Zeitschrift: Helvetica Physica Acta
Band: 72 (1999)
Heft: 5-6

Artikel: Feynman path integral and Toeplitz quantization
Autor: Charles, Laurent
DOI: https://doi.org/10.5169/seals-117184

Helv. Phys. Acta 72 (1999) 341 - 355
0018-0238/99/050341-15 $ 1.50+0.20/0

Helvetica Physica Acta

Feynman Path Integral and Toeplitz Quantization

by

Laurent CHARLES

Centre de Recherche en Mathématiques de la Décision
Université de Paris IX- Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris cedex 16, France

CEA/Saclay, Service de Physique Théorique
F-91 191 Gif sur Yvette Cedex, FRANCE

ABSTRACT. - We present a Feynman path integral in the setting of geometric quantization of symplectic manifolds with Kähler polarization, where the Hamiltonian operator is given by Toeplitz quantization. We compute the quantum propagator as a limit of path integrals, involving Brownian motion in the phase space and geometricaly meaningful stochastic processes.

RÉSUMÉ. - Nous présentons un formalisme pour l'intégrale de Feynman, dans le cadre de la quantification géométrique des variétés symplectiques munies d'une polarisation kählerienne, où l'hamiltonien est donné par une quantification de Toeplitz. Nous calculons le propagateur quantique comme une limite d'intégrales stochastiques, en introduisant un mouvement brownien sur l'espace des phases et d'autres processus liés à la structure géométrique.

1 Introduction

Let us describe formally what is Feynman's path integral. We consider a particle of mass m moving in a potential $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$. The quantum state space is $L^2(\mathbb{R}^3)$. The time evolution of a state Ψ_0 is given by the Schrödinger equation

$$-\frac{\hbar}{i}\frac{d}{dt}\Psi_t = H\,\Psi_t$$

where $H = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) + V$ is the Hamiltonian operator.

Feynman's idea is to express Ψ_t as a sum over paths

$$\Psi_t(x) = \int_{\Omega_{x,t}} e^{\frac{i}{\hbar}\int_0^t L(c(s),c'(s))ds}\Psi_0(c(0))dc \tag{1}$$

where $\Omega_{x,t}$ is the set of paths $c : [0,t] \longrightarrow \mathbb{R}^3$ such that $c(t) = x$, dc is a measure on $\Omega_{x,t}$, and L denotes the Lagrangian of the system defined by

$$L(x, \dot{x}) = \frac{1}{2} m |\dot{x}|^2 - V(x)$$

With formula (1), Feynman developed many concepts of quantum mechanics (see [1]). Path integral methods were used by mathematicians to derive solutions of heat equations. They have also played a major role in quantum field theory. In non-relativistic quantum mechanics, the framework of this article, physicists have exploited Feynman's integral formula to study semi-classical properties. Mathematical works on the subject are reviewed in ([2]).

In this article, we propose a Feynman path integral formula in the setting of Toeplitz quantization of compact Kähler manifolds, with an explicit dependence on $\hbar$. The results we obtain generalise works of I.Daubechies, J.R.Klauder and T.Paul ([3],[4],[5]), treating of Hamiltonians defined on the euclidean space, the Lobachevsky half-plane, and the sphere. We compute the quantum propagator by well-defined path-integrals involving Wiener measure on phase-space in the limit of diverging diffusion constant. Not only does this formulation give a rigorous computation of the solution of the Shrödinger equation, but it also allows a natural geometrical formulation of the problem in terms of the symplectic form, prequantum bundle, and Kähler metric.

In the first section, using the equivalence between the Lagrangian and the Hamiltonian formalisms on the tangent and cotangent bundle of a configuration space, we explain how the phase of the integral of the lagrangian along a path γ is defined on a symplectic manifold (M, ω) endowed with a prequantization (P, α). Recall that $P \longrightarrow M$ is a principal $(\mathbb{R}/2\pi\hbar\mathbb{Z})$ bundle, and α a connection one-form, with curvature ω. Let $H \in C^\infty(M, \mathbb{R})$ be a Hamiltonian function. In this setting, $\exp(\frac{i}{\hbar}\int_0^t L(c(s), c'(s))ds)$ is replaced by the product of two terms, the first being the parallel transport along γ, the second the phase of the integral of the hamiltonian along γ. In the second section, we introduce the prequantum hilbert space $L^2(M, L)$, where L is the Hermitian line bundle associated with P. Using the covariant derivative induced by α, we define the prequantum dynamics, and we give a description of its propagator in proposition 3.1, where the geometric objects introduced above appear.

The third section is devoted to the quantum setting. M is endowed with a Kähler metric g, with fundamental two-form ω. L has a natural holomorphic structure compatible with the covariant derivative. The quantum space $\mathcal{H}$ is the set of holomorphic sections of L. Let $\Pi : L^2(M, L) \longrightarrow L^2(M, L)$ be the orthogonal projector onto $\mathcal{H}$. The Hamiltonian operator is $\Pi L_H \Pi$, where L_H denotes multiplication by H. The fundamental estimate is proved in the fourth section. We show that the propagator $\exp(-i\frac{t}{\hbar}\Pi L_H \Pi)$ of the schrödinger equation is approximated by the heat semi-group of the generalised Laplacian $\nu\Delta_{hol} + \frac{i}{\hbar}L_H$, as ν tends to ∞, where Δ_{hol} is the Hodge Laplacian of L. Using a Weitzenböck formula and a generalised Feynman-Kac formula we express in the last section these heat kernels as path integrals, leading to a Feynman path integral.

Let x_t^ν be the Brownian motion on the Riemannian manifold $(M, \nu g)$, starting at $x_0 \in M$. The phase of the integral action of the sample path is a semi-martingale $\mathcal{P}_t^\nu$, defined as the parallel transport $\mathcal{P}_t^\nu : L_{x_t^\nu} \longrightarrow L_{x_0^\nu}$ along $x_s^\nu, 0 \leq s \leq t$. The main result we obtain is stated as follows

Theorem 1.1. *For every $\Psi \in C^\infty(M, L)$, let $\Psi_t^\nu \in C^\infty(M, L)$ be defined by the path integral*

$$\Psi_t^\nu(x_0) = e^{t\frac{\nu n}{2\hbar}} \, \mathrm{E}\left[e^{\frac{i}{\hbar}\int_t^0 H(x_s^\nu)ds} \mathcal{P}_t^\nu . \Psi(x_t^\nu) \right]$$

then

$$\Psi_t^\nu \longrightarrow e^{-i\frac{t}{\hbar}\Pi L_H \Pi}\Psi \text{ in } \mathcal{H} \text{ as } \nu \longrightarrow \infty$$

Acknowledgments We would like to thank Thierry Paul for suggesting the subject.

