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The Faddeev formula in the inverse scattering for Dirac operators

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ABSTRACT.We study the high energy limit of the Faddeev Scattering Amplitude for the Dirac operator associated with a potential Q . We prove that the Fourier transform of the potential and the limit of the scattering amplitude are related by an integral equation . Finally we apply these results to reconstruct the potential modulo a gauge transformation.

RESUME. On étudie le comportement à haute énergie de l'Amplitude de Diffusion de Faddeev pour l'opérateur de Dirac associée à un potentiel Q . Nous montrons que la transformée de Fourier du potentiel et la limite de l'amplitude de diffusion sont liées par une équation intégrale. Enfin nous appliquons ces résultats pour reconstruire le potentiel modulo une transformation de jauge.

0-Introduction.

The free Dirac operator $\mathcal{A}$ in $\mathbb{R}^3$ (see [T]), acts on 4-spinor fields according to :

$$\mathcal{A}u = -i\sum_{j=1}^{3}\alpha_j\frac{\partial u}{\partial x_j} + \alpha_4 u. \qquad (0-1)$$

where $(\alpha_j)_{j=1}^4$ are the Dirac matrices, they are 4x4 Hermitian matrices which satisfy the following relations :

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}. \qquad (0-2)$$

Let Q be the multiplication operator by a 4x4 matrix valued function $Q(.)$, the operator

$$\mathcal{H} = \mathcal{A} + Q. \qquad (0-3)$$

is thought as a perturbed operator of $\mathcal{A}$. The scattering theory for the pair $(\mathcal{A},\mathcal{H})$ is well studied, see [G,S], [E], [Y], [B,H]. In an unpublished work [H1], we investigated the scattering eigenfunctions for the Dirac operator $\mathcal{H}$ and the analytic properties of the scattering operator . In this paper we consider the high energy behavior of the scattering amplitude. While the Faddeev formula for Schrödinger operator shows that the Fourier transform of the potential is obtained as a limit of the scattering amplitude , we show that for Dirac operator the Fourier transform of the potential and the limit of the scattering amplitude are related by an integral equation. This different high energy behavior was first observed in the case of scattering by a spherically symetric potential by Parzen who showed heuritically that $\lim_{k\to\infty}\delta_l(k) = -\int_0^\infty V(r)dr$ where $\delta_l(k)$ is the phase shift at energy k^2

for the l partial wave, while $\lim_{k\to\infty} \delta_l(k) = 0$ in the case of Schrödinger operator. Recently [B,G,W] established this result by studying the high energy asymptotics for the solutions of the radial Dirac equation. For non radial potentials and in the framework of the $\bar{\partial}$ approach,see [H2] we obtained a related result for an analytic extension of the scattering amplitude. Related works on the inverse problem for Dirac operator is [Is].In the papers [J], [It] the high energy asymptotics of the scattering operator are considered, while [J] uses a time dependant method , [It] focuses on the high energy behavior of the resolvent. The results we present are equivalent to those obtained by these authors, however our method is based on the properties of the eigenfunctions and on the use of a Green function introduced in inverse scattering by Faddeev.
I wish to thank the referee for calling my attention to the paper [It] and for his usefull remarks.

In the standard representation we have :

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}; \; \alpha_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Where the $(\sigma_j)_{j=1}^3$ are the Pauli matrices and I the identity matrix. We set $\beta = \alpha_4$. We denote by $\mathbf{S}$ the spinors space which we identify with $\mathbb{C}^4$ with the Hermitian scalar product and by $\mathbf{M}$ the algebra of 4x4 matrices over $\mathbb{C}$. If $A, B \in \mathbf{M}$ then A^* is Hermitian adjoint of A and

$$\{A, B\} = AB + BA.$$

A matrix $A \in \mathbf{M}$ is formed with 2×2 matrices as blocks, denote by A_Δ the matirx obtained from A upon replacing the off diagonal blocks by 0 .We have $\{A, \beta\} = 2\beta A_\Delta$.

The Fourier transformation is denoted by $\mathcal{F}$ and is extended to $\mathbf{S}$ and $\mathbf{M}$ valued functions in the usual way. We also denote by $\hat{f}$ the transform of f . If $\mathbf{H}, \mathbf{K}$ are Hilbert spaces, $\mathbf{L}(\mathbf{H}, \mathbf{K})$ is the space of bounded operators from $\mathbf{H}$ to $\mathbf{K}$ .We denote by $\star$ the convolution.

I-The scattering theory for the pair $(\mathcal{A}, \mathcal{H})$.

The operator $\mathcal{A}$ acts in $\mathbf{K} = \mathbf{L}^2(\mathbb{R}^3, \mathbf{S})$ with $\mathbf{H}^1(\mathbb{R}^3, \mathbf{S})$ as domain and

$$\sigma(\mathcal{A}) =]-\infty, -1] \cup [1, +\infty[.$$

The symbol of $\mathcal{A}$ is :

$$A(\xi) = \xi\alpha + \beta, \tag{1-1}$$

and we can find a unitary matrix $U_0(\xi)$ such that $\forall \xi \in \mathbb{R}^3$

$$U_0^*(\xi) A(\xi) U_0(\xi) = w(\xi)\beta. \tag{1-2}$$

where the function $w(\xi) = (|\ \xi\ |^2 + 1)^{\frac{1}{2}}$ is the energy. The $\mathbf{M}$ valued function $U_0(.)$ is $\mathbf{C}^\infty$. A canonical choice is

$$U_0(\xi) = (2w^2 + 2w)^{-\frac{1}{2}} \begin{pmatrix} (1+w)I & -\xi.\sigma \\ \xi.\sigma & (1+w)I \end{pmatrix}$$

