, \delta g) = V(\psi, g) \tag{4.18}$$

so that $X(\psi, g, 0) = Z(\psi, g)$. □

A cost of this representation is that we have to repeat the stability analysis for the interpolated interaction. In the linear approximation, this follows from an explicit calculation. The following bound is uniform in the interpolation parameter.

Lemma 4.6 *For all $(\phi, g, t) \in \mathbb{R} \times \mathbb{G} \times [0, 1]$, we have that*

$$\int \mathrm{d}\mu_{(1-t)\gamma}(\xi) \, V(\phi + \xi, g) \geq \frac{g}{2} \phi^4 - 15 v^2 g. \tag{4.19}$$

Proof

$$\begin{aligned} \int \mathrm{d}\mu_{(1-t)\gamma}(\xi) \, V(\phi + \xi, g) &= g \int \mathrm{d}\mu_{(1-t)\gamma}(\xi) \, P_4(\phi + \xi, v) \\ &= g \, P_4\big(\phi, v - (1-t)\gamma\big) \\ &\geq g \left\{ \frac{\phi^2}{2} - 15 \left(v - (1-t)\gamma \right)^2 \right\} \end{aligned} \tag{4.20}$$

and

$$\frac{\beta^2 \gamma}{1 - \beta^2} = v - \gamma \leq v - (1-t)\gamma \leq v = \frac{\gamma}{1 - \beta^2} \tag{4.21}$$

show the assertion, since $\beta^2 = L^{2-D} < 1$. □

4.2.1 Large field domination

We are now in the position to estimate the downstairs factor in (4.14), using up a fraction, say one half, of the large field behavior of (4.19).

Lemma 4.7 *Let V be given by (4.1). For all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$, we have that*

$$\begin{aligned} &\mathrm{e}^{-\int \mathrm{d}\mu_{(1-t)\gamma}(\xi) \, V(\phi+\xi, g)} \, \frac{\gamma}{2} \left(\frac{\partial}{\partial \phi} \int \mathrm{d}\mu_{(1-t)\gamma}(\xi) \, V(\phi + \xi, g) \right)^2 \leq \\ &\qquad C(g) \sqrt{g} \, \exp\!\left(-\frac{g}{4} \phi^4 + 15 \, g \, v^2 \right) \end{aligned} \tag{4.22}$$

with

$$C(g) \le 8\,\gamma \left(A_6 + 9\,v^2\,A_2\,g \right) \tag{4.23}$$

where

$$A_{2n} = \sup_{\phi\in\mathbb{R}} \left(\mathrm{e}^{-\frac{\phi^4}{4}}\,\phi^{2n} \right). \tag{4.24}$$

Proof The stability bound (4.19), in conjunction with the elementary estimates

$$\left(\frac{\partial}{\partial\phi} P_4(\phi, v) \right)^2 = 16\,\left(\phi^6 - 6\,v\,\phi^4 + 9\,v^2\,\phi^2\right) \le 16\,\left(\phi^6 + 9\,v^2\,\phi^2\right) \tag{4.25}$$

and

$$\exp\left(-\tfrac{g}{4}\,\phi^4\right)\,\phi^{2n} = \exp\left\{-\tfrac{1}{4}\left(g^{\frac{1}{4}}\,\phi\right)^4\right\}\,\left(g^{\frac{1}{4}}\,\phi\right)^{2n}\,g^{-\frac{n}{2}} \le A_{2n}\,g^{-\frac{n}{2}}, \tag{4.26}$$

implies that

$$\begin{aligned} &\exp\left(-\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\,V(\phi+\xi,g)\right)\,\frac{\gamma}{2}\left(\frac{\partial}{\partial\phi}\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\,V(\phi+\xi,g)\right)^2 \\ &\le \exp\left(-\frac{g\,\phi^4}{2} + 15\,v\,g\right)\,8\,\gamma\,g^2\left(\phi^6 + 9\,v^2\,\phi^2\right) \\ &\le \exp\left(-\frac{g\,\phi^4}{4} + 15\,v\,g\right)\,8\,\gamma\,g^{\frac{1}{2}}\left(A_6 + 9\,g\,v^2\,A_2\right). \quad \square \end{aligned} \tag{4.27}$$

Eq. (4.26) shows that each ϕ in the downstairs factor kills $g^{\frac{1}{4}}$. Therefore, each of the two ϕ-derivatives in its calculation yields $g^{\frac{1}{4}}$. Two ϕ-derivatives give a total factor of $g^{\frac{1}{2}}$.

4.2.2 Fluctuation integral

Half of the stability estimate has now been used up for the control of the downstairs factor. The other half suffices to do the fluctuation integral.

Lemma 4.8 *Let b and c be positive constants. For all $\psi \in \mathbb{R}$, we have that*

$$\int \mathrm{d}\mu_\gamma(\zeta)\,\mathrm{e}^{-b\,(\psi+\zeta)^4} \le \exp\left(-\frac{b\,c}{1+b\,c\,\gamma}\,\frac{\psi^2}{2} + \frac{b\,c^2}{16}\right). \tag{4.28}$$

Proof ¿From (4.6), we deduce that

.

$$b\,\phi^4 \ge \frac{b\,c}{2}\phi^2 - \frac{b\,c^2}{16}. \tag{4.29}$$

The Gaussian convolution of a Gauss function is again a Gauss function. From its explicit form, we find (4.28). $\square$

Lemma 4.9 *For all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$, we have that*

$$\int \mathrm{d}\mu_{t\gamma}(\zeta)\ \mathrm{e}^{-\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\ V(\phi+\zeta+\xi)}\ \frac{\gamma}{2}\ \left(\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\ V(\phi+\zeta+\xi)\right)^2 \le$$
$$C(g)\ \sqrt{g}\ \exp\left(-\frac{1}{2}\frac{g^\rho}{1+g^\rho\gamma}\,\phi^2 + 15\,v^2\,g + \frac{1}{4}\,g^{2\rho-1}\right). \tag{4.30}$$

Proof ¿From (4.22) and (4.28), with $b = \frac{g}{4}$ and $c = 4\,g^{\rho-1}$, it follows that

$$\int \mathrm{d}\mu_{t\gamma}(\zeta)\ \mathrm{e}^{-\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\ V(\phi+\zeta+\xi)}\ \frac{\gamma}{2}\ \left(\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\ V(\phi+\zeta+\xi)\right)^2$$
$$\le \int \mathrm{d}\mu_{t\gamma}(\zeta)\ C(g)\ \sqrt{g}\ \exp\left(-\frac{g}{4}(\phi+\zeta)^4 + 15\,v^2\,g\right)$$
$$\le C(g)\ \sqrt{g}\ \exp\left(-\frac{1}{2}\frac{g^\rho}{1+g^\rho\gamma}\cdot\phi^2 + 15\,v^2\,g + \frac{1}{4}\,g^{2\rho-1}\right). \quad \square \tag{4.31}$$

In eq. (4.14), we also encounter another fluctuation integrals without downstairs factors. It is estimated in the same manner.