2 The phase of the action integral

In this section, we explain how the integrand of the Feynman path integral is defined on a symplectic manifold. Let us review the Lagrangian formulation of classical mechanics on a tangent bundle TQ of a configuration manifold Q (see [6], 3.5,3.6,3.8). Let $L : TQ \longrightarrow \mathbb{R}$ be a smooth function called the Lagrangian. The fiber derivative FL of L is a fiber preserving map from TQ to T^*Q, sometimes called the Legendre transformation. Let us denote by β the canonical one-form of T^*Q defined by $< \beta, v >=< \pi_{T^*Q} v, (\pi_Q)_* v >, \quad \forall v \in T(TQ)$ where $\pi_Q : T^*Q \longrightarrow Q$ and $\pi_{T^*Q} : T(T^*Q) \longrightarrow T^*Q$ denote the canonical projections. The Lagrange one-form β_L and the Lagrange two-form ω_L are defined by

$$\beta_L = FL^*\beta \qquad\qquad \omega_L = FL^* d\beta$$

We assume that FL is a local diffeomorphism, hence (TQ, ω_L) is a symplectic manifold.

We define the action $A : TQ \longrightarrow \mathbb{R}$ by $A(v) =< FL(v), v >$ and the energy by $E = A - L$. The Lagrangian vector field of L is the unique vector field X_E on TQ such that $\iota_{X_E}\omega_L + dE = 0$. X_E is a second order equation (i.e. $(\pi_Q)_* \circ X_E = Id$) and its integral curves $\gamma(t)$ satisfy Lagrange's equation. Using coordinates $(q^i, \dot{q}^i)$ on TQ we recover the classical Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(\gamma(t))\right) = \frac{\partial L}{\partial q^i}(\gamma(t))$$

Let $\Omega(q_1, q_2, T)$ be the set of C^∞ curves $\{c : [0,T] \longrightarrow Q / c(0) = q_1, c(T) = q_2\}$. The Lagrangian integral is a functional defined on $\Omega(q_1, q_2, T)$ by

$$\mathcal{L}(c) = \int_0^T L(c(t), \dot{c}(t))dt$$

The variational principle of Hamilton says that the solutions $\gamma(t) = (c(t), \dot{c}(t))$ of Lagrange equation with $c \in \Omega(q_1, q_2, T)$ are the extremals of the functional $\mathcal{L}$. To prepare the Hamiltonian point of view, we introduce the action integral $\mathcal{A}(c) = \int_0^T A((c(t), \dot{c}(t))dt$ and prove the following

Lemma 2.1. *Let $\gamma : [0,T] \longrightarrow TQ$ be the curve defined by $\gamma(t) = (c(t), \dot{c}(t))$, we have*

$$\mathcal{A}(c) = \int_\gamma \beta_L$$

Proof. It suffices to prove that $A(\gamma(t)) =< \beta_L, \gamma_* \partial_t >$. We have

$$\begin{aligned} < \beta_L, \gamma_*\partial_t > &= < \beta, FL_*\gamma_*\partial_t > && \text{since } \beta_L = FL^*\beta, \\ &= < FL(\gamma(t)), (\pi_Q)_* FL_* \gamma_* \partial_t > && \text{by definition of } \beta \text{ ,} \\ &= < FL(\gamma(t)), \gamma(t) > \end{aligned}$$

since FL is fiber preserving which implies $(\pi_Q)_* \circ FL_* = (\pi_Q)_*$, and $(\pi_Q)_*\gamma_*\partial_t = c_*\partial_t = \gamma(t)$,

$$= A(\gamma(t))$$

□

In the Hamiltonian formalism, the data are a symplectic manifold (M,ω) and a hamiltonian function $H \in C^{\infty}(M,\mathbb{R})$, instead of the Lagrangian on the tangent space of a configuration space. The classical dynamic is the flow of the Hamiltonian vector field X_H, defined by $\iota_{X_H}\omega + dH = 0$. Let (p_i, q^i) be canonical coordinates on M, an integral curve $t \longrightarrow (p_i(t), q^i(t))$ of X_H satisfies Hamilton's equation

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

Recall that M is the cotangent space T^*Q with $\omega = d\beta$, and assume that the Legendre transform FL is a diffeomorphism. The relationship between Hamiltonian and Lagrangian formulations is given by $H = E \circ FL^{-1}$. The integral curves of X_E are mapped by FL onto integral curves of X_H. If $\gamma : [O,T] \longrightarrow T^*Q$ is a path on T^*Q, a natural definition of its Lagrangian integral is $\mathcal{L}(\gamma) = \int_0^T L(FL^{-1}(\gamma(t)))dt$. It follows from lemma (2.1) that

$$\mathcal{L}(\gamma) = \int_\gamma \beta - \int_0^T H(\gamma(t))dt$$

The first term of the sum, which we denote $\mathcal{A}(\gamma)$ and call action integral, is not defined on a general symplectic manifold (M,ω), because ω is not necessarily exact. If γ is a contractible loop on M, we can set $\mathcal{A}(\gamma) = \int_S \omega$, where S is a surface of deformation of γ. This depend on S unless the integral of ω over any two-sphere in M is zero. Since the action integral appears in the Feynman path integral as the argument of the function $x \longrightarrow \exp(\frac{i}{\hbar}x)$, we just need to define the phase of the action integral $\mathcal{P}(\gamma) = \exp(\frac{i}{\hbar}\mathcal{A}(\gamma))$. Observe that $\gamma \longrightarrow \exp(\frac{i}{\hbar}\int_S \omega)$ is well-defined, if we assume that the integrals of ω over the two-sphere in M are multiples of $2\pi\hbar$.

Let us try to define the phase of the action integral of any loop of a symplectic manifold (M,ω). Let $x \in M$. We introduce $C_1(M,x)$ (resp. $C_2(M,x)$) the free abelian group generated by the set of differentiable singular one-simplices (resp. two-simplices) which have all their vertices mapped into the basepoint x. Observe that the generators of $C_1(M,x)$ are the loops based at x. Let $\partial : C_2(M,x) \longrightarrow C_1(M,x)$ be the boundary operator, $Z_2(M,x) = \operatorname{Ker}\partial$ the group of two-cycles, $B_1(M,x) = \operatorname{Im}\partial$ the group of one-boundaries. Assume that M is connected. The first homology group $H_1(M)$ with integer coefficient is $C_1(M,x)/B_1(M,x)$. Let $\mathbb{T}_\hbar$ be the quotient group $\mathbb{R}/2\pi\hbar\mathbb{Z}$ and $I : C_2(M,x) \longrightarrow \mathbb{T}_\hbar$ the morphism of group defined by $I(S) = \int_S \omega \mod 2\pi\hbar\mathbb{Z}$. We formulate the problem of the definition of the phase of the integral action of the loops based at x in the following manne! ! r: find a group homomorphism $\mathcal{P} : C_1(M,x) \longrightarrow \mathbb{T}_\hbar$, such that

$$\begin{tikzcd} C_2 \arrow[r, "\partial"] \arrow[dr, "I"'] & C_1 \arrow[d, "\mathcal{P}"] \\ & \mathbb{T}_\hbar \end{tikzcd} \tag{2}$$

commutes. This problem has a solution if and only if $Z_2(M,x) \subset \operatorname{Ker} I$, that is the periods of ω are multiples of $2\pi\hbar$. If this is the case, then the set of solutions is a principal homogeneous space for the morphisms group $\operatorname{Mor}(H_1(M), \mathbb{T}_\hbar) \cong H^1(M, \mathbb{T}_\hbar)$. We can also solve this problem in a geometric manner with Souriau-Kostant prequantization.

