Zeitschrift: Helvetica Physica Acta
Band: 72 (1999)
Heft: 5-6

Artikel: The sl_q(2)-covariant oscillator algebra with q a root of unity
Autor: Chung, W.-S.
DOI: https://doi.org/10.5169/seals-117188

Helv. Phys. Acta 72 (1999) 408 - 418
0018-0238/99/050408-11 \$ 1.50+0.20/0

Helvetica Physica Acta

The $sl_q(2)$-Covariant Oscillator Algebra with q a Root of Unity.

W-S.Chung
Department of Physics and Research Institute of Natural Science,
Gyeongsang National University,
Jinju,660-701,Korea

Abstract

When q is a root of unity, rwo mode extension of a single q-oscillator algebra was constructed. It was shown that this algebra ($sl_q(2)$-covariant oscillator algebra) is covariant under $sl_q(2)$. The coherent states and coherence factor were computed.

1 Introduction

The theory of quantum groups [1-3] has led to the generalization (deformation) of the oscillator (boson, fermion) algebras in several directions. The development of differential calculus in non-commutative (quantized) spaces has identified multimode systems of deformed creation and annihilation operators covariant under the actions of quantum groups [4-6].Generalization of the usual boson-fermion realizations to quantized Lie algebras and superalgebras have resulted in the study of single-mode deformed bosons [7,8] and fermions [9,10].

A single-mode q-oscillator with the creation ($a^\dagger$), annihilation (a) and number operators (N) operators obeying the relations

$$aa^\dagger - qa^\dagger a = 1, \quad [N,a] = -a, \quad [N,a^\dagger] = a^\dagger$$

has been the subject of study by some authors [11-13] in the past, independent of the recent developments due to the theory of quantum groups.

When the deformation parameter q is real, the first relation of the above equation is invariant under the hermitian conjugation. So, $a^\dagger$ can be interpreted as a hermitian conjugate operator of a. But, the situation is different when q is a complex number.

In this paper, we restrict our discussion to the case that q is a root of unity. In this case, the first relation of the above equation is not invariant under the hermitian conjugation any more. Thus, we should rewrite the q-oscillator algebra as follows;

$$aa_+ - qa_+a = 1,$$

where a_+ is not a hermitain conjugate of a. But, a and a_+ play roles of lowering and raising operators, respectively, if the following relations maintain

$$[N, a_+] = a_+, \quad [N, a] = -a.$$

Some representation theory of the q-oscillator algebra with q a root of unity is discussed elsewhere [14,15].

In this paper, we discuss two-mode q-oscillator system which is covariant under some quatum group $sl_q(2)$. In the following we restrict our discussion to the case that q is a k-th primitive root of unity

$$q^k = 1, \quad or \quad q = e^{\frac{2\pi}{k}i}.$$

2 $sl_q(2)$-covariant Oscillator Algebra

When q is real, quantum group covariant oscillator algebra was firstly introduced by Pusz and Woronowicz [4-6]. They demanded the $gl_q(n)$-covariance among step operators. However, when q is a root of unity, $gl_q(n)$-covariance should be replaced with $sl_q(n)$-covariance. Following their technique, we can write the $sl_q(2)$-covariant two-mode oscillator algebra as follows;

$$\begin{aligned}
a_1 a_2 &= q a_2 a_1, \\
a_{1+} a_{2+} &= q^{-1} a_{2+} a_{1+}, \\
a_1 a_{2+} &= q a_{2+} a_1, \\
a_2 a_{1+} &= q a_{1+} a_2, \\
a_1 a_{1+} - q^2 a_{1+} a_1 &= 1, \\
a_2 a_{2+} - q^2 a_{2+} a_2 &= 1 + (q^2 - 1) a_{1+} a_1.
\end{aligned} \tag{1}$$

When $q^k = 1$, from the algebra (1), we see that both a_i^k and a_{i+}^k commute with all operators of algebra (1), which means that they are central elements of algebra (1). So we can set

$$a_{i+}^k = a_i^k = 0 \tag{2}$$

and we have the finite dimensional representation.

Now we will prove the $sl_q(2)$-covariance of the algebra (1) explicitly. In order to do so, we should introduce the $sl_q(2)$-matrix. An $sl_q(2)$-matrix can be written in the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the following commutation relations hold

$$ad - da = (q - \frac{1}{q})bc,$$

$$ab = qba, \qquad cd = qdc,$$

$$ac = qca, \qquad bd = qdb,$$

$$bc = cb, \qquad det_q M = ad - qbc = 1. \tag{3}$$

By the $sl_q(2)$-covariance of the system, it is meant that the linear trns-formations

$$M\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix},$$

$$(\, a_{1+} \quad a_{2+} \,) M^{-1} = (\, a_{1+} \quad a_{2+} \,)\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = (\, a_{1+}' \quad a_{2+}' \,) \tag{4}$$

lead to the same commutation relations (1) for (a_1', a_{1+}') and (a_2', a_{2+}'). It should be noted that the particular coupling between the two modes is completely dictated by the required $sl_q(2)$-covariance.