Lemma 4.10 *For all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$, we have that*

$$\int \mathrm{d}\mu_{t\gamma}(\zeta)\ \exp\left(-\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\ V(\phi+\zeta+\xi)\right) \le$$
$$\exp\left(-\frac{1}{2}\frac{g^\rho}{1+g^\rho\gamma}\,\phi^2 + 15\,v^2\,g + \frac{1}{8}\,g^{2\rho-1}\right). \tag{4.32}$$

Proof ¿From (4.22) and (4.28), with $b = \frac{g}{2}$ and $c = 2\,g^{\rho-1}$, we find that

$$\int \mathrm{d}\mu_{t\gamma}(\zeta)\ \exp\left(-\int \mathrm{d}\mu_{(1-t)\gamma}(\xi)\ V(\phi+\zeta+\xi)\right)$$
$$\le \int \mathrm{d}\mu_{t\gamma}(\zeta)\ \exp\left(-\frac{g}{2}\phi^4 + 15\,v^2\,g\right)$$
$$\le \exp\left(-\frac{1}{2}\frac{g^\rho}{1+g^\rho\gamma}\,\phi^2 + 15\,v^2\,g + \frac{1}{8}\,g^{2\rho-1}\right). \quad \square \tag{4.33}$$

4.2.3 Scale transformation

The remaining task is to combine (4.30) with (4.32) and to rescale the field and the coupling. We insert these estimates into (4.14) to obtain the following error bound.

Lemma 4.11 *Let F_1 be given by (4.9). For all $\epsilon \in \left(0, \frac{1}{2}\right)$ and all $L \in \{2, 3, 4, \dots\}$ there exists a maximal coupling $g_{\max}$, depending on L, ϵ, such that for all $(\psi, g, t) \in \mathbb{R} \times \mathbb{G} \times [0, 1]$, we have that*

$$\left|\frac{\partial}{\partial t}X(\psi, g, t)\right| \le g^{\frac{1}{2}-\epsilon}\ F_1(g)\ Z_{QU}(\phi, g). \tag{4.34}$$

Proof Insert (4.30) and (4.32) into (4.14) to conclude that

$$\left|\frac{\partial}{\partial t} X(\psi, g, t)\right| \leq \alpha\,(\delta\,g)^{\frac{1}{2}}\,C(\delta\,g)\,\exp\left(-\frac{\alpha}{2}\,\frac{(\delta\,g)^{\rho}}{1+(\delta\,g)^{\rho}}\,(\beta\,\psi)^{2}\right) \exp\left(15\,\alpha\,v^{2}\,\delta\,g+\left(\frac{\alpha-1}{8}+\frac{1}{4}\right)\,(\delta\,g)^{2\,\rho-1}\right). \tag{4.35}$$

We choose $g_{\max}$ such that

$$\alpha\,\delta^{\frac{1}{2}}\,g^{\epsilon}\,C(\delta\,g)\,\exp\left\{15\,(\alpha\,\delta-1)\,v^{2}\,g\right\} \exp\left\{\left(\frac{\alpha-1}{8}+\frac{1}{4}\right)\,(\delta\,g)^{2\rho-1}-\frac{g^{2\rho-1}}{8}\right\}\,\exp\left\{-a_{QU}(g)\right\} \leq 1 \tag{4.36}$$

and

$$\alpha\,\beta^{2}\,\frac{(\delta\,g)^{\rho}}{1+(\delta\,g)^{\rho}} \geq b_{QU}(g). \quad \square \tag{4.37}$$

The condition (4.37) is easy to fulfill because $\alpha\,\beta^{2} = L^{2}$ is on our side. The condition (4.36) is also easy to fulfill, but it requires $g_{\max}$ to be exponentially small as a function on L.

Lemma 4.12 *Let ϵ, L, and $g_{\max}$ be as in Lemma 4.11. Put*

$$F_{2}(g) = g^{\frac{1}{2}-\epsilon}\,F_{1}(g). \tag{4.38}$$

Then $\|T_{1}(Z)\|_{F_{2}} \leq 1$.

Lemma (4.11) is the first instant in this section where we need a small coupling argument. (Additionally, this F_1 limits the contraction mapping to small couplings.) To deal with large couplings case, we have to look for a modification of (3.9). Since also (4.9) would require a modification, and since the exponent $\sigma = \frac{1}{2} - \epsilon$ is anyway too small to meet the condition (3.24), we will not eleborate on this possibility here. Instead, we will replace (4.1) by an approximant from higher order perturbation theory and modify the estimate of this section for this case.

5 Perturbation theory in g and $g^2\ln(g)$

In this section, we first recall the formal power series solution to the fixed point problem $T(V) = V$, where T is defined by $S(Z) = \exp(-T(V))$ with $Z = \exp(-V)$. As in [41], we develop V into a double perturbation expansion in both g and $g^2\ln(g)$. We then prove a stability bound for the perturbative approximants (of odd order in g), extending the analysis in [44].

5.1 Formal power series representation

In three dimensions, the renormalized ϕ^4-trajectory is not expandable into a formal power series in g. However, it does admit a formal power series representation in both g and $g^2\ln(g)$. See [41].

To simplify the bookkeeping, we prefer g and $\kappa = \ln(g)$ (instead of $g^2 \ln(g)$) as formal expansion parameters. Let $V(\phi, g, \kappa)$ be given by a double formal power series

$$V(\phi, g) = \sum_{r=1}^{\infty} \sum_{a=0}^{\left[\frac{r}{2}\right]} V^{(r,a)}(\phi)\, g^r\, \kappa^a \tag{5.1}$$

with polynomial coefficients

$$V^{(r,a)}(\phi) = \sum_{n=0}^{N(r,a)} P_{2n}(\phi, v)\, V_{2n}^{(r,a)}, \qquad N(r,a) = r - 2a + 1. \tag{5.2}$$

The maximal number of fields N at a given order (r, a) is peculiar to ϕ^4-theory. For safety reasons, we define

$$V_{2n}^{(r,a)} = 0 \qquad n > N(r,a). \tag{5.3}$$

Also, we set the order zero to zero. To first order, the trajectory is defined to be a pure normal ordered ϕ^4-vertex,

$$V_{2n}^{(1,a)} = \delta_{a,0}\, \delta_{n,2}. \tag{5.4}$$

The perturbative fixed point turns out to have two free parameters one for each resonance. See [41]. All of these solutions are suitable approximants for the contraction mapping. We set both parameters to zero,

$$V_2^{(2,0)} = V_0^{(3,0)} = 0. \tag{5.5}$$

The choice (5.5) has the advantage is to have a minimal number of vertices.