Definition 2.1. Let (M,ω) be a symplectic manifold. A prequantization of (M,ω) is a principal $\mathbb{T}_\hbar$ bundle $\pi : P \longrightarrow M$, together with a connection form $\alpha \in \Omega^1(P,\mathbb{R})$ such that $\pi^*\omega = -d\alpha$

Remark 2.1. The connection is $\mathbb{R}$ valued, because we identify the Lie algebra of $\mathbb{T}_\hbar$ with $\mathbb{R}$ by means of the canonical projection $p : \mathbb{R} \longrightarrow \mathbb{T}_\hbar$

A symplectic manifold (M, ω) admits a prequantization if and only if $\omega/2\pi\hbar$ defines an integral cohomology class, and if this condition is verified, the set of prequantizations is a principal homogeneous space for $H^1(M, \mathbb{T}_\hbar)$ (see [7]). In this context the phase of the action integral is defined as follows

Definition 2.2. Let (M, ω) be a symplectic manifold with prequantization (P, α). The phase of the action integral of a loop γ is the holonomy $\mathcal{P}(\gamma) \in \mathbb{T}_\hbar$ of γ. If γ is a path of M joining x_1 to x_2, the phase of its action integral is the parallel transport along γ: it is a $\mathbb{T}_\hbar$-isomorphism from P_{x_1} onto P_{x_2}.

Remark 2.2. The introduction of parallel transport, surprising at the first view, is appropriate for the following sections, see remark 3.1.

Remark 2.3. If M is a cotangent bundle T^*Q, the trivial bundle $T^*Q \times \mathbb{T}_\hbar$ with the connection form α is $-\pi^*\beta + d\theta$ is a prequantization. The parallel transport along a curve γ is multiplication by $p(\int_\gamma \beta)$.

Remark 2.4. Considering the different prequantizations of (M, ω), we obtain all the group homomorphisms such that diagram 2 commutes.

To complete the parallel between Lagrangian and Hamiltonian formalism, we mention that Weinstein stated variational principles which relate the extremals of the multivalued functional $\gamma \longrightarrow \operatorname{Log} \mathcal{P}(\gamma) - \int H(\gamma(t))dt$ to the integral curves of the Hamiltonian vector field X_H (see [8]).

3 Prequantization

Let (M, ω) be a symplectic compact manifold with prequantization (P, α). Let ρ denote the representation of $\mathbb{T}_\hbar$ on $\mathbb{C}$ whose character is $[\theta] \longrightarrow \exp(\frac{i}{\hbar}\theta)$, and let $L = P \times_\rho \mathbb{C}$ be the associated Hermitian line bundle. The connection α induces a covariant derivation $\nabla : C^\infty(M, L) \longrightarrow \Omega^1(M, L)$ which is compatible with the Hermitian structure $h \in C^\infty(M, L^* \otimes \bar{L}^*)$. The scalar product of two sections Ψ and Ψ' in $C^\infty(M, L)$ is defined by

$$< \Psi, \Psi' >= \int_M h(\Psi, \Psi') \, |\omega^{\wedge n}|$$

The Hilbert space $\mathcal{H}_p = L^2(M, L)$ of prequantisation is the completion of $(C^\infty(M, L), <,>)$. Each classical observable H in $C^\infty(M, \mathbb{R})$ acts on $\mathcal{H}_p$ as an unbounded operator $O_{preq}(H)$, with domain $C^\infty(M, L)$ according to

$$O_{preq}(H)\Psi = (L_H - i\hbar\nabla_{X_H})\Psi, \qquad \forall \Psi \in C^\infty(M, L)$$

where L_H is multiplication by H, and X_H is the Hamiltonian vector field of H. The linear mapping which sends H into the formally selfadjoint differential operator $O_{preq}(H)$ satifies the commutation rules of P.A.M. Dirac. That is,

$$O_{preq}(1) = \mathrm{Id} \tag{3}$$

$$O_{preq}(\{F, G\}) = \frac{i}{\hbar}[O_{preq}(F), O_{preq}(G)] \tag{4}$$

where $\{F, G\} = \omega(X_F, X_G)$ denotes the Poisson bracket of F and G. These properties are generally presented as a motivation for introducing the prequantum bundles. The next step of the quantization procedure is the definition of the Hilbert quantum space as a subspace of $\mathcal{H}_p$ using polarization. Nevertheless, until the end of this section, we continue without adding any structure, because it allows us to introduce some important ideas for following sections.

The dynamics in a prequantum system is given by the Schrödinger equation

$$\frac{d}{dt}(\Psi) = -\frac{i}{\hbar}O_{preq}(H)\Psi \qquad (E_{preq})$$

Proposition 3.1. *Let $M \times \mathbb{R} \longrightarrow M, (y,t) \longrightarrow x_y(t)$ denote the flow of the Hamiltonian vector field X_H. The section $\Psi : M \times \mathbb{R} \longrightarrow L, (x,t) \longrightarrow \Psi_t(x)$ defined by*

$$\Psi_t(x_y(t)) = e^{-\frac{i}{\hbar}\int_0^t H(x_y(s))ds}\mathcal{P}(x_y|_{[0,t]}).\Psi_0(y) \qquad (5)$$

is a solution of equation (E_{preq}) with initial condition Ψ_0 in $C^\infty(M,L)$

Remark 3.1. $x_y|_{[0,t]}$ is the integral curve of X_H joining y to $x_y(t)$. By definition, $\mathcal{P}(x_y|_{[0,t]}) : P_y \longrightarrow P_{x_y(t)}$ is the parallel transport along $x_y|_{[0,t]}$. Every $u \in P$ can be seen as a $\mathbb{C}$–isomorphism $u : L_{\pi(u)} \longrightarrow \mathbb{C}$. We consider that $\mathcal{P}(x_y|_{[0,t]})$ is a $\mathbb{C}$–isomorphism from L_y to $L_{x_y(t)}$ defined by $[\mathcal{P}(x_y|_{[0,t]}).u]^{-1} \circ u$, where $u \in P_y$. This definition is of course independant of the choice of u.

Remark 3.2. Observe that the right part of (5) is the integrand of the Feynman Path integral evaluated on the classical trajectory. This gives the prequantum dynamics as a Feynman path integral where all the mass is concentratred on the path corresponding to the classical trajectory.

Proof. There is a natural identification between sections of L and the functions $\Psi \in C^\infty(P,\mathbb{C})$ such that $R^*_{[\theta]}\tilde{\Psi} = e^{-\frac{i}{\hbar}\theta}\tilde{\Psi}, \forall\theta \in \mathbb{R}$, where R_g denote right multiplication in P by $g \in \mathbb{T}_\hbar$. Namely every section $\Psi \in C^\infty(M,L)$ is associated to the function $\tilde{\Psi}$ by

$$u.\tilde{\Psi}(u) = \Psi(\pi(u)), \quad \forall u \in P \qquad (6)$$

On $C^\infty(P,\mathbb{C})$, equation (E_{preq}) reads as: $-(X_H^\# - \pi^* H\partial_\theta)\tilde{\Psi} = \frac{d}{dt}\tilde{\Psi}$ where $X_H^\#$ is the horizontal lift of X_H and ∂_θ is the vector field of P associated with the one parameter group $t \longrightarrow R_{[t]}$. The pull-back of $\tilde{\Psi}_0$ by the flow of $X_H^\# - \pi^* H\partial_\theta$ will be a smooth solution of this equation. Since M is compact, X_H is complete. Let u_0 be a point of P. Let $(u_t)_{t\in\mathbb{R}}$ denote the integral curve of $X_H^\#$ starting from u_0. It is the horizontal lift of the integral curve x_t of X_H through the point $x_0 = \pi(u_0)$. We shall look for a curve g_t of $\mathbb{T}_\hbar$ which makes $R_{g_t}u_t$ an integral curve of $X_H^\# - H\partial_\theta$. Applying Leibniz formula leads to the differential equation $\frac{d}{dt}(g_t) = -H(x_t)$. A solution $g_t = [\theta_t]$ is given by :

$$\theta_t = -\int_0^t H(x_s)ds \qquad (7)$$

The solution obtained on $C^\infty(P,\mathbb{C})$, seen on $C^\infty(M,L)$, gives the result. □

Let $U_t : C^\infty(M,L) \longrightarrow C^\infty(M,L)$ be the operator sending Ψ_0 to Ψ_t defined in (5).