Let $U_{0,l}(.), l = 1, \ldots 4$, be the columns of $U_0(.)$. For $f \in \mathbf{C}_0^\infty(\mathbb{R}^3, \mathbf{S})$ we set

$$(\Phi_0 f)(\xi) = U_0^*(\xi)(\mathcal{F}f)(\xi). \qquad (1-3)$$

Then the map Φ_0 extends to $\mathbf{K}$ as a unitary operator and

$$\Phi_0 \mathcal{A} \Phi_0^* = w\beta, \qquad (1-4)$$

where w is the multiplication operator by the function $w(.)$. Thus $\mathcal{A}$ is diagonalized by Φ_0 . For $k \in \mathbb{C}$, $Im\, k > 0$, let $g(.,k)$ be the kernel of $(-\Delta - k^2)^{-1}$

$$g(x,k) = \frac{1}{4\pi|x|} \exp{(ik|x|)}.$$

For $k > 0$, $g(.,\pm k)$ are the outgoing $(+)$ and incoming $(-)$ fundamental solutions of $(-\Delta - k^2)$. If $z \notin \sigma(\mathcal{A})$, and $k(z) = (z^2 - 1)^{\frac{1}{2}}$ with $Im\, k(z) \geq 0$, then the kernel of $(\mathcal{A} - z)^{-1}$ is :

$$G_{\mathcal{A}}(.,z) = (\mathcal{A} + z) g(., k(z)). \qquad (1-5-a)$$

The limits

$$G_{\mathcal{A}}(., \lambda \pm i0) = (\mathcal{A} + \lambda) g(., k(\lambda \pm i0)) \qquad \text{if} \quad |\lambda| > 1 \qquad (1-5-b)$$

are well defined as distributions. Note that $k(\lambda \pm i0) = \pm \mathrm{sgn}(\lambda)(\lambda^2 - 1)^{\frac{1}{2}}$,where sgn is the sign function.

Definition 1-1. — *Let $\rho(x) = (1 + |x|^2)^{\frac{1}{2}}$, $s \in \mathbb{R}$ and $k \geq 0$. If $\mathbf{X}$ is a finite dimensional vector space, the weighted Sobolev spaces $\mathbf{K}_s(\mathbf{X})$ and $\mathbf{W}_s^k(\mathbf{X})$ are defined by*

$$\mathbf{K}_s(\mathbf{X}) = \{f | \quad \rho^s f \in \mathbf{L}^2(\mathbb{R}^3, \mathbf{X}) \quad \},$$
$$\mathbf{W}_s^k(\mathbf{X}) = \{f | \quad (I - \Delta)^{\frac{k}{2}} f \in \mathbf{K}_s(\mathbf{X}) \quad \},$$

where $(-\Delta)$ is the Laplacian. These spaces are endowed with their natural norms, denoted by $\|.\|_s$, $\|.\|_{k,s}$ respectively; we omit s when $s = 0$. As usual we denote by $\mathcal{S}$ the space of rapidly decreasing functions in $\mathbf{C}^\infty(\mathbb{R}^3, \mathbb{C})$ and for $m, r \in \mathbb{N}$ we set

$$\mathcal{S}_r^m = \{f | f \in \mathbf{C}^m(\mathbb{R}^3, \mathbb{C}); \rho^r \partial^t f \in \mathbf{L}^\infty; |t| \leq m\},$$

endowed with the natural norm, then $\mathcal{S} = \cap_{m,r \in \mathbb{N}} \mathcal{S}_r^m$, with the projective limit topology. The dual $\mathcal{S}'$ is the space of tempered distributions. We will use also the spaces $\mathcal{S}_r^\infty = \cap_{m \in \mathbb{N}} \mathcal{S}_r^m$ and $\mathcal{S}_\infty^m = \cap_{r \in \mathbb{N}} \mathcal{S}_r^m$. Function spaces of vector or matrix valued functions are denoted similarly whenever their components belong to the above spaces.

Let $\mathcal{R}(z, \mathcal{A})$ be the resolvent of $\mathcal{A}$. Then $\mathcal{A}$ satisfies the Limiting Absorption Principle, i.e., the limits

$$\lim_{\epsilon \to 0} \mathcal{R}(\lambda \pm i\epsilon, \mathcal{A}) = \mathcal{R}(\lambda \pm i0, \mathcal{A}) \qquad (1-6)$$

exist in $\mathbf{L}(\mathbf{K}_s(\mathbf{S}), \mathbf{K}_{-s}(\mathbf{S}))$ for $s > \frac{1}{2}$, uniformly for $|\lambda| \geq \delta$, $\delta > 1$.

Hypothesis 1-2.. — *Suppose the potential* Q *is in* $\mathcal{S}$ *and of the form* $Q = a.\alpha + \varphi.I$ *where the vector function* $a(.)$ *and* $\varphi(.)$ *are respectively the magnetic and electric potential. Denote by* $|Q|_{m,r}$ *the norm of* Q *in* $\mathcal{S}_r^m$.

Proposition 1-3. — *The operator* $\mathcal{H} = \mathcal{A} + Q$ *has a unique self adjoint extension in* $\mathbf{K}(\mathbf{S})$ *with domain* $\mathbf{W}^1(\mathbf{S})$ *. We have*

$$\sigma_{\text{ess}}(\mathcal{H}) =]-\infty, -1] \cup [1, +\infty[.$$

The operator $\mathcal{H}$ *has a pure point spectrum in the interval* $]-1, 1[$ *and*

$$\lim_{\epsilon \to 0} \mathcal{R}(\lambda \pm i\epsilon, \mathcal{H}) = \mathcal{R}(\lambda \pm i0, \mathcal{H}) \tag{1 - 7}$$

exists in $\mathbf{L}(\mathbf{K}_s(\mathbf{S}), \mathbf{K}_{-s}(\mathbf{S}))$ $s > \frac{1}{2}$ *uniformly for* $\lambda, |\lambda| \geq \delta$, $\delta > 1$ *. In particular* $\mathcal{H}$ *is absolutely continuous .*

The spectral properties of $\mathcal{H}$ are well known, see [E],[H 1],[Y],[G-S],[B-H],[It],[T].