The formal power series (5.1) supplies us with a sequence of polynomial approximants

$$V^{(r_{\max})}(\phi, g) = \sum_{r=1}^{r_{\max}} \sum_{a=0}^{\left[\frac{r}{2}\right]} V^{(r,a)}(\phi)\, g^r\, \ln(g)^a \tag{5.6}$$

labeled by the maximal power $r_{\max}$ of g. The first of which, $r_{\max} = 1$, is the above linear approximant. The default value of $r_{\max}$ will be seven in the following.

5.1.1 Recursion relation

To be a fixed point of the extended renormalization group transformation T (the transformation for the interaction V) in the sense of a double formal power series, the coefficients have to satisfy the following recursion relation. Let $\langle \mathcal{O}_1; \ldots ; \mathcal{O}_n \rangle_{\gamma,\Phi}^T$ denote the cumulants associated with the Gaussian moments

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\gamma,\Phi} = \int \mathrm{d}\mu_\gamma(\zeta)\, \mathcal{O}_1(\Phi + \zeta) \cdots \mathcal{O}_n(\Phi + \zeta). \tag{5.7}$$

Lemma 5.1 *Let V be given by (5.1). Let $K(V)_{2n}^{(r,a)}$ be the coefficients defined by*[7]

$$K(V)_{2n}^{(r,a)} = \alpha\,\delta^r \sum_{i=2}^{r} \frac{(-1)^{i+1}}{i!} \sum_{r_1=1}^{r} \sum_{a_1=0}^{[\frac{r_1}{2}]} \cdots \sum_{r_i=1}^{r} \sum_{a_i=0}^{[\frac{r_i}{2}]} \delta_{r,r_1+\cdots+r_i}\, \delta_{a,a_1+\cdots+a_i} \times \frac{1}{(2n)!\,v^n} \int \mathrm{d}\mu_v(\psi)\, P_{2n}(\psi,v) \left\langle V^{(r_1,a_1)}; \dots ; V^{(r_i,a_i)} \right\rangle^T_{\gamma,\beta\psi}. \tag{5.8}$$

Then $Z = \mathrm{e}^{-V}$ is a fixed point of S in the sense of a formal double power series if and only if

$$\left(1 - L^{3-n-r}\right) V_{2n}^{(r,a)} = -L^{-r} \sum_{b=a+1}^{[\frac{r}{2}]} \binom{b}{a} \ln(L)^{b-a}\, V_{2n}^{(r,b)} + K(V)_{2n}^{(r,a)}. \tag{5.9}$$

To derive this set of equations, one performs a cumulant expansion for the hierarchical renormalization group, rescales the coupling, and compares equal double orders (r,a).

Lemma 5.2 *The system of equations (5.9) has a unique solution of the form (5.1) with the properties (5.4) and (5.5).*

Proof The set of equations (5.9) can be solved recursively. The condition (5.3) iterates through the recursion.[8] One proceeds forwards in the order $r-1 \to r$ and, at the order r, backwards in $a \to a-1$. Suppose that we have computed $V_{2n}^{(s,b)}$ both

1. for all (s,b) with $1 \le s \le r-1$ and $0 \le b \le [\frac{s}{2}]$ and
2. for all (s,b) with $s = r$ and $a+1 \le b \le \frac{r}{2}$.

Then this data determines the right hand side of (5.9). Therefrom, we compute $V_{2n}^{(r,a)}$ for all $n \le N(r,a)$. We find two cases.

1. Non-resonant case: If $3-n-r \ne 0$, then (5.9) determines $V_{2n}^{(r,a)}$.
2. Resonant case: If $(r,n) \in \{(2,1),(3,0)\}$, then the left hand side of (5.9) is zero. In both cases, we find a constraint on the right hand side of (5.9).

The two resonances are resolved by logarithmic corrections. Consider the mass resonance $(2,1)$. Since[9]

$$K(V)_2^{(2,1)} = 0, \tag{5.10}$$

[7] The Gaussian integral projects onto the $P_{2n}(\psi)$-component of the cumulant.

[8] One cannot build connected diagrams with more than $2(r-2a+1)$ external legs from $r-2a$ vertices $g : \phi^4 :_v$ and a vertices $g^2 \ln(g) : \phi^2 :_v$.

[9] $g^2 \ln(g) : \phi^2 :_v$ is not generated in the contraction of two vertices $g : \phi^4 :_v$.

the equation labeled by $(r,a,n) = (2,1,1)$ is automatically satisfied. The equation with $(r,a,n) = (2,0,1)$ becomes

$$0 = -L^{-2}\ \ln(L)\ V_2^{(2,1)} + K(V)_2^{(2,0)}. \tag{5.11}$$

We use it to determine $V_2^{(2,1)}$. The other parameter $V_2^{(2,0)}$ is unconstrained. We put it to zero. The vacuum resonance $(3,0)$ is analogously resolved. □

5.2 Stability bound

To any finite order, perturbation theory furnishes approximate solutions to our fixed point problem, which are polynomials in ϕ with coefficients that are polynomials in both g and $g^2 \ln(g)$. We intend to a polynomial of this kind as the approximate fixed point in our contraction mapping. To this aim, we need to prove two properties, a stability bound and an error bound. Both bounds will be proved analogously to [44].

5.2.1 Tree approximation

We first prove stability for the tree approximation, which is defined as the polynomial in ϕ, whose coefficients are simplified to their leading powers in g. This bound extends by continuity to a stability bound for the complete perturbative approximant in a small coupling region.

The set of coefficients $V_{tree}^{(r)} = V_{2(r+1)}^{(r,0)}$ can be computed independently of the others. They define a tree approximation

$$V_{tree}(\phi,g) = \sum_{r=1}^{\infty} \phi^{2(r+1)}\ g^r\ V_{tree}^{(r)} \tag{5.12}$$

to the renormalized ϕ^4-trajectory[10]. Notice that we have replaced $:\phi^{2n}:_v$ by its highest term ϕ^{2n}.