Corollary 3.1. *U_t extends to $\mathcal{H}_p$ as a unitary operator. The unbounded operator $O_{preq}(f)$ with domain $C^\infty(M,L)$ is essentially selfadjoint. The closure of $\frac{1}{\hbar}O_{preq}(f)$ is the infinitesimal generator of $(U_t)_t$.*

Proof. Since $h(U_t\Psi, U_t\Psi)(y) = h(\Psi,\Psi)(x_y(-t))$ and the flow leaves ω invariant, we have $\| U_t\Psi \|_{\mathcal{H}_p} = \| \Psi \|_{\mathcal{H}_p}$. So U_t admits a unique continuous isometric extension to $\mathcal{H}_p$ and $U_t \circ U_{-t} = U_{-t} \circ U_t = \mathrm{Id}$ implies that U_t is a unitary operator of $\mathcal{H}_p$. $(U_t)_t$ is a one-parameter group. Let us prove that it is strongly continuous. Using the denseness of $C^\infty(M,L)$ in $\mathcal{H}_p$, we just have to show that

$$U_t\Psi \underset{\mathcal{H}_p}{\longrightarrow} \Psi \text{ as } t \to 0, \qquad \forall \Psi \in C^\infty(M,L) \tag{8}$$

$M \times \mathbb{R} \longrightarrow \mathbb{R}, (x,t) \longrightarrow h(U_t\Psi - \Psi, U_t\Psi - \Psi)(x)$ is continuous. Using uniform continuity on the compact $M \times [0,1]$, we see that $h(U_t\Psi - \Psi, U_t\Psi - \Psi)(x)$ converges uniformly in x to 0 and (8) follows. In the same way, the preceeding proposition implies that

$$\frac{U_t\Psi - \Psi}{t} \underset{\mathcal{H}_p}{\longrightarrow} -\frac{i}{\hbar} O^k_{preq}(H).\Psi \text{ as } t \to 0 \qquad \forall \Psi \in C^\infty(M,L)$$

So U_t is a strongly continuous one-parameter unitary group, which is strongly differentiable on a dense domain. The result follows (see Thm VIII.10 of [9]). □

Semi-classical mechanics deals with the limit $\hbar \to 0$. It permits us to make the link between classical and quantum mechanics. When we attempt to introduce quantum mechanics in a geometric formalism, it becomes a justification of the construction. In the preceeding discussion, $\hbar$ was given and we assumed that the periods of ω were multiples of $2\pi\hbar$. Let us regard the inverse approach, that is, (M,ω) is given and we consider the set $\mathcal{Q}$ of $\hbar$ such that (M,ω) admit a prequantification $(P, \mathbb{T}_\hbar, \alpha)$. Let Per $\subset \mathbb{R}$ be the group of periods of ω. We have : $\hbar \in \mathcal{Q} \iff \text{Per} \subset 2\pi\hbar\mathbb{Z}$. If Per is a dense subgroup of $\mathbb{R}$, $\mathcal{Q}$ is empty. If Per is reduced to zero, $\mathcal{Q}$ is the set of real numbers. We are not interested in this situation, since we assume that M is compact. The last possibility is that Per is a cyclic group. If we denote d the positive generator of Per, we have $\mathcal{Q} = \left\{ \hbar_k | \hbar_k = \frac{d}{2\pi k}, k \in \mathbb{N} \right\}$.

To each prequantization $(P_1, \mathbb{T}_{\hbar_1}, \alpha_1)$ of (M,ω) is associated a family of prequantizations $(P_k, \mathbb{T}_{\hbar_k}, \alpha_k)$. Namely, if $\mathbb{Z}_k$ denote the subgroup of $\mathbb{T}_{\hbar_1}$ of order k, we set $P_k = P_1/\mathbb{Z}_k$ and denote p_k the projection from P_1 to P_k. $\pi_k : P_k \longrightarrow M$ is a principal $\mathbb{T}_{\hbar_k}$-bundle with π_k defined by $\pi_k \circ p_k = \pi_1$ and the action R^k by $p_k \circ R^1_{[\theta]_1} = R^k_{[\theta]_k} \circ p_k$. Let $\alpha_k \in \Omega^1(P_k, \mathbb{R})$ be defined by $p_k^* \alpha_k = \alpha_1$. With our convention (see remark 2.1), this is a connection one-form defining a prequantization of M. If we denote by ρ_k the representation of $\mathbb{T}_{\hbar_k}$ on $\mathbb{C}$ whose character is $[\theta] \longrightarrow \exp(\frac{i}{\hbar_k}\theta)$, the line bundle associated to each prequantisation is $L_k = P_k \times_{\rho_k} \mathbb{C}$. Observe that

$$L_k \sim P_1 \times_{\rho_1^{\otimes k}} \mathbb{C} \sim L_1^{\otimes k}$$

where $\rho_1^{\otimes k}$ denotes the representation of $\mathbb{T}_{\hbar_1}$ on $\mathbb{C}$ whose character is $[\theta] \longrightarrow \exp(i\frac{k}{\hbar_1}\theta)$. The covariant derivations defined on the equivalent line bundles are equivalent.

Instead of introducing different prequantizations, we consider in the following the k-th tensor product of a line bundle L associated to one prequantization $(P, \mathbb{T}_\hbar, \alpha)$ with $\hbar = \frac{d}{2\pi}$. In this setting we have $O^k_{preq}(H) = L_H - i\frac{\hbar}{k}\nabla^k_{X_H}$ with ∇^k the covariant derivation on $L^{\otimes k}$, and to express the different result on $L^{\otimes k}$, we just need to replace $\hbar$ by $\frac{\hbar}{k}$. In the same way as in the proof of proposition 3.1, the sections of $L^{\otimes k}$ lift to functions on P and using this identification, we have

$$L^2(P, \mathbb{C}) = \bigoplus_{k=-\infty}^{k=\infty} \mathcal{H}_p^k$$

with $\mathcal{H}_p^k = L^2(M, L^{\otimes k})$ and the hermitian product on $L^2(P, \mathbb{C})$ defined by $< \tilde{\Psi}, \tilde{\Psi}' >= \int_P \tilde{\Psi}\bar{\tilde{\Psi}}'\mu_P$ where $\mu_P = \frac{1}{2\pi\hbar}|\alpha \wedge d\alpha^{\wedge n}|$. We consider then that $L^2(P, \mathbb{C})$ is the prequantum semi-classical space. The semi-classical properties will appear in the next section, once we would introduce the quantum spaces.