The Fock space representation of the algebra (1) can be easily constructed by introducing the hermitian number operators $\{N_1, N_2\}$ obeying

$$[N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_{j+}] = \delta_{ij} a_{j+}, \quad (i, j = 1, 2). \tag{5}$$

Let $|0, 0>$ be the unique ground state of this system satisfying

$$N_i|0,0>= 0, \quad a_i|0,0>= 0, \quad (i = 1, 2) \tag{6}$$

and $\{|n, m> |n, m = 0, 1, 2, \cdots, s\}$ be the set of the orthonormal number eigenstates

$$N_1|n, m>= n|n, m>, \qquad N_2|n, m>= m|n, m>,$$

$$< n, m|n', m' >= \delta_{nn'}\delta_{mm'}. \tag{7}$$

From the algebra (1) the representation is given by

$$a_1|n, m >= \sqrt{[n]}|n-1, m >, \quad a_2|n, m >= q^n\sqrt{[m]}|n, m-1 >,$$

$$a_{1+}|n, m >= \sqrt{[n+1]}|n+1, m >, \quad a_{2+}|n, m >= q^n\sqrt{[m+1]}|n, m+1 >, \tag{8}$$

where the q-number $[x]$ is defined as

$$[x] = \frac{q^{2x}-1}{q^2-1}.$$

Since a_{i+} is not a hermitian conjugate operator of a_i, we introduce two hermitian conjugate operators a_i^+ and $a_{i+}^\dagger$, where both are hermitian conjugate operators of a_i and a_{i+}, respectively. The representation of $a_i^\dagger$ and $a_{i+}^\dagger$ are given by

$$a_1^\dagger|n, m >= \sqrt{[n+1]^*}|n-1, m >,$$

$$a_2^\dagger|n, m >= q^{-n}\sqrt{[m+1]^*}|n, m+1 >,$$

$$a_{1+}^\dagger|n, m >= \sqrt{[n]^*}|n-1, m >,$$

$$a_{2+}^\dagger|n, m >= q^{-n}\sqrt{[m]^*}|n, m-1 >, \tag{9}$$

where * implies a complex conjugation. Then, $a_{i+}^\dagger$ and $a_i^\dagger$ play roles of annihilation and creation operators, respectively and they satisfy

$$(a_{i+}^\dagger)^k = (a_i^\dagger)^k = 0. \tag{10}$$

At this point, it is worth noticing that the following relations hold;

$$a_i a_i^\dagger - q a_i^\dagger a_i = q^{-N_i},$$

$$a_1 a_2^\dagger = q^{-1} a_2^\dagger a_1, \quad a_2 a_1^\dagger = q a_1^\dagger a_2,$$

$$a_{i+}^\dagger a_{i+} - q a_{i+} a_{i+}^\dagger = q^{-N_i},$$

$$a_{1+}^\dagger a_{2+} = q a_{2+} a_{1+}^\dagger, \quad a_{2+}^\dagger a_{1+} = q^{-1} a_{1+} a_{2+}^\dagger,$$

$$a_{i+} a_i^\dagger = q a_i^\dagger a_{i+},$$

$$a_{i+} a_j^\dagger = q a_j^\dagger a_{i+}, \quad (i \neq j), \tag{11}$$

where we used the formulas

$$[x][x]^* = q^{2-2x}[x]^2,$$

$$\sqrt{[x+1][x+1]^*} - q\sqrt{[x][x]^*} = q^{-x},$$

$$\sqrt{[x+1][x]^*} = q\sqrt{[x+1]^*[x]}.$$

3 Coherent States

There exists several methods for constructing coherent states. In this section, we construct coherent states as eigenstates for annihilation operators a_1 and a_2. From the definition of coherent states

$$a_1|z) = z_1|z),$$

$$a_2|z) = z_2|z),$$

where $|z) = |z_1, z_2)$ and z_i's are called coherent variables. In the case of the ordinary harmonic oscillator, the coherent variables are complex numbers. Here, the situation is a little bit different. From the generalized nilpotency of annihilation operators

$$a_1^k = a_2^k = 0,$$

the coherent variables should be generalized Grassmann variables obeying

$$z_1^k = z_2^k = 0.$$

From the commutation relations of two annihilation operators, we have

$$z_1 z_2 = q^{-1} z_2 z_1. \tag{12}$$

If we introduce the complex conjugation of z_i as z_i^* and assume that $(ab)^* = b^* a^*$, we have

$$z_1^* z_2^* = q^{-1} z_2^* z_1^* \tag{13}$$

and

$$(z_1^*)^k = (z_2^*)^k = 0. \tag{14}$$

Then, we obtain the unnormaized coherent state as follows;

$$|z) = \sum_{n,m=0}^{k-1} \frac{z_1^n z_2^m}{\sqrt{[n]![m]!}} |n, m> . \tag{15}$$