Lemma 5.3 *The recursion relation for $V_{tree}^{(r)}$ decouples. It reads*

$$\begin{aligned}
\left(1 - L^{2(1-r)}\right)\ V_{tree}^{(r)} = \\
L^{3-r} \sum_{i=2}^{r} \frac{(-1)^{i+1}}{i!} \sum_{r_1=1}^{r} \cdots \sum_{r_i=1}^{r} \delta_{r,r_1+\cdots+r_i}\ V_{tree}^{(r_1)} \cdots V_{tree}^{(r_i)} \\
\times \frac{1}{(2(r+1))!\, v^{2(r+1)}} \int d\mu_v(\psi)\ P_{2(r+1)}(\psi,v) \left\langle P_{2(r_1+1)}; \cdots ; P_{2(r_i+1)} \right\rangle^T_{\gamma,\beta\psi}
\end{aligned} \tag{5.13}$$

All vertices in the tree approximation are irrelevant in the extended powercounting. In particular, there are no resonances in the tree approximation. The tree coefficients have a simple sign pattern.

[10]The sum of tree graph contributions is in fact convergent. We will not use this fact, since we are dealing with finite order approximants, which are polynomials in ϕ.

Lemma 5.4 *For all* $r \geq 1$, $V^{(r)}_{tree} = (-1)^{r+1} \left| V^{(r)}_{tree} \right|$.

Proof An induction on the order r. □

It follows that all tree approximants with even maximal order $r_{\max}$ are unstable. Therefore, we will restrict our attention to the friendly approximants with odd maximal order.

The sign pattern remains valid at sufficiently small couplings. To realize this, assemble the perturbative approximant with loop contributions.

Lemma 5.5 *Let* $V^{(-1)}_{tree} = V^{(0)}_{tree} = 0$. *Then we have that*

$$\sum_{r=1}^{r_{\max}} \sum_{a=0}^{\left[\frac{r}{2}\right]} V^{(r,a)}(\phi)\, g^r \, \ln(g)^a = \sum_{n=0}^{r_{\max}+1} \phi^{2n}\, g^{n-1} \left(V^{(n-1)}_{tree} + \lambda_{2n}(g) \right) \tag{5.14}$$

with $\lambda_{2n}(g) = O(g, g^2 \ln(g))$.

The tree coefficients are the leading terms of the perturbative vertex functions at small couplings. The loop corrections are continuous functions of g (since they are polynomials in g and $g^2 \ln(g)$). Therefore, properties like the sign pattern at $g = 0$ extend to a finite region $g \in [0, g_{\max}]$ of small couplings.

5.2.2 Effective ϕ^4-coupling

All $g^2 \ln(g)$-terms are subleading. For this reason, the tree graph bound of [44] applies also to the three dimensional model.

Let $\mu_{2n}(g) = V^{(n-1)}_{tree} + \lambda_{2n}(g)$ (a polynomial in g and $g^2 \ln(g)$) so that the perturbative approximant becomes

$$V^{(r_{\max})}(\phi, g) = \sum_{n=0}^{r_{\max}+1} \phi^{2n}\, g^{n-1}\, \mu_{2n}(g). \tag{5.15}$$

Lemma 5.6 *For all* $r_{\max} \geq 1$, *there exists a maximal coupling* $g_{\max} > 0$ *(depending on* $r_{\max}$*) such that for all* $g \in \mathbb{G}$ *and* $n \in \{2, 3, \ldots, r_{\max}+1\}$,

$$\mu_{2n}(g) = (-1)^n \, |\mu_{2n}(g)|. \tag{5.16}$$

In the following, we will assume that $g_{\max}$ is sufficiently small such that (5.16) holds.

The following statements are presumably true for any finite order $r_{\max} \in 2\mathbb{N}+1$. As a part of their proofs, we will have to compute certain coefficients recursively. I have only done this up to the (already ridiculously high) order $r_{\max} = 99$.

Lemma 5.7 *Let $r_{\max} \in \{1,3,5,\dots,99\}$. Then there exists a maximal coupling $g_{\max} > 0$ such that, for all $g \in \mathbb{G}$, (5.15) is bounded from below by*

$$V^{(r_{\max})}(\phi, g) \geq \sum_{n=0}^{1} \phi^{2n}\, g^{n-1}\, \mu_{2n}(g) + \phi^4\, g\, \rho_4(g), \tag{5.17}$$

where $\rho_4(g)$ is the solution to the recursion relation

$$\rho_{4n}(g) = \mu_{4n}(g) - \frac{\mu_{4n+2}(g)^2}{4\,\rho_{4n+4}(g)} \tag{5.18}$$

with the initial condition

$$\rho_{2(r_{\max}+1)}(g) = \mu_{2(r_{\max}+1)}(g). \tag{5.19}$$

Proof The proof is an induction on powers of ϕ^4. (Notice that $2(r_{\max}+1) \in 4\mathbb{N}$.) The induction step follows from

$$\begin{aligned} &\phi^{4n}\, g^{2n-1} \left\{ \mu_{4n}(g) + \phi^2\, g\, \mu_{4n+2}(g) + \phi^4\, g^2\, \rho_{4n+4}(g) \right\} \\ &= \phi^{4n}\, g^{2n-1} \left\{ \mu_{4n}(g) - \frac{\mu_{4n+2}(g)^2}{4\,\rho_{4n+4}(g)} + \rho_{4n+4}(g) \left(\phi^2\, g + \frac{\mu_{4n+2}}{2\,\rho_{4n+4}} \right)^2 \right\} \end{aligned} \tag{5.20}$$

since $\rho_{4n+4}(g)$ is positive for small couplings. □

A proof of the positivity of the effective ϕ^{4n}-couplings is given below. The solution of the recursion relation (5.18) is a rational function

$$\rho_4(g) = \frac{P(g, g^2 \ln(g))}{Q(g, g^2 \ln(g))}, \tag{5.21}$$

where P and Q are polynomials in g and $g^2 \ln(g)$.