4 Toeplitz Quantization

We assume that M is provided with a Kähler structure such that the fundamental two-form is ω. Let J in $C^\infty(M, \mathrm{End}(TM))$ denote the complex structure. Since the (0,2) component of the curvature tensor $-i\frac{k}{\hbar}\omega$ is vanishing, the C^∞ line bundle $L^{\otimes k}$ has a unique structure of a holomorphic line bundle, whose local holomorphic sections Ψ are characterised by

$$\nabla^k_{X+iJX}\Psi = 0, \quad \forall X \in C^\infty(M, TM)$$

The quantum space $\mathcal{H}^k$ consists of the global holomorphic sections of $L^{\otimes k}$. It is the kernel of the differential operator $\nabla^{k(0,1)} : C^\infty(L^{\otimes k}) \longrightarrow \Omega^{0,1}(M, L^{\otimes k})$ defined by

$$\nabla^{k(0,1)} = (Id + iJ)\nabla^k.$$

Since $L^{\otimes k}$ and $L^{\otimes k} \otimes \Lambda^{0,1}T^*M$ are Hermitian bundles, we can define $(\nabla^{k(0,1)})^*$ as the formal adjoint of $\nabla^{k(0,1)}$. We set

$$\Delta^k_{hol} = (\nabla^{k(0,1)})^* \circ \nabla^{k(0,1)}$$

$\Delta^k_{hol} : C^{-\infty}(M, L^{\otimes k}) \longrightarrow C^{-\infty}(M, L^{\otimes k})$ is an elliptic operator, thus its kernel consists of smooth sections. Since M is compact, it is finite dimensional. From the definition of Δ^k_{hol}, it follows that :

$$< \Delta^k_{hol}\Psi, \Psi >= 0 \Leftrightarrow \nabla^{k(0,1)}\Psi = 0, \qquad \forall \Psi \in C^\infty(M, L^{\otimes k})$$

Thus the quantum space $\mathcal{H}^k$ is the kernel of Δ^k_{hol}. As a finite dimensional subspace of $\mathcal{H}_p^k$ and hence closed, it is a Hilbert space. As a first semi-classical result it follows from Riemann-Roch-Hirzebruch formula and Kodaira's vanishing theorem that

$$\dim \mathcal{H}^k \sim \left(\frac{k}{2\pi\hbar}\right)^n \int_M \omega^{\wedge n} \qquad \text{as } k \longrightarrow \infty \tag{9}$$

and consequently, the quantum spaces $\mathcal{H}^k$ are not trivial.

Let us define the family of coherent states $\{e^k_u\}$ indexed by P. Since $\mathcal{H}^k$ is a finite dimensional vector space consisting of smooth functions defined on P, the map sending $\Psi \in \mathcal{H}^k$ to $\tilde{\Psi}(u)$ is a continuous functional, for every $u \in P$. By Riesz lemma, there is a unique quantum state $e^k_u \in \mathcal{H}^k$ such that $< \Psi, e^k_u >= \tilde{\Psi}(u)$. Let Π^k denote the orthogonal projection onto $\mathcal{H}^k$, whose domain of definition is $\mathcal{H}_p^k$ or $L^2(P, \mathbb{C})$ according to the context. The Schwartz kernel of Π^k, seen as an operator acting on $C^\infty(P, \mathbb{C})$, is called the reproducing kernel and is given by

$$\Pi^k(u, u') = < e^k_u, e^k_{u'} > \mu_P(u) \otimes \mu_P(u') \tag{10}$$

An important class of examples is the set of integral coadjoint orbits of compact Lie groups. By Kostant's version of the Borel-Weil-Bott theorem, the Hilbert quantum spaces defined in this way are the irreducible representations. The coherent states were introduced in this context by Perelomov in a different manner.

To complete the quantization, we associate to each classical observable $H \in C^\infty(M, \mathbb{R})$ an operator of $\mathcal{H}^k$. The operator $O^k_{preq}(H)$ defined in the preceeding section is not suitable because it does not preserve $\mathcal{H}^k$. Following Toeplitz quantization we define $O^k_{Toep}(H) : \mathcal{H}^k \longrightarrow \mathcal{H}^k$ by

$$O^k_{Toep}(H).\Psi = \Pi^k L_H \Psi \tag{11}$$

It is sometimes convenient to see $O^k_{Toep}(H)$ as an operator acting on $\mathcal{H}^k_p$ or $L^2(P, \mathbb{C})$, defined in these cases by $O^k_{Toep}(H) = \Pi^k L_H \Pi^k$. The following property draws an analogy between coherent states and Dirac functions.

Proposition 4.1. *The Schwartz kernel of $O^k_{Toep}(H)$, seen as an operator acting on $C^\infty(P, \mathbb{C})$, is given by*

$$O^k_{Toep}(H)(u, u') = < L_H e^k_u, e^k_{u'} > \mu_P(u) \otimes \mu_P(u')$$

Consequently

$$\mathrm{Tr}\left(O^k_{Toep}(H)\right) = \int_P < L_H e^k_u, e^k_u > \mu_P(u)$$

The commutation rules are no larger satisfied. Nevertheless the following deformation quantization result is proved in [11].

Theorem 4.1. *For all $F, G \in C^\infty(M, \mathbb{R})$, we have*

$$\begin{aligned} \|O^k_{Toep}(F)\|_{\mathcal{H}^k} &= \|F\|_\infty + O(1/k) \\ \|O^k_{Toep}(F) O^k_{Toep}(G) - O^k_{Toep}(FG)\|_{\mathcal{H}^k} &= O(1/k) \\ \|k[O^k_{Toep}(F), O^k_{Toep}(G)] - O^k_{Toep}(\{F, G\})\|_{\mathcal{H}^k} &= O(1/k) \end{aligned}$$

Other semiclassical properties were developed in [10]. These results were proved using the microlocal analysis of the Szego projector $\oplus \Pi^k$ and the symbolic calculus of Hermite operators.

The propagator for the Schrödinger equation

$$\frac{d}{dt}(\Psi) = -i \frac{k}{\hbar} O^k_{Toep}(H)\Psi \tag{E^k}$$

is the one parameter group $\exp(-t\, \frac{ik}{\hbar} O^k_{Toep}(H))$. The exponential is easily defined because $\mathcal{H}^k$ is a finite dimensional vector space. Let us consider the one parameter group of isomorphisms of $\mathcal{H}^k_p$:

$$\exp(-t\, \frac{ik}{\hbar} \Pi^k L_H \Pi^k) = \begin{cases} \exp(-t\, \frac{ik}{\hbar} O^k_{Toep}(H)) & \text{on } \mathcal{H}^k \\ 0 & \text{on } (\mathcal{H}^k)^\perp \end{cases}$$

Since $\mathcal{H}^k$ is a finite dimensional vector space consisting of smooth sections, $\exp(-t\, \frac{ik}{\hbar} \Pi^k L_H \Pi^k)$ is a smoothing operator for every $t > 0$. In the following section, we approximate this semi-group by semi-groups generated by second order differential operators.