Proposition 1-4. — *For every* $\xi \in \mathbb{R}^3 \setminus \{0\}$ *the Lippmann-Schwinger equations*

$$U_l(.,\xi) = U_{0,l}(\xi)e^{i<.,\xi>} - \mathcal{R}(\delta_l(w(\xi) + i0), \mathcal{A})(QU_l(.,\xi)) \tag{1 - 8}$$

for $l = 1, \dots, 4$, $\delta_l = 1$ *for* $l = 1, 2$ *and* $\delta_l = -1$ *for* $l = 3, 4$ *, have unique solutions* $U_l(.,\xi) \in \mathbf{W}^k_{-s}(\mathbf{S})$ *,* k *arbitrary,* $s > 3/2$ *the map* $\xi \mapsto U_l(.,\xi)$ *is continuous and bounded from the set* $|\xi| \geq \delta_0 > 0$ *into* $\mathbf{W}^0_{-s}(\mathbf{S})$ *.*

Proof.The existence of the solution is classical, the boundedness statement results from the relation

$$U_l(.,\xi) = U_{0,l}(\xi)e^{i<.,\xi>} - \mathcal{R}(\delta_l(w(\xi) + i0), \mathcal{H})(QU_{0,l}(.,\xi)e^{i<.,\xi>})$$

and Proposition 1-3. ◊

Let $U(.,\xi)$ be the matrix whose columns are $\{U_l(.,\xi) | l = 1, \dots, 4\}$, and set for $f \in \mathbf{C}_0^\infty(\mathbb{R}^3, \mathbf{S})$

$$\Phi f(\xi) = \int U^*(x,\xi) f(x)\, dx. \tag{1 - 9}$$

Then Φ extends to a unitary operator in $\mathbf{K}(\mathbf{S})$ and

$$\Phi \mathcal{H}_{ac} \Phi^* = w\beta.$$

Equations (1-8) may be written as a matrix equation for $U(.,\xi)$ using (1-5-b) :

$$U(.,\xi) = U_0(\xi)e^{i<.,\xi>} - g(.,|\xi|) \star (\mathcal{A}\, Q\, U(.,\xi) + w(\xi)\, Q\, U(.,\xi)\, \beta). \tag{1 - 10}$$

We designate by S^2 the unit sphere in $\mathbb{R}^3$ and introduce polar coordinates for ξ : $k = |\xi|$ and $\omega = |\xi|^{-1}\xi$ and for x : $r = |x|$ and $\theta = |x|^{-1}x$.

DEFINITION 1-5. — *The Faddeev Scattering Amplitude. For every* $\xi \in \mathbb{R}^3$ *set*

$$F(\eta, \xi) = U_0^{-1}(\eta)\, \mathcal{F}(Q(.)\, U(., \xi))(\eta). \qquad (1-11)$$

Since $Q(.)\,U(.,\xi)$ is in $\mathcal{S}$, the map $\xi \mapsto F(.,\xi)$ is well defined and continuous from $\mathbb{R}^3 \setminus 0$ to $\mathcal{S}$. The relation of F to the Scattering Amplitude is illustrated by the following.

PROPOSITION 1-6. — *For* $k > 0$ *, we have as* $r \to \infty$

$$U(r\theta, k\omega) = U_0(k\omega)e^{ikr\langle\omega,\theta\rangle} - (2\pi)^{\frac{1}{2}} w(k) U_0(k\theta)\beta F_\Delta(k\theta, k\omega)\frac{e^{ikr}}{r} + O(r^{-2}) \qquad (1-12)$$

uniformly in $\omega, \theta \in S^2$ *and* k *in a compact .*

Proof.We have following relations :

$$|x-y|^{-1}e^{ik|x-y|} = r^{-1}e^{ikr}e^{-ik\langle\theta,y\rangle} + a(x,y,k)r^{-2},$$
$$\nabla_x|x-y|^{-1}e^{ik|x-y|} = ik\theta r^{-1}e^{ikr}e^{-ik\langle\theta,y\rangle} + b(x,y,k)r^{-2}$$

with $|a(x,y,k)| \leq C\rho(y)^5$, $|b(x,y,k)| \leq C\rho(y)^7$ uniformly $\theta \in S^2$ and k in a compact. With $V(.,\xi) = QU(.,\xi)$ we have by integration over y

$$(g \star V)(r\theta) = (\pi/2)^{1/2}\hat{V}(k\theta, k\omega)r^{-1}e^{ikr} + O(r^{-2}),$$
$$(A(D)g \star V)(r\theta) = (\pi/2)^{1/2}A(k\theta)\hat{V}(k\theta, k\omega)r^{-1}e^{ikr} + O(r^{-2}),$$

insert these relations in (1-10) and use the relations

$$F(k\theta, k\omega) = U_0^{-1}(k\theta)\hat{V}(., k\omega)(k\theta), \qquad U_0^{-1}(k\theta)A(k\theta)U_0(k\theta) = w(k)\beta$$

by noting that $\{F, \beta\} = 2\beta F_\Delta$ we get relation (1-12). ◊

So the scattering amplitude matrix is :

$$T(k,\theta,\omega) = -(2\pi)^{\frac{1}{2}} w(k)\beta F_\Delta(k\theta, k\omega).$$

As in the scattering theory for Schrödinger operators [N],[A,J,S], we can show that $T(k,\theta,\omega)$ is simply related to the kernel of the S-matrix for the pair $(\mathcal{A}, \mathcal{H})$, [H1],[It]. The scattering operator commutes with $\mathcal{A}$ it is then diagonal in the spectral representation of $\mathcal{A}$, this implies that the S- matrix commutes with β , hence $T(k,.,.)$ is block diagonal.