Lemma 5.8 *Let $r_{\max} \in \{1,3,5,\dots,99\}$. There exists a positive number $c > 0$ and a maximal coupling $g_{\max} > 0$ such that for all $g \in \mathbb{G}$, we have that*

$$\rho_4(g) \geq c. \tag{5.22}$$

Proof The value $\rho_4(0)$ is determined as the solution of the recursion relation

$$\rho_{4n}(0) = V_{tree}^{(2n-1)} - \frac{V_{tree}^{(2n)}}{4\,\rho_{4n+4}(0)}, \tag{5.23}$$

with the initial condition $\rho_{2(r_{\max}+1)}(0) = V_{tree}^{(r_{\max})}$. Therefore, it depends only on the tree graph coefficients. An explicit computation shows that $\rho_4(0)$ is a positive number. Since (5.21) is a continuous function of g, the assertion follows. □

The effective ϕ^{4n}-couplings at order seven will be listed below. In particular, the value of $\rho_4(0)$ at order seven is

$$\rho_4(0) = \frac{4306}{5627}. \tag{5.24}$$

As a side remark, we mention that the effective ϕ^4-coupling in the tree approximation is not a small number at large orders. (It presumably converges as the order is taken to infinity.)

Lemma 5.9 *Let $r_{\max} \in \{1, 3, 5, \ldots, 99\}$. There exist positive numbers $g_{\max}$, c, and a (all strictly larger than zero) such that for all $(\phi, g) \in \mathbb{R} \times \mathbb{G}$, we have that*

$$V^{(r_{\max})}(\phi, g) \;\geq\; g\,\left(\frac{c}{2}\,\phi^4 - a\right). \tag{5.25}$$

Proof ¿From (5.17) and (5.19), it follows that there exist two polynomials A and B and a positive number c such that

$$V^{(r_{\max})}(\phi, g) \;\geq\; g\,\left\{A\big(g, g^2\ln(g)\big) + B\big(g, g^2\ln(g)\big)\,\phi^2 + c\,\phi^4\right\}. \tag{5.26}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_{\max}]$, where $g_{\max}$ is a certain positive number. For this maximal coupling, define $\|A\|_\infty = \sup_{g \in \mathbb{G}} |A(g, g^2\ln(g)|$ and analogously $\|B\|_\infty$. Then we have that

$$\begin{aligned} V^{(r_{\max})}(\phi, g) \;&\geq\; g\,\left\{-\|A\|_\infty - \|B\|_\infty\,\phi^2 + c\,\phi^4\right\} \\ &\geq\; g\,\left\{-\|A\|_\infty - \frac{\|B\|_\infty^2}{2\,c} + \frac{c}{2}\,\phi^4\right\}. \end{aligned} \tag{5.27}$$

Put $a = \|A\|_\infty + \frac{\|B\|_\infty^2}{2\,c}$ to obtain the assertion. □

The remaining stability analysis is completely analogous to the linear case.

Lemma 5.10 *Let $r_{\max} \in \{1, 3, 5, \ldots, 99\}$. Let $g_{\max}$ be as in Lemma 5.9. Let $F_1 : \mathbb{G} \to \mathbb{R}^+$ be given by*

$$F_1(g) \;=\; \exp\left(a\,g + \frac{1}{8\,c}\,g^{2\rho-1}\right). \tag{5.28}$$

Then $Z^{(r_{\max})} = \exp\{-V^{(r_{\max})}\}$ is bounded in the norm associated with F_1. We have that

$$\|Z^{(r_{\max})}\|_{F_1} \;\leq\; 1. \tag{5.29}$$

This shows that the perturbative approximants are indeed in the domain of the extended renormalization group. The stability bound is complete aside of a proof of the positivity of $\rho_{4n+4}(g)$. By continuity, it suffices to prove the positivity of $\rho_{4n+4}(0)$, which depends only on the tree coefficients.

5.2.3 Computation of tree coefficients

There is another way to compute the tree coefficients than by the recursion relation (5.13), which uses a Hamilton-Jacobi differential equation. This other way is both simpler than to iterate (5.13) and it also relies on an interpolation formula, which we will need independently in the error bound.

The perturbative renormalization group is the formal power series solution of the non-linear transformation

$$T(V)(\psi, g) \;=\; -\alpha\,\ln\left[\int d\mu_\gamma(\zeta)\,\exp\{-V(\beta\psi + \zeta, \delta g)\}\right]. \tag{5.30}$$

It can be computed in two steps. Step one is the fluctuation integral

$$W(\phi,g,t) \;=\; -\ln\left[\int d\mu_{t\gamma}(\zeta)\;\exp\{-V(\phi+\zeta,\delta g)\}\right], \tag{5.31}$$

evaluated at $t = 1$. The interpolated quantity satisfies the renormalization group differential equation

$$\frac{\partial}{\partial t}W(\phi,g,t) \;=\; \frac{\gamma}{2}\left[\frac{\partial^2}{\partial\phi^2}W(\phi,g,t)-\left\{\frac{\partial}{\partial\phi}W(\phi,g,t)\right\}^2\right] \tag{5.32}$$

in the sense of a formal powerseries, with the inital condition

$$W(\phi,g,0) \;=\; V(\phi,g). \tag{5.33}$$

Step two is the scale transformation of the result of step one,

$$T(V)(\psi,g) \;=\; \alpha\, W\big(\beta\psi,\delta g,1\big). \tag{5.34}$$

Consider the tree approximation hereof. The tree approximation affects only step one. Eq. (5.32) has to be replaced by the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}W_{tree}(\phi,g,t) \;=\; -\frac{\gamma}{2}\left\{\frac{\partial}{\partial\phi}W_{tree}(\phi,g,t)\right\}^2 \tag{5.35}$$

with the initial condition

$$W_{tree}(\phi,g,0) \;=\; V_{tree}(\phi,g). \tag{5.36}$$

The condition of renormalization invariance becomes

$$V_{tree}(\phi,g) \;=\; \alpha\, W_{tree}\big(\beta\psi,\delta g,1\big). \tag{5.37}$$

Lemma 5.11 *The Hamilton-Jacobi equation (5.35) has a unique formal power series solution*

$$W(\phi,g,t) \;=\; \sum_{n=2}^{\infty} B_{2n}(t)\;\phi^{2n}\,g^{n-1} \tag{5.38}$$

with the boundary condition

$$W(\phi,g,1) \;=\; \alpha^{-1}\,W\big(\beta^{-1}\phi,\delta^{-1}g\big) \tag{5.39}$$

and $B_4(0) = 1$. It reads

$$B_{2n}(t) \;=\; b_{2n}\left\{-\gamma\left(\frac{1}{L^2-1}+t\right)\right\}^{n-2}, \tag{5.40}$$

where the coefficients b_{2n} are recursively determined by

$$(n-2)\,b_{2n} \;=\; 2\sum_{m+l=n+1} m\,l\,b_{2m}\,b_{2l} \tag{5.41}$$

with the initial condition $b_4 = 1$.