5 An approximation by the heat semi-group

We will need the following result about the heat kernel of a generalised Laplacian. Let E be a fiber bundle over a Riemannian manifold (M, g). A second order differential operator $\Delta^E : C^\infty(M, E) \longrightarrow C^\infty(M, E)$ is a generalised Laplacian, if its symbol

$$\sigma(\Delta^E) \in C^\infty(T^*M, \pi^\sharp_{T^*M} \operatorname{End}(E))$$

satisfies $\sigma(\Delta^E)(x, \xi) = -|\xi|^2_g Id$, where $|\ |_g$ denote the metric of T^*M associated with a Riemannian metric g.

Theorem 5.1. *Let (M, g) be a compact Riemannian manifold and $E \longrightarrow M$ a fiber bundle. If Δ^E is a generalised Laplacian, then there exists a unique section $k \in C^\infty((0, \infty) \times M \times M, \pi_L^* E^* \otimes \pi_R^* E)$ which satisfies :*

i) $(\partial_t + \Delta_x)k = 0$

ii) $\lim_{t \to 0} \int_M k(t, x, y) \otimes s(y)\ |dg|(y) = s(x), \qquad \forall s \in C^\infty(M, E)$

where $|dg| \in |\Omega|^n(M)$ is the Riemannian density. The section k is called the heat kernel of Δ^E.

Remark 5.1. A generalised Laplacian Δ^E needs not to be formally selfadjoint.

Let $e^{-t\Delta^E}$ denote the smoothing operator whose Schwartz kernel is $k(t, .)$. From theorem 5.1, it follows that the family $(e^{-t\Delta^E})_{t>0}$ form a one parameter semi-group. The following corollary is more adapted for the proof of theorem 5.2.

Corollary 5.1. *Let (M, g) be a compact Riemannian manifold and $E \longrightarrow M$ a Hermitian bundle. If Δ^E is a generalised Laplacian, then there exists a unique, strongly continuous family $(Q(t))_{t>0}$ of smoothing operators of $L^2(M, E)$ which satisfy:*

i) $(Q(t))_{t>0}$ *is strongly differentiable on* $C^\infty(M, E)$ *and* $\frac{d}{dt}[Q(t)s] + \Delta Q(t)s = 0$ *on* $(0, \infty)$, $\forall s \in C^\infty(M, E)$.

ii) $\lim_{t \to 0} Q(t)s = s$ *in* $L^2(M, E)$, *for every* s *in* $C^\infty(M, E)$

Of course, $Q(t) = e^{-t\Delta^E}$. Observing that $\Delta^k_{hol} + i\frac{k}{\hbar}L_H$ is a generalised Laplacian associated with the Kähler metric, we can state the main result of this section.

Theorem 5.2. *For every $t > 0$, $e^{-t(\nu\Delta^k_{hol} + i\frac{k}{\hbar}L_H)}$ tends to $e^{-t\frac{ik}{\hbar}\Pi^k L_H \Pi^k}$ as $\nu \longrightarrow \infty$ in the uniform operator topology.*

We prove this result by estimating $(e^{-t(\nu\Delta^k_{hol} + i\frac{k}{\hbar}L_H)} - e^{-t\frac{ik}{\hbar}\Pi^k L_H \Pi^k})\Psi$ with Ψ in $\mathcal{H}^k$ and in $(\mathcal{H}^k)^\perp$. We set

$$R^k_\nu(t, H) = e^{-t(\nu\Delta^k_{hol} + i\frac{k}{\hbar}L_H)}$$

and

$$Q^k(t, H) = e^{-t\frac{ik}{\hbar}\Pi^k L_H \Pi^k}$$

We will establish the estimates implying the theorem in three lemmas.

Lemma 5.1. *The semi-group $R_\nu^k(t,H)$ is contractive, that is $\|R_\nu^k(t,H)\| \leq 1, \quad \forall t > 0$.*

Proof. Since $C^\infty(M, L^{\otimes k})$ is dense in $\mathcal{H}_p^k$, it is enough to prove that $\|R_\nu^k(t,H)s\| \leq \|s\|$ for every section s in $C^\infty(M, L^{\otimes k})$.

$$\begin{aligned}\frac{d}{dt}\|R_\nu^k(t,H)s\|^2 &= < -(\nu\Delta_{hol}^k + ik\hbar^{-1}L_H)R_\nu^k(t,H)s, R_\nu^k(t,H)s > \\ &\quad + < R_\nu^k(t,H)s, -(\nu\Delta_{hol}^k + ik\hbar^{-1}L_H)R_\nu^k(t,H)s > \\ &= -2 < \nu\Delta_{hol}^k R_\nu^k(t,H)s, R_\nu^k(t,H)s > \\ &= -2\nu\|\nabla^{k(0,1)}R_\nu^k(t,H)s\|^2 \\ &\leq 0\end{aligned}$$

Integrating the inequality gives the result. □

As an elliptic, formally selfadjoint operator on a compact manifold, the unbounded operator Δ_{hol}^k with domain $C^\infty(M, L^{\otimes k})$ is essentially selfadjoint, its spectrum is discrete and each eigenspace is finite dimensional (see [12]). Since $\Delta_{hol}^k = (\nabla^{k(0,1)})^* \circ \nabla^{k(0,1)} \geq 0$, the eigenvalues are non negative. Let us denote λ_k^1 the first positive eigenvalue.

Lemma 5.2. $\|R_\nu^k(t,H)\circ(Id - \Pi^k)\| \leq e^{-\lambda_k^1\nu t} + 2k\hbar^{-1}\frac{\|L_H\|}{\lambda_k^1\nu}$

Proof. Using the decomposition of $\mathcal{H}_p^k$ into the orthogonal sum of eigenspace of Δ_{hol}^k, the inequality holds if $H = 0$. Let us introduce the semi-group of bounded operators of $\mathcal{H}_p^k$:

$$\tilde{R}_\nu^k(t,H) = \begin{cases} Q^k(t,H) & \text{on } \mathcal{H}^k \\ R_\nu^k(t,0) & \text{on } (\mathcal{H}^k)^\perp \end{cases}$$

Since $\tilde{R}_\nu^k(t,H) = R_\nu^k(t,0) - R_\nu^k(t,0)\circ\Pi^k + Q^k(t,H)$, it is a smoothing operator which is strongly differentiable on $C^\infty(M, L^{\otimes k})$. Observing that $\Delta_{hol}^k \circ \Pi^k = 0$ and $\Pi^k \circ R_\nu^k(t,0)\circ(Id - \Pi^k) = 0$, we calculate the derivative

$$\frac{d}{dt}[\tilde{R}_\nu^k(t,H)\Psi] = -(\nu\Delta_{hol}^k + ik\hbar^{-1}\Pi^k L_H \Pi^k)\tilde{R}_\nu^k(t,H)\Psi \quad \forall s \in C^\infty(M, L^{\otimes k}) \tag{12}$$

We also have

$$\lim_{t\to 0}\tilde{R}_\nu^k(t,H)\Psi = \Psi \text{ in } \mathcal{H}_p^k \quad \forall \Psi \in C^\infty(M, L^{\otimes k}) \tag{13}$$

We claim that :

$$R_\nu^k(t,H)\Psi = \tilde{R}_\nu^k(t,H)\Psi - ik\hbar^{-1}\int_0^t R_\nu^k(t-s,H)\circ(L_H - \Pi^k L_H \Pi^k)\circ\tilde{R}_\nu^k(s,H)\Psi ds \tag{14}$$

for every section Ψ in $\mathcal{H}_p^k$. Since the function we integrate is continuous, we need only use the Riemann integral. To prove this equality, it suffices to show that the operator defined on the right side satisfies the conditions *i)* and *ii)* of corollary 5.1, which are consequences of (12) and (13).