II-The Faddeev Formula

Let $V(.,\xi) = e_{(-\xi)}U(.,\xi)U_0^{-1}(\xi)$, where e_ξ is the operator of multiplication by the function $e_\xi(x) = e^{i<x,\xi>}$ and set

$$h(.,\xi) = e_{(-\xi)}g(.,|\xi|). \qquad (2-1)$$

After multiplication of the Lippman-Schwinger equation (1-10) on the left by $e_{(-\xi)}$ and on the right by $U_0^{-1}(\xi)$ we see that $V(.,\xi)$ satisfy the equation

$$V(.,\xi) = I - h(.,\xi) \star (A(D+\xi)\,Q\,V(.,\xi) + Q\,V(.,\xi)\,A(\xi)) \qquad (2-2)$$

where we have used (1-2) and the relation

$$e_{(-\xi)}(K\star)e_\xi = (e_{(-\xi)}K)\star; \qquad \forall K \in \mathcal{S}'.$$

Define the operator $\mathcal{S}(\xi)$ on matrix valued functions by

$$\mathcal{S}(\xi)X = h(.,\xi) \star (A(D+\xi)\,Q\,X(.) + Q\,X(.)\,A(\xi)),$$

equation (2-2) is then written as $V = I - \mathcal{S}(\xi)V$, the kernel (2-1) is a boundary value of Faddeev fundamental solution of $-\Delta - 2i\zeta\nabla$, $\zeta \in \mathbb{C}^3$ and we know that $\mathcal{S}(\xi)$ is a compact operator $\mathbf{W}^l_{-s}$, $s > 1/2$ with norm $O(|Q|_{l,r})$, $r > 2s$,uniformly in $\{\xi, |\xi| \geq \delta\}$. (See [H2] especially Remark 2-14, [W1], [N2]). Equation (2-2) is equivalent to (1-10) and we have

PROPOSITION 2-1. — *For* $\xi \in \mathbb{R}^3 \setminus \{0\}$ *equation (2-2) has a unique solutions in* $\mathbf{W}^l_{-s}$, $s > 3/2$ *and the map* $\xi \mapsto V(.,\xi)$ *is continuous and bounded on the set* $\{\xi, |\zeta| \geq \delta\}$.

DEFINITION 2-2. — *Let* $\omega \in S^2$ *and* $\mathbb{R}^3 = \mathbb{R}\omega \oplus (\mathbb{R}\omega)^\perp$ *the decomposition of* $\mathbb{R}^3$ *into subspaces parallel and orthoganal to* ω *in this decompsition define the distribution*

$$h_\infty(x'_\omega, x''_\omega, \omega) = \frac{i}{2}\Theta(x'_\omega) \otimes \delta(x''_\omega) \qquad (2-3)$$

where Θ *is the one dimensional Heaviside distribution and* δ *the two dimensional Dirac distribution .*

The proof of the following two Lemma is given in the appendix.

LEMMA 2-3. — *We have* $\lim_{k\to+\infty} k.h(.,k\omega) = h_\infty(.,\omega)$ *in* $(\mathcal{S}^2_\theta)^*$, $\theta > 2$ *in the strong dual topology uniformly in* ω .

LEMMA 2-4. — *Let* $h \in (\mathcal{S}_r^m)^*$, $r > 3$ *,then the map* $f \mapsto h \star f$ *is continuous from* $\mathbf{W}_s^l$ *into* $\mathbf{W}_{-s}^{-l'}$, $s > r - 3/2$, $l + l' \geq m$. *The map* $h \mapsto h\star$ *is continuous from* $(\mathcal{S}_r^m)^*$ *with strong topology into* $\mathbf{L}(\mathbf{W}_s^l, \mathbf{W}_{-s}^{-l'})$.

LEMMA 2-5. — *For* $k > 0$, $\omega \in S^2$ *define the operator* $\mathcal{S}_\infty(\omega)$ *operating on matrix valued functions by*

$$\mathcal{S}_\infty(\omega)X = h_\infty(.,\omega) \star \{\omega.\alpha\, , Q\, X\},$$

Let $l \geq 2$ *and* $s > 3/2$ *then the operator* $\mathcal{S}_\infty(\omega)$ *is bounded from* $\mathbf{W}_{-s}^l$ *into* $\mathbf{W}_{-s}^l$ *,with operator norm* $O(|Q|_{l,r})$ *for* $r > 2s$ *and*

$$\lim_{k \to +\infty} \mathcal{S}(k\omega) = \mathcal{S}_\infty(\omega),$$

in $\mathbf{L}(\mathbf{W}_{-s}^l, \mathbf{W}_{-s}^0)$ *uniformly in* ω .

Proof. By Lemma 2-3,2-4 $\lim_{k \to +\infty} k.h(.,k\omega) = h_\infty(.,\omega)$ in $(\mathcal{S}_\theta^2)^*$, $\theta > 2$ and $\lim_{k \to +\infty} k.h(.,k\omega)\star = h_\infty(.,\omega)\star$ in $\mathbf{L}(\mathbf{W}_s^l, \mathbf{W}_{-s}^0)$ uniformly in ω with $l \geq 2$, $s > 3/2$. But from their definition $\mathcal{S} - \mathcal{S}_\infty = \mathcal{T}_1 + \mathcal{T}_2$ where

$$\mathcal{T}_1(k,\omega)X = (k.h(.,k\omega) - h_\infty(.,\omega)) \star \{\omega.\alpha\, Q\, X\},$$

$$\mathcal{T}_2(k,\omega)X = h(.,k\omega) \star ((-i\alpha.\nabla)Q\, X + \{\beta, Q\, X\}),$$

since multiplication by Q is bounded from $\mathbf{W}_{-s}^l$ into $\mathbf{W}_s^l$, we have $\lim_{k \to +\infty} \mathcal{T}_1(k,\omega) = 0$ $\mathbf{L}(\mathbf{W}_{-s}^l, W_{-s}^0)$ uniformly in ω . On the other hand it is well known, [W1], that the operator $h(.,k\omega)\star$ is bounded from $\mathbf{W}_s^l$ into $\mathbf{W}_{-s}^l$ with norm $O(k^{-1})$ uniformly in ω , hence $\mathcal{T}_2$ is in $\mathbf{L}(\mathbf{W}_{-s}^l, \mathbf{W}_{-s}^0)$ with norm $O(k^{-1})$. from Lemma 2-4 and the fact that multiplication by Q is bounded from $\mathbf{W}_{-s}^l$ into $\mathbf{W}_s^l$ with norm $O(|Q|_{l,r})$,we get the estimate for the operator norm of $\mathcal{S}(\xi)$ and $\mathcal{S}_\infty(\omega)$. ◇