We remark that this solution makes sense beyond a formal power series. To see this, one writes

$$W(\phi, g, t) = g^{-1} W(\sqrt{g}\,\phi, 1, t) \tag{5.42}$$

and shows inductively a bound on the positive coefficients b_{2n}. However, since we only need the formal power series solution, we leave this issue aside.

Corollary 5.1 *The tree graph coefficients are given by*

$$V_{tree}^{(r)} = b_{2(r+1)} \left(\frac{\gamma}{1 - L^2} \right)^{r-1}. \tag{5.43}$$

This confirms the sign pattern of the tree coefficients.

Once the tree coefficients are written on the blackboard, we proceed to compute the effective ϕ^{4n}-couplings in the tree approximation. With the t-dependence switched on, their recursion relation reads

$$\rho_{4n}(t) = B_{4n}(t) - \frac{B_{4n+2}(t)^2}{\rho_{4n+4}(t)} \tag{5.44}$$

starting at

$$\rho_{2(r_{\max}+1)}(t) = B_{2(r_{\max}+1)}(t). \tag{5.45}$$

Lemma 5.12 *The effective ϕ^{4n}-couplings are given by*

$$\rho_{4n}(t) = r_{4n} \left\{ \gamma \left(\frac{1}{L^2 - 1} + t \right) \right\}^{2(n-1)} \tag{5.46}$$

with (t-independent) numbers r_{4n} determined recursively by

$$r_{4n} = b_{4n} - \frac{b_{4n+2}^2}{r_{4n+4}} \tag{5.47}$$

where

$$r_{2(r_{\max}+1)} = b_{2(r_{\max}+1)}. \tag{5.48}$$

Remarkably, the effective ϕ^4-coupling comes out to be independent of the interpolation parameter t. To seventh order we find the following numbers.

Tree coefficients	
n	$(-1)^n\, b_{2n}$
2	1
3	-8
4	96
5	-1408
6	23296
7	-417792
8	7938048

Seventh order	
n	r_{4n}
4	7938048
3	33817/19
2	90032/1321
1	4306/5627

The error bound is complete for the order seven approximant. As a sidedish, we find the following useful bound which is uniform in the interpolation parameter.

Lemma 5.13 *The tree approximant of order seven*

$$W^{(7)}_{tree}(\phi,g,t) \;=\; \sum_{n=2}^{8} B_{2n}(t)\;\phi^{2n}\;g^{n-1} \tag{5.49}$$

satisfies for all $(\phi,g,t)\in\mathbb{R}\times\mathbb{R}^+\times[0,1]$ *the lower bound*

$$W^{(7)}_{tree}(\phi,g,t) \;\geq\; \frac{4306}{5627}\,g\phi^4. \tag{5.50}$$

The recursion relations (5.41) and (5.47) can be solved by computer algebra. Their solution proves the positivity of the tree approximation (at least) up to the order 99.

5.3 Error bound

To prove an error bound for the higher order approximants, we proceed analogous to the linear case. The main tool is a generalization of the interpolation formula (4.17). ¿From perturbation theory, we have a polynomial

$$V^{(r_{\max})}(\phi,g) \;=\; \sum_{r=1}^{r_{\max}}\sum_{a=0}^{\left[\frac{r}{2}\right]} V^{(r,a)}(\phi)\;g^r\;\ln(g)^a \tag{5.51}$$

which satisfies the scaling relation

$$V^{(r_{\max})}(\psi,g) \;=\; \alpha\sum_{n=1}^{r_{\max}}\frac{(-1)^{n+1}}{n!}\;\mathcal{P}^{(r_{\max})}\left\langle\left[V^{(r_{\max})}(\cdot,\delta g);\right]^n\right\rangle^T_{\gamma,\beta\psi}. \tag{5.52}$$

Here $\mathcal{P}^{(r_{\max})}$ denotes a projector

$$\mathcal{P}^{(r_{\max})}\left(g^r\ln(g)^a\right) \;=\; \begin{cases} g^r\ln(g)^a & \text{if } r\leq r_{\max} \text{ and}\\ 0 & \text{else.}\end{cases} \tag{5.53}$$

The truncated cumulant expansion in (5.52) contains terms of higher order than $g^{r_{\max}}$. These are projected out by means of $\mathcal{P}^{(r_{\max})}$.

Definition 5.1 *Let* $W^{(r_{\max})}:\mathbb{R}\times\mathbb{G}\times[0,1]$ *be defined by*

$$W^{(r_{\max})}(\psi,g,t) \;=\; \sum_{n=1}^{r_{\max}}\frac{(-1)^{n+1}}{n!}\;\mathcal{P}^{(r_{\max})}\left\langle\left[V^{(r_{\max})}(\cdot,\delta g);\right]^n\right\rangle^T_{t\gamma,\psi}. \tag{5.54}$$

This interpolation is identical with the formal power series solution of (5.31), projected to $\mathcal{P}^{(r_{\max})}$. Its boundary values are

$$W^{(r_{\max})}(\psi,g,0) \;=\; V^{(r_{\max})}(\psi,g) \;=\; \alpha\,W^{(r_{\max})}(\beta\psi,\delta g,1). \tag{5.55}$$

We use it to define the following generalization of (4.11) (the case $r_{\max}=1$).

Definition 5.2 *Let $X^{(r_{\max})} : \mathbb{R} \times \mathbb{G} \times [0,1]$ be defined by*

$$X^{(r_{\max})}(\psi, g, t) = \left\{ \int \mathrm{d}\mu_{t\gamma}(\zeta) \, \exp\Big(-W^{(r_{\max})}(\beta\psi + \zeta, \delta g, 1-t) \Big) \right\}^{\alpha} \tag{5.56}$$

To be well defined, eq. (5.56) calls for a stability bound for the interpolation (5.55). Postpone this issue for a short while. Eq. (5.56) yields the following representation for the error term.

Lemma 5.14 *Let $r_{\max} \in \{1, 3, 5, \ldots, 99\}$. Then (5.56) is well defined for the perturbative approximant (5.51). We have that*

$$T_1\big(Z^{(r_{\max})}\big)(\psi, g) = X^{(r_{\max})}(\psi, g, 1) - X^{(r_{\max})}(\psi, g, 0). \tag{5.57}$$

The usefulness of this representation relies on the following differential formula.