Using that $\|\tilde{R}^k_\nu(t,H)\circ[Id-\Pi^k]\| = \|R^k_\nu(t,0)\circ[Id-\Pi^k]\| \leq e^{-\lambda^1_k\nu t}$, it follows from (14) and lemma 5.1 that

$$\begin{aligned}\|R^k_\nu(t,H)\circ[Id-\Pi^k]\Psi\| &\leq \left(e^{-\lambda^1_k\nu t} + k\hbar^{-1}\int_0^t \|L_H - \Pi^k L_H \Pi^k\| e^{-\lambda^1_k \nu u} du\right)\|\Psi\| \\ &\leq \left(e^{-\lambda^1_k\nu t} + 2k\hbar^{-1}\frac{\|L_H\|}{\lambda^1_k\nu}\right)\|\Psi\|\end{aligned}$$

□

Lemma 5.3. $\|[Q^k(t,H) - R^k_\nu(t,H)]\circ\Pi^k\| \leq k\hbar^{-1}\frac{\|L_H\|}{\lambda^1_k\nu}(1+2tk\hbar^{-1}\|L_H\|)$

Proof. Note that $Q^k(t,H)\Pi^k = \tilde{R}^k_\nu(t,H)\Pi^k$. Thus equation (14) implies

$$\begin{aligned}[Q^k(t,H) - R^k_\nu(t,H)]\circ\Pi^k =& ik\hbar^{-1}\int_0^t R^k_\nu(t-s,H)\circ(L_H - \Pi^k L_H\Pi^k)\circ\tilde{R}^k_\nu(s,H)\circ\Pi^k ds \\ =& ik\hbar^{-1}\int_0^t R^k_\nu(t-s,H)\circ(Id-\Pi^k)\circ L_H\Pi^k\circ\tilde{R}^k_\nu(s,H)ds\end{aligned}$$

since Π^k and $\tilde{R}^k_\nu(s,H)$ commute.

Observing that $\|\Pi^k\circ\tilde{R}^k_\nu(s,H)\| = 1$, we get the estimate

$$\begin{aligned}\|[Q^k(t,H) - R^k_\nu(t,H)]\circ\Pi^k\| \leq& k\hbar^{-1}\int_0^t \|R^k_\nu(s,H)\circ(Id-\Pi^k)\|\|L_H\| ds \\ \leq& k\hbar^{-1}\frac{\|L_H\|}{\lambda^1_k\nu}(1+2tk\hbar^{-1}\|L_H\|)\end{aligned}$$

Here we have used lemma 5.2. □

Proof of theorem 5.2. Since $Q^k(t,H) - R^k_\nu(t,H) = (Q^k(t,H) - R^k_\nu(t,H))\circ\Pi^k - R^k_\nu(t,H)\circ(Id-\Pi^k)$, lemmas 5.2 and 5.3 imply

$$\|Q^k(t,H) - R^k_\nu(t,H)\| \leq e^{-\lambda^1_k\nu t} + k\hbar^{-1}\frac{\|L_H\|}{\lambda^1_k\nu}(3+2tk\hbar^{-1}\|L_H\|) \tag{15}$$

□

Remark 5.2. The dependence of the estimate (15) on k can be specified. It is proved in [14] that there exists a constant C such that $\lambda^1_k \geq C + k$. It follows that there exists $C_1, C_2 \geq 0$ which do not depend on k, ν and t such that

$$\|e^{-t(\nu\Delta^k_{hol} + i\frac{k}{\hbar}L_H)} - e^{-t\frac{ik}{\hbar}\Pi^k L_H\Pi^k}\| \leq C_1 e^{-k\nu t} + \frac{C_2}{\nu}(1+kt) \tag{16}$$

6 Feynman Path Integral

In the preceding part, we saw how the propagator for the Schrödinger equation can be approximated by heat semi-group. Using a generalised Feynman-Kač formula, we will express the solution of Schrödinger's equation as a limit of path integrals. This will be our Feynman's integral formula.

In [13], Norris proved a Feynman-Kač formula for a generalised Laplacian Δ^E which acts on sections of a vector bundle E. The first step consists in decomposing Δ^E as

$$\Delta^E = -\frac{1}{2}\operatorname{Tr}\circ\nabla^{E\otimes\, T^*M}\circ\nabla^E + V$$

where $\nabla^E : C^\infty(M,E) \longrightarrow \Omega^1(M,E)$ is a covariant derivative, $\operatorname{Tr} : C^\infty(M, E\otimes T^*M\otimes T^*M) \longrightarrow C^\infty(M,E)$ is the contraction with a metric g and $V \in C^\infty(M, \operatorname{End} TM)$ acts linearly on each fiber.

To pursue this program, we need the following Weitzenböck formula (see [15]):

$$\Delta^k_{hol} = \frac{1}{2}\Delta^k - \frac{nk}{2\hbar}$$

where $\Delta^k = (\nabla^k)^*\circ\nabla^k = -\operatorname{Tr}\circ\nabla^{L^k\otimes T^*M}\circ\nabla^k$, and $2n$ is the dimension of M. It follows that

$$e^{-t(\nu\Delta^k_{hol}+i\frac{k}{\hbar}L_H)} = e^{t\frac{\nu nk}{2\hbar}}e^{-t(\frac{\nu}{2}\Delta^k+i\frac{k}{\hbar}L_H)}$$

In the following we will apply Feynman-Kač formula to

$$\frac{\nu}{2}\Delta^k + i\frac{k}{\hbar}L_H = -\frac{\nu}{2}\operatorname{Tr}\circ\nabla^{L^{\otimes k}\otimes\, T^*M} + i\frac{k}{\hbar}L_H$$

As in section 3, all the objects we will introduce depend on k only through the representation $\rho^{\otimes k}$: we will construct stochastic flows on M, P and $\mathbb{T}_\hbar$ which do not depend on k, and then using $\rho^{\otimes k}$ deduce the heat propagation on L^k.

Given x_0 in M, let us consider the Brownian motion on M starting from x_0 associated to the Riemannian metric νg. It consists of a probability space $(\Omega^\nu, \mathcal{F}^\nu, \mathbb{P}^\nu)$ equipped with a right continuous filtration $(\mathcal{F}^\nu_t)_{t\geq 0}$ such that $\mathcal{F}^\nu_0$ contains all the $\mathbb{P}^\nu$-null sets, together with a martingale

$$\begin{aligned} x^\nu :\ \Omega^\nu\times[0,\infty) &\longrightarrow M \\ (\omega,t) &\longrightarrow x^\nu_t(\omega) \end{aligned}$$

which satisfies

$$b(\partial x^\nu_t, \partial x^\nu_t) = \nu\operatorname{Tr}\circ b(x^\nu_t)\partial t, \quad \forall\, b\in C^\infty(M, T^*M\otimes T^*M)$$

where $b(\partial x^\nu_t, \partial x^\nu_t)$ denotes the b-quadratic variation of x^ν_t. This definition is the Lévy characterisation of Brownian motion adapted to M-valued semimartingales. A construction is obtained by solving a stochastic differential equation and using stochastic development (see [16]).