We will show that the operator $(I + S_\infty)$ is invertible $\mathbf{W}_{-s}^l$, to this effect we solve for X the equation

$$X = K - h_\infty(.,\omega) \star \{\omega.\alpha, QX\}. \qquad (2-4)$$

where K is a given matrix valued function. Let

$$P = (2)^{-\frac{1}{2}} \begin{pmatrix} I & -\omega.\sigma \\ \omega.\sigma & I \end{pmatrix} \qquad (2-5)$$

then $P^{-1}\omega.\alpha P = \beta$, let $Y = P^{-1}XP$, $L = P^{-1}KP$ and $R = P^{-1}QP$. By multiplying (2-4) on the left by P^{-1} and on the right by P we get

$$Y = L - h_\infty \star \{\beta, RY\}, \qquad (2-6)$$

Any 4×4 matrix has the 2×2 block representation

$$M = \begin{pmatrix} M_{++} & M_{-+} \\ M_{+-} & M_{--} \end{pmatrix}$$

and we have

$$\{\beta, M\} = \begin{pmatrix} 2M_{++} & 0 \\ 0 & -2M_{--} \end{pmatrix}$$

By using the relations $\sigma_i.\sigma_j = i\varepsilon_{ijk}\sigma_k$,where ε_{ijk} is the signature of the permutation $(123) \to (ijk)$ we have $(\omega.\sigma)(a.\sigma) = \omega.a + (\omega \wedge a)\sigma$ we compute

$$R = \begin{pmatrix} q_+ I & a_\perp.\sigma \\ a_\perp.\sigma & q_- I \end{pmatrix}, \qquad (2-7)$$

with the notations $q_\pm = \varphi \pm a_\|$ where $a_\| = a.\omega$ and $a_\perp = a - a_\|\omega$. Clearly $a_\|$ is the magnitude of the projection of a on ω and $a_\perp$ its component orthogonal to ω .

We have

$$[RY]_{++} = q_+ Y_{++} + a_\perp.\sigma Y_{+-}, \qquad [RY]_{--} = q_- Y_{--} + a_\perp.\sigma Y_{-+}.$$

By expliciting the block structure of equation (2-6) we get $Y_{+-} = L_{+-}$, $Y_{-+} = L_{-+}$, the equations decouple

$$Y_{++} = L_{++} - 2h_\infty \star (q_+ Y_{++} + a_\perp.\sigma Y_{+-}), \qquad Y_{--} = L_{--} + 2h_\infty \star (q_- Y_{--} + a_\perp.\sigma Y_{-+}).$$

These two equation are similar, so it suffices to consider the first. Recall now that $h_\infty(x', x'', \omega) = \frac{i}{2}\Theta(x') \otimes \delta(x'')$,(we omit the reference to ω) to see that the integral equation is equivalent to

$$\partial_{x'}(Y_{++} - L_{++}) = -iq_+(x).(Y_{++} - L_{++}) + M_+$$

where $M_+ = -iq_+ L_{++} - a_\perp.\sigma L_{+-}$ with the initial conditions at $-\infty$

$$\lim_{x' \to -\infty} (Y_{++} - L_{++})(x', x'') = 0$$

So we get

$$Y_{++}(x', x'') = L_{++}(x', x'') + \Big(\int_{-\infty}^{x'} e^{i\tilde{q}_+(t,x'')} M_+(t, x'') dt\Big) e^{-i\tilde{q}_+(x',x'')}, \qquad (2-8)$$

where $\tilde{q}_+(x', x'') = \int_{-\infty}^{x'} q_+(t, x'')$ .We can check that if $L \in \mathbf{W}^l_{-s}$ then the solution $Y \in \mathbf{W}^l_{-s}$ and that the map $L \mapsto Y$ is bounded.

PROPOSITION 2-6. — *The limit* $\lim_{k\to+\infty} V(.,k\omega) = V_\infty(.,\omega)$ *exist in* $\mathbf{W}^0_{-s}$, $s > 3/2$, *uniformly in* $\omega \in S^2$ *and is the unique solution of the equation*

$$V_\infty(.,\omega) = I - h_\infty(.,\omega) \star \{\omega.\alpha, QV_\infty(.,\omega)\}. \qquad (2-9).$$

Proof. Let $\delta > 0$ as in Prop.2-1, and ω fixed. For $k \geq \delta$ we have

$$V(.,k\omega) - V_\infty(.,\omega) = [I + \mathcal{S}_\infty(\omega)]^{-1} (\mathcal{S}_\infty(\omega) - \mathcal{S}(k\omega))\, V(.,k\omega),$$

by Lemma 2-5,

$$\lim_{k\to+\infty} \mathcal{S}(k\omega) = \mathcal{S}_\infty(\omega),$$

in $\mathbf{L}(\mathbf{W}^l_{-s}, \mathbf{W}^0_{-s})$ uniformly in ω . Since $[I + \mathcal{S}_\infty(\omega)]^{-1}$ is bounded in $\mathbf{L}(\mathbf{W}^0_{-s}, \mathbf{W}^0_{-s})$ and $\{V(.,k\omega), k \geq \delta\}$ is bounded in $\mathbf{W}^l_{-s}$ we have $\lim_{k\to+\infty}(V(k\omega) = V_\infty(.,\omega)$. $\diamondsuit$

THEOREM 2-7. — *The Faddeev Formula. The limit* $\lim_{k\to\infty} F(\eta + k\omega, k\omega) = F_\infty(\eta,\omega)$ *exist in* $\mathcal{S}_0^\infty$ *,uniformly in* $\omega \in S^2$, $F_\infty(.,\omega) \in \mathcal{S}$ *and the following equation holds*

$$F_\infty(.,\omega) = \hat{R}(.,\omega) - \hat{R}(.,\omega) \star (\hat{h}_\infty(.,\omega).\{\beta, F_\infty(.,\omega)\}), \qquad (2-10)$$

where $R(.,\omega)$ *is given by* (2-7). *The limit* F_∞ *is given by relations* (2-11),(2-12).