Lemma 5.15 *Let $r_{\max} \in \{1, 3, 5, \ldots, 99\}$. Then (5.56) is continuously differentiable in $t \in (0,1)$. We have that*

$$\begin{aligned}
&\frac{\partial}{\partial t} X^{(r_{\max})}(\psi, g, t) = \\
&\alpha \left\{ \int \mathrm{d}\mu_{t\gamma}(\zeta) \, \exp\Big(-W^{(r_{\max})}(\beta\psi + \zeta, \delta g, 1-t) \Big) \right\}^{\alpha - 1} \\
&\quad \times \int \mathrm{d}\mu_{t\gamma}(\zeta) \, \exp\Big(-W^{(r_{\max})}(\beta\psi + \zeta, \delta g, 1-t) \Big) \\
&\quad \times \frac{\gamma}{2} \left[1 - \mathcal{P}^{(r_{\max})} \right] \left(\frac{\partial}{\partial \zeta} W^{(r_{\max})}(\beta\psi + \zeta, \delta g, 1-t) \right)^2.
\end{aligned} \tag{5.58}$$

Proof

$$\begin{aligned}
&\frac{\partial}{\partial t} \int \mathrm{d}\mu_{t\gamma}(\zeta) \, \exp\left[-\mathcal{P} \left\{ \ln \int \mathrm{d}\mu_{(1-t)\gamma}(\xi) \, \exp(-V(\phi + \zeta + \xi, g)) \right\} \right] \\
&= \int \mathrm{d}\mu_{t\gamma}(\zeta) \left(\frac{\partial}{\partial t} + \frac{\gamma}{2} \frac{\partial^2}{\partial \zeta^2} \right) \exp\left[-\mathcal{P} \left\{ \qquad \right\} \right] \\
&= \int \mathrm{d}\mu_{t\gamma}(\zeta) \, \exp\left[-\mathcal{P} \left\{ \qquad \right\} \right] \\
&\quad \times \left[-\left(\frac{\partial}{\partial t} + \frac{\gamma}{2} \frac{\partial^2}{\partial \zeta^2} \right) \mathcal{P} \left\{ \qquad \right\} + \frac{\gamma}{2} \left(\frac{\partial}{\partial \zeta} \mathcal{P} \left\{ \qquad \right\} \right)^2 \right]
\end{aligned} \tag{5.59}$$

and

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{\gamma}{2} \frac{\partial^2}{\partial \zeta^2} \right) \mathcal{P} \left\{ \qquad \right\} &= \mathcal{P} \left(\frac{\partial}{\partial t} + \frac{\gamma}{2} \frac{\partial^2}{\partial \zeta^2} \right) \left\{ \qquad \right\} \\
&= \mathcal{P} \frac{\gamma}{2} \left(\frac{\partial}{\partial \zeta} \left\{ \qquad \right\} \right)^2 \\
&= \frac{\gamma}{2} \mathcal{P} \left(\frac{\partial}{\partial \zeta} \mathcal{P} \left\{ \qquad \right\} \right)^2. \quad \square
\end{aligned} \tag{5.60}$$

The upstairs factor is understood as a synonym for the truncated cumulant expansion. It is a polynomial expression. The downstairs factor in (5.59) is in particular a polynomial expression and (5.60) is a manipulation of a polynomial expression.

The important feature of (5.58) is that in the downstairs factor all orders lower than $r_{\max}$ are cancelled by the t-dependent upstairs factor. To be well defined for all values of the interpolation parameter, (5.56) requires an additional stability bound.

Lemma 5.16 *Let $r_{\max} \in \{1, 3, 5, \ldots, 99\}$. There exist positive numbers $g_{\max}$, c, and a (all dependent on $r_{\max}$) such that the following stability bound holds for all $(\phi, g, t) \in \mathbb{R} \times \mathbb{G} \times [0, 1]$:*

$$W^{(r_{\max})}(\phi, g, t) \;\geq\; g\, \left(\frac{c}{2}\phi^4 - a\right). \tag{5.61}$$

Proof

Since $W^{(r_{\max})}(\phi, g, 0) = V^{(r_{\max})}(\phi, g)$, we know that the bound (5.61) is valid at $t = 0$. Furthermore, we have shown that the effective ϕ^4-coupling is independent of t in the tree approximation. The assertion now follows from the uniform continuity of the effective ϕ^4-coupling of the complete perturbative approximants.

As the result of a truncated cumulant expansion, we have that

$$W^{(r_{\max})}(\phi, g, t) \;=\; \sum_{n=0}^{r_{\max}+1} \phi^{2n}\, g^{n-1}\, \mu_{2n}(g, g^2, t). \tag{5.62}$$

For $n \geq 2$, each coupling $\mu_{2n}(g, g^2, t)$ is the sum of a tree term and loop contributions

$$\mu_{2n}(g, g^2, t) \;=\; B_{2n}(t) + \lambda_{2n}(g, g^2 \ln(g), t). \tag{5.63}$$

The tree term is given by (5.40). The loop contributions are higher order corrections

$$\lambda_{2n}(g, g^2 \ln(g), t) \;=\; O(g, g^2 \ln(g)) \tag{5.64}$$

as they are polynomials in g and $g^2 \ln(g)$ whose coefficients are polynomials in t.

Consider the effective ϕ^4-coupling $\rho_4(g, t)$ defined as above.[11] As it is a continued fraction of couplings (5.63), it is a rational function (of g, $g^2 \ln(g)$, and t) on some rectangle $[0, g'_{\max}] \times [0, 1]$. Since the tree approximation has this particular t-dependence, we have that $\rho_4(0, t) = r_4$ for all $t \in [0, 1]$ at $g = 0$. Let $c = r_4/2$. By continuity, there exists a positive number $g_{\max}$ (with $0 < g_{\max} \leq g'_{\max}$) such that for all $(g, t) \in [0, g_{\max}] \times [0, 1]$, we have that

$$\rho_4(g, t) \geq c. \tag{5.65}$$

Taking care of the constant and quadratic term in ϕ analogously to (5.26) and (5.27), the assertion follows. □

As in the linear case, the stability bound on the interpolated interaction is independent of the interpolation parameter. The remaining analysis is completely analogous to the linear case.

[11] To be precise, we should consider the collections of all ϕ^{4n}-couplings $\rho_{4n}(g, t)$ and repeat the following reasoning for all of them.

5.3.1 Large field domination

The harvest of the higher order perturbation theory is a higher power of g in the bound after dominating the large fields by part of the stability estimate.