Its dual state is given by

$$(z| = \sum_{n,m=0}^{k-1} < nm| \frac{(z_2^*)^m (z_1^*)^n}{\sqrt{[n]^*![m]^*!}}. \tag{16}$$

From the algebra (1), we can obtain the remaining commutation relations;

$$z_1^* z_2 = q z_2 z_1^*,$$

$$z_2^* z_1 = q^{-1} z_1 z_2^*,$$

$$z_1^* z_1 = z_1 z_1^*,$$

$$z_2^* z_2 = z_2 z_2^*, \tag{17}$$

The length $z_i^* z_i$ commutes with all coherent variables and their complex conjugates. Using this fact, the norm of coherent states is given by

$$(z|z) = E_k(z_1^* z_1) E_k(z_2^* z_2), \tag{18}$$

where

$$E_k(x) = \sum_{n=0}^{k-1} \frac{q^{\frac{1}{2}n(n-1)}}{[n]!} x^n$$

and we used the formula

$$[n]^*! = q^{-n(n-1)}[n]!.$$

4 Coherence Factor

The coherence factor turns out to be very important in quantum optics and multiphoton spectroscopy. It occurs in the expression of the absorption or emission multiphoton intensity. We define here the coherence factor $g_i^{(m)}$ of order m by

$$g_i^{(m)} = \frac{< (a_i^\dagger)^m a_i^m >}{< a_i^\dagger a_i >^m}, \tag{19}$$

where we adopted the notation

$$< X > = \frac{(z|X|z)}{(z|z)}$$

for denoting the expectation value of the operator X on the coherent state $|z)$.

It is then a simple problem to prove that

$$g_1^{(m)} = \theta(k - 1 - m),$$

$$g_2^{(m)} = \frac{[E_k(q^{2m} z_1^* z_1)]}{[E_k(q^2 z_1^* z_1)]^m} \theta(k-1-m), \tag{20}$$

where the step function θ is defined by

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} .$$

For example, when $k = 3$, the non-vanishing coherent factors are

$$g_1^{(1)} = g_1^{(2)} = 1,$$

$$g_2^{(1)} = 1,$$

$$g_2^{(2)} = [1 + (2+3q) z_1^* z_1 + (6q-1)(z_1^* z_1)^2],$$

where $q = e^{\frac{2\pi i}{3}}$.

When k goes to ∞ (boson limit), we have $q \longrightarrow 1$ and $E_k(x)$ becomes an ordinary exponential function. So, we have

$$g_i^{(m)} = 1 \quad for \quad m = 1, 2, \cdots, \tag{21}$$

which generalizes well-known results for ordinary bosons. In the case where k is an arbitrary integer greater than two, the vanishing of the coherent factor $g_i^{(m)}$ for $m > k-1$ indicates that a given quantum state cannot be occupied by more than $k-1$ identical particles, which generalizes the Pauli's exclusion principle.

5 Conclusion

In this paper, I have obtained two mode oscillator algebra which is covariant under $sl_q(2)$ transform when q is a k-th primitive root of unity. In this case, it was shown that there exist two types of creation (annihilation) operators. The Fock representation for this algebra was obtained and the coherent state for this algebra was explicitly constructed by introducing the generalization of the Grassmann variables. Using these results, the coherence factor of order m was computed and it was shown that for this algebra a quantum state cannot be occupied by more than $k-1$ identical particles. These results can be easily extended to more general case, multimode oscillator algebra which is covariant under a quantum group $sl_q(n)$. Therefore, in this paper, I restricted my concern to two mode case.

Acknowledgement

This paper was supported by the KOSEF (981-0201-003-2).

References

[1] M.Jimbo,Lett.Math.Phys.10 (1985) 63.

[2] V.Drinfeld,Proc.Intern.Congress of Mathematicians (Berkeley, 1986) 798.

[3] S.Woronowicz, Commun.Math.Phys.111(1987) 613.

[4] W.Pusz and S.Woronowicz, Reports on Math.Phys.27 (1989) 231.

[5] W.Pusz , Reports on Math.Phys.27 (1989) 349.

[6] J.Wess and B.Zumino, Nucl.Phys.Proc.Suppl. B18 (1990) 302.

[7] A.Macfarlane,J.Phys.A22(1989) 4581.

[8] L.Biedenharn, J.Phys.A22(1989)L873.

[9] M.Chaichian and P.Kulish, Phys.Lett. B234 (1990) 72.

[10] R.Parthasarathy and K.Viswanathan, J.Phys.A24 (1991) 613.

[11] M.Arik and D.Coon, J.Math.Phys.17 (1976) 524.

[12] D.Coon, S.Yu and M.Baker, Phys.Rev.D15 (1972) 1429.

[13] V.Kuryshkin, Ann.Found.Loise de Broglie 5 (1980) 111.

[14] W.S.Chung, Helv.Phys.Acta 70 (1997) 367.

[15] M.Kibler, " The k-fermions as objects interpolating between fermions and bosons", *talk given in VIII International Conference on Symmetry Methods in Physics,1997.*