Proof.The proof results easily from the Proposition 2-5, by noting that

$$F(\eta+\xi,\xi) = U_0(\eta+\xi)^{-1}.(\mathcal{F}Q(.)V(\xi,.))(\eta).U_0(\xi).$$

$\diamondsuit$

For the Schrödinger operator we have

$$\hat{Q}(\eta) = -4\pi \lim_{k\to\infty} A(\eta + k\omega, k\omega)$$

where $A(\xi,\eta)$ is the scattering amplitude and the limit is taken with fixed momentum transfer η . This is the Born approximation to the scattering amplitude , see [N1]pg 282, [N2] pg 25, this relation is known as the Faddeev formula . In the case of the Dirac operator the high energy limit of F at fixed momentum transfer is linked to the potential Fourier transform by equation (2-10).

Equation (2-10) leads to the solution for the inverse problem, in order to see this we analyse the equation to find the structure of F_∞ , which we denote simply by F omitting the reference to ω . Decopmpose the equation into blocks after performing the inverse Fourier transform

$$\check{F}_{++} = R_{++}(1 - 2h_\infty(.,\omega) \star \check{F}_{++}) \quad , \quad \check{F}_{-+} = R_{-+}(1 + 2h_\infty(.,\omega) \star \check{F}_{--})$$
$$\check{F}_{+-} = R_{+-}(1 - 2h_\infty(.,\omega) \star \check{F}_{++}) \quad , \quad \check{F}_{--} = R_{--}(1 + 2h_\infty(.,\omega) \star \check{F}_{--})$$

Note that $R_{++} = q_+ I$, $R_{--} = q_- I$ and $R_{+-} = R_{-+} = a_\perp.\sigma$. Observe that for $w \in \mathcal{S}$ the scalar equation $u = w(1 \pm 2h_\infty(.,\omega) \star u)$ has a unique solution in $u \in \mathcal{S}$ so we must have $\check{F}_{++} = f_+ I$ and $\check{F}_{--} = f_- I$, f_+ , f_- beeing numerical function in $\mathcal{S}$, since $\operatorname{Tr} R_{+-} = \operatorname{Tr} R_{-+} = 0$ we conclude that $\operatorname{Tr} \check{F}_{+-} = \operatorname{Tr} \check{F}_{-+} = 0$. Denote by g_+, g_- the functions $g_\pm = (1 \mp 2h_\infty(.,\omega) \star f_\pm)$, from Definition 2-2, we see that

$$g_\pm(x', x'') = 1 \mp i \int_{-\infty}^{x'} f_\pm(t, x'')dt \qquad (2-11)$$

so equation (2-10) is recast into

$$\check{F}_{++} = f_+ I \quad , \quad \check{F}_{--} = f_- I \qquad (2-12-i)$$

$$\check{F}_{+-} = g_+.a_\perp.\sigma \quad , \quad \check{F}_{-+} = g_-.a_\perp.\sigma \qquad (2-12-ii)$$

where :

$$g_\pm = \exp(\mp i \int_{-\infty}^{x'} q_\pm(t, x'')dt), \qquad f_\pm = q_\pm \exp(\mp i \int_{-\infty}^{x'} q_\pm(t, x'')dt). \qquad (2-12-iii)$$

This shows that $g_\pm \in \mathcal{S}_0^\infty$, $f_\pm \in \mathcal{S}$ are such that $f_+ = \bar{f}_-$, $|g_\pm| = 1$ and we have $q_\pm = f_\pm \bar{g}_\pm$. Now (2-12-ii) shows that $a_\perp.\sigma = \bar{g}_+ \check{F}_{+-} = \bar{g}_- \check{F}_{-+}$ is a traceless self-adjoint matrix that anticommutes with $\omega.\sigma$. Note conversely that any such a matrix may be written uniquely $M = b.\sigma$ with $2b_k = \{M, \sigma_k.\}$ and b orthogonal to ω . This proves the unicity of the solution of (2-10) and gives a characterization of F_∞ . The following theorem gives the solution to the inverse problem : the reconstruction of the potential knowing F_∞ .

THEOREM 2-8. — *Let $F \in \mathcal{S}$ be a matrix valued function and ω a unit vector. Suppose that $\check{F}_{++} = f_+ I$ and $\check{F}_{--} = f_- I$ where $f_\pm$ are scalar functions such that $\bar{f}_+ = f_-$ and $|g_\pm| = 1$, with $g_\pm$ defined by* (2-11), *suppose further that $\bar{g}_+ \check{F}_{+-} = \bar{g}_- \check{F}_{-+}$ are self adjoint with trace zero and anticommutes with $\omega.\sigma$. Then there exist a real valued functions φ ,and a real vector field a such that if $Q = \varphi I + a.\alpha$ then $F_\infty(.,\omega) = F(.)$.*

Proof.The condition $|g_\pm| = 1$ implies that $\bar{g}_+ f_+$ and $\bar{g}_- f_-$ are real valued functions.Define φ and $a_\|$ by

$$2\varphi = \bar{g}_+ f_+ + \bar{g}_- f_- \quad , \quad 2a_\| = \bar{g}_+ f_+ - \bar{g}_- f_-$$

and let $a_\perp$ be the unique real vector field such

$$\bar{g}_+ \check{F}_{+-} = a_\perp.\sigma.$$

then ω orthogonal to $a_\perp$, let $a = a_\|.\omega + a_\perp$ and $Q = \varphi + a.\alpha$ then $Q \in \mathcal{S}$ is self-adjoint, consider the direct scattering problem then by Theorem 2-7, $F_\infty(.,\omega)$ exists and is the solution to 2-10, hence $F_\infty(.,\omega) = F(.)$ by the unicity of the solution. $\diamond$