Lemma 5.17 *Let* $r_{\max} \in \{1,3,5,\ldots,99\}$. *For all* $(\phi,g,t) \in \mathbb{R}\times\mathbb{G}\times[0,1]$, *we have that*

$$\exp\Big\{-W^{(rmax)}(\phi,g,t)\Big\}\,\frac{\gamma}{2}\,\Big[1-\mathcal{P}^{(r_{\max})}\Big]\,\left(\frac{\partial}{\partial\phi}\,W^{(rmax)}(\phi,g,t)\right)^2 \leq C(g)\,g^{r_{\max}/2}\,\exp\left(-\frac{c}{4}\,g\,\phi^4+a\,g\right) \tag{5.66}$$

for some polynomial $C \in \mathbb{R}^+[g,g^2\ln(g)]$ *(with positive coefficients).*

Proof

The downstairs factor is a polynomial of the form

$$\begin{aligned}
&\frac{\gamma}{2}\,\Big[1-\mathcal{P}^{(r_{\max})}\Big]\,\left(\frac{\partial}{\partial\phi}\,W^{(rmax)}(\phi,g,t)\right)^2 \\
&= \sum_{n=0}^{r_{\max}+1}\phi^{2n}\,g^{r_{\max}+1}\,B_{2n}(g,g^2\ln(g),t) \\
&+ \sum_{n=r_{\max}+2}^{2r_{\max}+1}\phi^{2n}\,g^{n-1}\,B_{2n}(g,g^2\ln(g),t)
\end{aligned} \tag{5.67}$$

with certain polynomials B_{2n}. Notice that the projector affected only the first sum in (5.67). Notice also that the highest power of ϕ is $2(2(r_{\max}+1))-2$, where -2 comes from the two ϕ-derivatives. With the help of the stability bound, we find the upper bound

$$\begin{aligned}
&\left|\exp\Big\{-W^{(rmax)}(\phi,g,t)\Big\}\,\frac{\gamma}{2}\,\Big[1-\mathcal{P}^{(r_{\max})}\Big]\,\left(\frac{\partial}{\partial\phi}\,W^{(rmax)}(\phi,g,t)\right)^2\right| \\
&\leq \exp\left(-\frac{c}{4}\,g\,\phi^4+g\,a\right)\left\{\sum_{n=0}^{r_{\max}+1} g^{r_{\max}-\frac{n}{2}+1}\,A_{2n}\,|B_{2n}(g,g^2\ln(g),t)|\right. \\
&\left.+\sum_{n=r_{\max}+2}^{2r_{\max}+1} g^{\frac{n}{2}-1}\,A_{2n}\,|B_{2n}(g,g^2\ln(g),t)|\right\}
\end{aligned} \tag{5.68}$$

where

$$A_{2n} = \sup_{\phi\in\mathbb{R}}\left|\exp\left(-\frac{c}{4}\phi^4\right)\,\phi^{2n}\right|. \tag{5.69}$$

Expand the polynomials B and take the supremum of $t \in [0,1]$ in each term to arrive at the bound (5.66). □

5.3.2 Fluctuation integral and scale transformation

The fluctuation integral and the scale transformation are now identical to the linear case aside of a minimal cosmetic modification to include the constant c. We therefore do not repeat them here and jump to the following conclusion.

Lemma 5.18 *Let $r_{\max} \in \{1, 3, 5, \ldots, 99\}$. Let F_1 be the function (5.28) from the stability bound. For all $\epsilon \in \left(0, \frac{r_{\max}}{2}\right)$ and all $L \in \{2, 3, 4, \ldots\}$, there exists a maximal coupling $g_{\max}$ such that for $(\psi, g, t) \in \mathbb{R} \times \mathbb{G} \times [0, 1]$, we have that*

$$\left| \frac{\partial}{\partial t} X^{(r_{\max})}(\psi, g, t) \right| \leq g^{\frac{r_{\max}}{2} - \epsilon} F_1(g)\, Z_{QU}(\psi, g). \tag{5.70}$$

The error bound is an immediate consequence hereof.

Corollary 5.2 *Let ϵ, L, and $g_{\max}$ be as in Lemma 5.18. Put*

$$F_2(g) = g^{\frac{r_{\max}}{2} - \epsilon} F_1(g). \tag{5.71}$$

Then we have that $Z^{(r_{\max})} = \exp\left(-V^{(r_{\max})}\right)$ satisfies the bound

$$\| T_1 \left(Z^{r_{\max}}\right) \| \leq 1. \tag{5.72}$$

We have computed $V^{(r_{\max})}(g)$ as a polynomial approximant of a formal double expansion in g and $g^2 \ln(g)$. Then we have shown that, for sufficiently small (but finite) couplings, $Z^{(r_{\max})} = \exp\left(-V^{(r_{\max})}\right)$ satisfies both

1. the stability bound $\|Z^{(r_{\max})}\|_{F_1} \leq 1$, where F_1 is a function of the form (3.32), and
2. the error bound $\|T_1(Z_1)\|_{F_2} \leq 1$, where F_2 is given by (3.23), with exponent $\sigma = \frac{r_{\max}}{2} - \epsilon$.

For $r_{\max} \geq 7$ and ϵ not too large, all assumptions of the contraction mapping are satisfied. The construction is complete.

6 Conclusions and outlook

The iteration of the contraction mapping provides a convergent representation for the ϕ_3^4-trajectory. It can be used to study the properties of the fixed point $Z_\star(\phi, g)$. One important problem, which can be shown, but which will not be shown here, is that $Z_\star(\phi, g)$ is positive. A brief discussion of its positivity is contained in [44]. Other questions about $Z_\star(\phi, g)$ could also be studied in principal, for instance the summability of perturbation theory, and analyticity properties of its Borel transform.

A very interesting question is the behavior of $Z_\star(\phi, g)$ at large couplings. Conceivably, $Z_\star(\phi, g)$ connects the trivial fixed point at $g = 0$ with the non-trivial infrared fixed point at $g = \infty$. The contraction mapping is potentially capable of a construction, which is uniform in the running coupling. But such an enterprise requires a better approximate fixed point $Z_1(\phi, g)$ than the one from perturbation theory. It is conceivable that one could extend the approximants from [31, 32, 33, 34] to achieve this aim.

The underlying scheme of this paper is to compute a renormalized trajectory as a renormalization invariant curve in the unstable manifold of a renormalization group fixed point. This scheme is certainly translatable to virtually every theory treated so far with the renormalization group. In particular, all hierarchical models mentioned in the introduction can be handled that way. We hope to present an extension of this method to the framework of polymer expansions and full models in future work. Another aspect of this theory is the question how traditionally computed renormalized actions converge to the renormalized trajectory. In other words, what is the domain of attraction of this extended fixed point of an extended renormalization group. This question is related to the problem of renormalization group improved actions and also to the question how to truncate a renormalization group such as to maintain control of the errors. We hope to make progress on these and other questions in this context in future work.

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