Next, given u_0 in P_{x_0}, we consider the horizontal lift $u^\nu : \Omega^\nu\times[0,\infty) \longrightarrow P$ of x^ν. This is the unique semimartingale u^ν_t in P over x^ν_t, that is $\pi\circ u^\nu_t = x^\nu_t$, such that

$$\partial((u^\nu_t)^{-1}.s(x^\nu_t)) = ((u^\nu_t)^{-1}.\nabla s)(\partial x^\nu_t) \quad \forall s\in C^\infty(M,L)$$

This means that, for all stopping times σ,τ, such that $\sigma\leq\tau$, we have

$$(u^\nu_\tau)^{-1}.s(x^\nu_\tau) - (u^\nu_\sigma)^{-1}.s(x^\nu_\sigma) = \int_\sigma^\tau ((u^\nu_t)^{-1}\nabla s)\partial x^\nu_t$$

where the right part denotes the Stratonovitch integral of the $T^*M\otimes\mathbb{C}$-valued semimartingale $(u^\nu_t)^{-1}.\nabla s$ against x^ν_t. The parallel transport $\mathcal{P}^\nu_t(\omega)$ is the $\mathbb{T}_\hbar$-morphism from $P_{x^\nu_t(\omega)}$ onto P_{x_0} such that $\mathcal{P}^\nu_t(\omega).u^\nu_t(\omega) = u_0$, this is the phase of the action integral of the path ω. As in proposition 3.1, $\mathcal{P}^{\nu,k}_t(\omega) = u_0\circ u^\nu_t(\omega)^{-1}$ acts as a bijective linear transformation from $L_{x^\nu_t(\omega)}$ to L_{x_0}.

Finally we define the stochastic exponential e_t^ν in $\mathbb{T}_\hbar$ by the following stochastic equation

$$de_t^\nu = e_t^\nu H(x_t^\nu)dt \tag{17}$$
$$e_0 = 1 \tag{18}$$

This means that for all stopping times σ, τ, such that $\sigma \leq \tau$, we have

$$s(e_\tau^\nu) = s(e_\sigma^\nu) + \int_\sigma^\tau (\partial_\theta s)(e_t^\nu)H(x_t^\nu)dt, \quad \forall s \in C^\infty(\mathbb{T}_\hbar, \mathbb{R})$$

Observe that equation (17) can be solved as (7) by integrating the stochastic process $H(x_t^\nu)$ and projecting it on $\mathbb{T}_\hbar$ using p. Details of stochastic exponentials in Lie groups can be found in [17]. We interpret $e_t^\nu(\omega)$ as the integral of the Hamiltonian H along the path ω.

We can write the generalised Feynman-Kač formula

$$\left(e^{-t(\frac{\nu}{2}\Delta^k + i\frac{k}{\hbar}L_H)}\Psi\right)(x_0) = \mathrm{E}\left[\rho^k(e_t^\nu)\mathcal{P}_t^{\nu,k}.\Psi(x_t^\nu)\right], \quad \forall \Psi \in C^\infty(M, L^{\otimes k})$$

Using proposition 5.2, we can state:

Theorem 6.1. *For every Ψ in $C^\infty(M, L^{\otimes k})$, let Ψ_t^ν be the section of $L^{\otimes k}$ defined by*

$$\Psi_t^\nu(x_0) = e^{t\frac{\nu n k}{2\hbar}}\,\mathrm{E}\left[\rho^k(e_t^\nu)\mathcal{P}_t^{\nu,k}.\Psi(x_t^\nu)\right] \tag{19}$$

then

$$\Psi_t^\nu \longrightarrow e^{-t\,\frac{ik}{\hbar}\Pi^k L_H \Pi^k}\Psi \text{ in } (\mathcal{H}^k, <,>) \text{ as } \nu \longrightarrow \infty$$

Remark 6.1. The construction of the propagator U_t in proposition 3.1 is very similar to (19). In the prequantum setting, we consider a deterministic smooth curve which is the integral curve of X_H starting at x_0 instead of a brownian motion. We lift it onto P, defining in this way the phase of its integral action, and introduce the integral of the Hamiltonian along the curve to solve a differential equation analogue to (17). The sign difference which appears in the parallel transport $\mathcal{P}_t^\nu(\omega)$ and equation (17) comes from the fact that the sample paths of x_t^ν are not ending, but starting, at x_0.

References

[1] R.P. Feynman and A.R. Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill,Inc., 1965.

[2] P. Cartier, C. DeWitt-Morette, A new perspective on functional integration, *J.Math.Phys*, Vol. **36**, 1995, pp. 2237-2312.

[3] I. Daubechies and J.R. Klauder, Quantum-mechanical path integrals with Wiener measure for all polynomial Hamiltonians, *J.Math.Phys.*, Vol. **26**, 1985, pp 2239-2256.

[4] I. Daubechies, J.R. Klauder and T. Paul, Wiener measures for path integrals with affine kinematic variables, *J.Math.Phys.*, Vol. **28**, 1987, pp 85-102

[5] T. Paul, Wavelets and Path Integral, in *Wavelets, time-frequency methods and phase space* Combes, Grossmann, Tchamitchian eds, Springer-Verlag, 1989, pp 204-208.

[6] R. Abraham and J. Marsden, *Foundations of mechanics*, Benjamin M.A., 1978.

[7] B. Kostant, Quantization and unitary representations, *Lect. Notes in Math.*, Vol. **170**, Springer-Verlag, 1970, pp 87-208.

[8] A. Weinstein, Bifurcations and Hamilton's principle, *Math.Z.*, Vol. **159**, 1978, pp 235-248.

[9] M. Reed and B. Simon, *Methods of modern mathematical physics, I. Functional analysis*, Academic Press Inc, 1970.

[10] D. Borthwick, T. Paul and A. Uribe, Semiclassical spectral estimates for Toeplitz operators, *Ann. Inst. Fourier*, Vol. **48**, 1998.

[11] M. Bordemann, E. Meinrenken and M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and $gl(N), N \longrightarrow \infty$ limit, Commun. Math. Phys., Vol. **165**, 1994, pp 281-296.

[12] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Springer Verlag, 1978.

[13] J.R. Norris, A complete differential formalism for stochastic calculus in manifolds, *Lect. Notes in Math.*, Vol. **1526**, 1992, pp. 189-209.

[14] V. Guillemin and A. Uribe, The laplace operator on the nth tensor power of a line bundle : Eigenvalues which are uniformly bounded in n, *Asympt. Anal.*, Vol. **1**, 1988, pp 105-113.

[15] A. Weil, *Variétés kähleriennes*, Hermann Paris, 1958.

[16] K.D.Elworthy *Stochastic differential equations in manifold*, Cambridge University Press, 1982.

[17] H. Dowek and D. Lépingle, L'exponentielle stochastique des groupes de Lie, *Lect. Notes in Math.*, Vol. **1204**, 1986, pp 352-374.