III. The high energy asymptotic of the scattering amplitude and the inverse problem.

In Section I we have introduced the scattering amplitude, it is modulo a factor depending only on $w(k)$ the restriction of the diagonal part of $F(\eta,\xi)$ to the cone $|\xi| = |\eta|$ identified to $\Gamma = \mathbb{R}_+^* \times S^2 \times S^2$ by setting $\eta = k\theta$ and $\xi = k\omega$. A path $k \mapsto (k, \theta(k), \omega(k))$ in Γ having a limit (∞, θ, ω) as $k \to \infty$ is said to be an Admissible Path if the momentum transfer $k(\theta(k) - \omega(k)) = \tau$ is constant for large k . It is easy to see that for such a path, $\omega = \theta$ and τ must be orthogonal to ω , in fact consider the pair $(\omega, \tau) \in S^2 \times \mathbb{R}^3$, let $k \mapsto \omega(k)$ be a path in S^2 with limit ω then $k \mapsto (k, \omega(k) + k^{-1}\tau, \omega(k))$ is admissible, all admissible pathes are generated in this manner.The following Lemma results from Theorem 2-6.

LEMMA 3-1. — *Let* $(\tau, \omega) \in \mathbb{R}^3 \times S^2$ *,such that* $\langle \tau, \omega \rangle = 0$ *then along any associated admissible path*

$$\lim_{k\to\infty} w(k)^{-1} T(k, \theta(k), \omega(k)) = -(2\pi)^{1/2} \beta F_{\infty,\Delta}(\tau, \omega).$$

The Lemma shows in particular that $\lim_{k\to\infty} w(k)^{-1} T(k, \omega, \omega)$ does not exist.

PROPOSITION 3-2. — *Let* $F_{\infty,\Delta}$ *be a given* $\mathbf{C}^\infty$ *function of* $(\tau, \omega) \in \mathbb{R}^3 \times S^2$ *on the set* $\langle \tau, \omega \rangle = 0$ *, rapidly decreasing as a function of* τ *, then the potentials* φ, a *are uniquely determined modulo a gauge trasformation. The direct problem with potentials* φ, a *with leads to* $F_{\infty,\Delta}$.

Proof.For a given ω take coordinates ξ' along ω , and ξ'' orthogonal to ω . By hypothesis the function $\xi'' \mapsto \check{F}_{\infty,\Delta}(0, \xi'')$ is given (omitting reference to ω). So $x'' \mapsto \int F_{\infty,\Delta}(x', x'')dx'$ is also given. ¿From relations (2-12) we have

$$\int \check{F}_{\infty,\Delta}(x', x'')dx' = \mathrm{diag}(\gamma_+ I, \gamma_- I)$$

$$\gamma_\pm(x'') = \int_{-\infty}^{+\infty} q_\pm(x', x'') \exp(\mp i \int_{-\infty}^{x'} q_\pm(t, x'')\, dt)\, dx'$$

this gives by integration

$$\gamma_\pm(x'') = \pm i(\exp(\mp i \int_{-\infty}^{+\infty} q_\pm(x', x'')dx') - 1),$$

since $q_\pm$ is real this relation is inverted

$$\int_{-\infty}^{+\infty} q_\pm(x', x'')dx' = \pm i \log(1 \mp i\gamma_\pm(x''))$$

with the principal determination of the complex log . This last relation means that $\xi'' \mapsto \hat{q}_\pm(0,\xi'')$ is determined uniquely.Recalling the definition of $q_\pm$, the above data determine for any ω the restrictions of $\hat{\varphi}$ and $\hat{a}_{\|}$ to the orthogonal of ω this suffices to determine φ . Let $\xi \in \mathbb{R}^3 \setminus \{0\}$ and take $\omega = |\xi|^{-1}\xi$, since $\hat{a}(\xi) - |\xi|^{-2}\langle \hat{a}(\xi), \xi\rangle \xi$ is orthogonal to ω , it is uniquely determined. Hence we have determined $a + \nabla\rho$ where $\rho = -\Delta^{-1}(\nabla . a)$. ◊

Appendix.

Proof of Lemma 2-2. With no loss of generality suppose that ω is along the x_1 axis, let $\chi \in \mathcal{S}^2_\theta$, $\theta > 2$ then

$$< kh(.,k\omega), \chi > = \int_{\mathbb{R}^3} k\frac{e^{-ik(x_1 - r)}}{4\pi r}\chi(x)dx$$

Take cylindrical coordinates x_1, ρ, φ , $\rho > 0$, $\varphi \in]0, 2\pi[$ then

$$< kh(.,k\omega), \chi > = \int_{\mathbb{R}} dx_1 e^{-ikx_1} \int_{\mathbb{R}_+} k\rho \frac{e^{ik\sqrt{\rho^2 + x_1^2}}}{\sqrt{\rho^2 + x_1^2}}\gamma(x_1, \rho)d\rho$$

where

$$\gamma(x_1, \rho) = \frac{1}{4\pi}\int_0^{2\pi} \chi(x_1, \rho\cos\varphi, \rho\sin\varphi)d\varphi$$

The function $\gamma(x_1, .)$ extends to an even function of class C^2 such that

$$|\partial^\alpha_\rho \partial^\beta_{x_1}\gamma| \le Cr^{-\theta}\|\chi\|$$

for any $\alpha, \beta \in \mathbb{N}, \alpha + \beta \le 2$. Where $\|\chi\|$ is the norm in $\mathcal{S}^2_\theta$. Integration by parts of the inner integral gives

$$\int_{\mathbb{R}_+} k\rho \frac{e^{ik\sqrt{\rho^2 + x_1^2}}}{\sqrt{\rho^2 + x_1^2}}\gamma(x_1, \rho)d\rho = \frac{i}{2}e^{ik|x_1|}\chi(x_1, 0) + i\int_{\mathbb{R}_+} e^{ik\sqrt{\rho^2 + x_1^2}}\gamma'_\rho(x_1, \rho)d\rho$$

by integration over x_1

$$< kh(.,k\omega), \chi > = \frac{i}{2}\int_{\mathbb{R}_+} \chi(x_1, 0)dx_1 + I_k(\chi) + J_k(\chi)$$

where

$$I_k(\chi) = \frac{i}{2}\int_{\mathbb{R}_-} e^{-2ikx_1}\chi(x_1, 0)dx_1$$

and

$$J_k(\chi) = i\int_{\mathbb{R}}\int_{\mathbb{R}_+} e^{-ik(x_1 - \sqrt{\rho^2 + x_1^2})}\gamma'_\rho(x_1, \rho)dx_1 d\rho$$

Observing that the first integral $< h_\infty(.,\omega), \chi >$, we conclude by showing that

$$|I_k(\chi)| \le Ck^{-1}\|\chi\|; \quad |J_k(\chi)| \le Ck^{-\delta}\|\chi\|, \quad \delta > 0.$$

The first estimate is easily obtained by integration by parts, the second estimate results from the following lemma with $f = \gamma'_\rho$. The uniformity of the limit in ω stems from the fact that the constants do not depend on ω independant. ◊

LEMMA.. — *Let* $f(x,\rho)$ *be a* C^1 *function on* $\mathbb{R}\times\mathbb{R}$ *,we suppose* $f(x,.)$ *is odd and that for some* $\theta > 2$

$$|\partial_\rho^\alpha \partial_x^\beta f| \leq m(f)(1+r)^{-\theta}$$

for $\alpha+\beta \leq 1$, $r = \sqrt{\rho^2+x^2}$ *and let*

$$J_k(f) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} e^{-ik(x-\sqrt{\rho^2+x^2})} f(x,\rho)dxd\rho,$$

then there exist $0 < \delta < (\theta-2)/3$ *such that*

$$|J_k(f)| \leq C.m(f).k^{-\delta}.$$

Proof. With the polar coordinates $x = r\cos\phi$, $\rho = r\sin\phi$, $r > 0$, $\phi \in]0,\pi[$ we have

$$J_k(f) = \int_{\mathbb{R}_+} rdr \int_0^\pi e^{-ikr(\cos\phi-1)} g(r,\phi)d\phi,$$

The function $g(r,\phi) = f(r\cos\phi, r\sin\phi)$ is of class C^1 and

$$|\partial_\phi^\alpha g| \leq C.m(f).(1+r)^{\alpha-\theta}$$

for $\alpha \leq 1$. Let $\lambda = kr$, then the inner integral is $I_\lambda(g(r,.))$ with

$$I_\lambda(u) = \int_0^\pi e^{-i\lambda(\cos\phi-1)} u(\phi)d\phi.$$

This oscillatory integral has stationnary points at $0,\pi$, the amplitude function u is the restriction to $[0,\pi]$ of a periodic odd function . Using results from [Hö], we can prove

$$|I_\lambda(u)| \leq C\lambda^{-1} \sum_{|\alpha|\leq 3} \sup_\phi |\partial^\alpha u|$$

if $u \in \mathbf{C}^3$ and by interpolation we prove that with δ, μ ,such that $4\delta < \mu < 1$,

$$|I_\lambda(u)| \leq C\lambda^{-\delta}|u|_{0,\mu},$$

the norm of u is in Hölder space $\mathbf{C}^{0,\mu}$. Taking into account the estimates on g we have

$$|I_\lambda(g(r,.)| \leq Cm(f)k^{-\delta}r^{-\delta}(1+r)^{1-\theta})$$

Inserting this estimate into $J_k(f)$ gives the desired estimate. ◇

Proof of Lemma 2-3. Observe first that if $t, \sigma > 3/2$ then

$$\mathcal{S}_{r+t}^m \subset \mathbf{W}_r^m \quad ; \quad \mathbf{W}_r^{m+\sigma} \subset \mathcal{S}_r^m$$

with continuous injections, this results from the definitions and the Sobolev embedding theorem. The Fourier transform maps $\mathcal{S}_r^m$ into $\mathcal{S}_m^{r-t}$ if $t > 3$ and by duality $(\mathcal{S}_r^m)^*$ into $(\mathcal{S}_m^{r-t})^*$. For $f \in \mathcal{S}$,the convolution $h * f$ is defined in $\mathcal{S}'$ as the distribution whose Fourier transform is $\hat{f}.\hat{h}$ and we have we $\forall f, g \in \mathcal{S}$

$$< h * f, \bar{g} > = < \hat{h}, \hat{f}\bar{\hat{g}} >$$

Suppose that $f \in \mathbf{W}_s^l$ and $g \in \mathbf{W}_s^{l'}$ with $s > 3/2$, then $\hat{f} \in \mathbf{W}_l^s$ and $\hat{g} \in \mathbf{W}_{l'}^s$. Since $\mathbf{W}_0^s = \mathbf{H}^s$ is an algebra when $s > 3/2$ we have then $\hat{f}\hat{g} \in \mathbf{W}_{l+l'}^s \subset \mathcal{S}_{l+l'}^{s-\sigma}$, with $\sigma > 3/2$. Let $\varepsilon = s - r - 3/2$ and take $t = 3 + \varepsilon, \sigma = 3/2 + 2\varepsilon$ and $l + l' = m$ then $s - \sigma = r - t$. Since $\hat{h} \in (\mathcal{S}_m^{r-t})^*$ we get

$$| < h * f, \bar{g} > | \leq C|\hat{h}|_{(\mathcal{S}_m^{r-t})^*}.|\hat{f}\hat{g}|_{\mathcal{S}_m^{r-t}} \leq C|h|_{(\mathcal{S}_r^m)^*}.|f|_{\mathbf{W}_s^l}|g|_{\mathbf{W}_s^{l'}}$$

this achieves the proof. ◇

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