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Global existence and scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds

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Abstract. We consider the nonlinear Schrödinger equation with a pure power repulsive nonlinearity on Schwarzschild manifolds. Equations of this type arise when a nonlinear wave equation on a Schwarzschild manifold is written in Hamiltonian form, cf. [2], [10]. For radial solutions with sufficiently localized initial data, we obtain global existence, L^p estimates, and the existence and asymptotic completeness of the wave operators. Our approach is based on a dilation identity and global space-time estimates.

1 Introduction

A *Schwarzschild manifold* is the space $\mathbf{R} \times \mathbf{R}^+ \times S^2$ equipped with the Schwarzschild metric:

$$g = g_{\mu\nu} dx^\mu dx^\nu,$$

which may be written in polar coordinates as:

$$g = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 \Delta_{S^2}; \tag{1.1}$$

$\Delta_{S^2} = d\theta^2 + \sin^2\theta d\phi^2$ is the Laplace-Beltrami operator on a 2-dimensional sphere. The parameter $M > 0$ is interpreted as the mass of the black hole. We restrict our attention to the *external* Schwarzschild solution ($r > 2M$).

Scattering theory for the wave and Klein-Gordon equations on Schwarzschild manifolds was first studied by Dimock [5] and Dimock and Kay [6]. Following these authors, we rewrite (1.1) as:

$$g = \Big(1 - \frac{2M}{r}\Big)(dt^2 - dr_*^2) - r^2\Delta_{S^2}, \tag{1.2}$$

where r_* is the Regge-Wheeler tortoise radial coordinate:

$$r_* = r + 2M\log(r - 2M) \tag{1.3}$$

(hence $\frac{dr}{dr_*} = 1 - \frac{2M}{r}$). The wave equation on the Schwarzschild manifold is:

$$\Box_g u = 0, \tag{1.4}$$

where:

$$\Box_g = |\det g|^{-1/2}\frac{\partial}{\partial x^\mu}\Big(|\det g|^{1/2}\, g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Big) \tag{1.5}$$

is the d'Alembertian associated to g. Using (1.2), we may write (1.5) as:

$$\Box_g = \Big(1 - \frac{2M}{r}\Big)^{-1}\Big(\frac{\partial^2}{\partial t^2} - r^{-2}\frac{\partial}{\partial r_*}r^2\frac{\partial}{\partial r_*}\Big) - r^{-2}\Delta_{S^2}. \tag{1.6}$$

(1.4)–(1.6) is equivalent to:

$$\frac{\partial^2}{\partial t^2}u + Hu = 0, \tag{1.7}$$

where:

$$H = -r^{-2}\frac{\partial}{\partial r_*}r^2\frac{\partial}{\partial r_*} - r^{-2}\Big(1 - \frac{2M}{r}\Big)\Delta_{S^2}. \tag{1.8}$$

In [5], it was proved that the wave operators for (1.7) exist and are complete. For the Klein-Gordon equation $\Box_g u + m^2 u = 0$ the existence of wave operators was proved in [6], and asymptotic completeness – in [1]. The results of [5] on scattering were recovered in [4], where the authors actually considered a more general class of noncompact manifolds; the proof in [4] relied on a Mourre estimate obtained for such manifolds in [7]. (See also [12], where certain techniques of geometric scattering theory were applied to the De Sitter model.)

An important open problem is to develop a scattering theory for equations such as (1.7), but with a nonlinear perturbation added. Partial results in that direction were obtained in [2], [10]; the class of metrics considered there is in fact more general than (1.1) and includes other black hole models. In particular, Nicolas [10] studied a nonlinear Klein-Gordon equation of the form:

$$\Box_g u + m^2 u + \lambda F(r)|u|^2 u = 0,$$

where $F(r)$ is an explicitly given factor vanishing as $r \to 2M$ and as $r \to \infty$. He obtained the global existence of solutions to the above equation and an outgoing radiation condition

for these solutions. We also remark that the Cauchy problem for the Yang-Mills equations on Schwarzschild manifolds was studied in [14].

In the present paper, we take a different route and study the scattering theory for the nonlinear Schrödinger equation:

$$i\frac{\partial u}{\partial t} = Hu + \lambda|u|^{p-1}u, \ \lambda > 0; \tag{1.9}$$

we will, moreover, restrict our attention to radially symmetric solutions, *i.e.,* assume that:

$$\Delta_{S^2} u \equiv 0. \tag{1.10}$$

The payoff for these simplifications is that we will be able to present a relatively elementary (modulo the estimates of [15]) proof of existence and completeness of the wave operators. As one might guess by considering the geometry of the manifold, (1.9) has two scattering channels: part of the outgoing wave escapes to the (spatial) infinity and becomes asymptotically free, while another part approaches the black hole horizon and therefore displays a different asymptotic behaviour. We prove that (in a suitable coordinate system) each part has the asymptotics of a solution to a simpler *linear* equation. Our analysis of (1.9) is based on a priori estimates, similar in spirit to the conformal and Morawetz identities; in particular, we obtain the local decay of solutions and suitable space-time L^p estimates.

We remark that the same proof, with only minor modifications, should work for slightly more general nonlinearities of the form $f(|u|)u$, where $f(s)$ is a suitable real-valued function; for the sake of brevity, we do not attempt here to find the exact conditions on f under which this can be done.

While our results do not imply anything directly about scattering for a nonlinear *wave* equation (which would be more interesting than (1.9) from the point of view of physics), we believe that the methods presented in this article may be developed further to yield progress in that direction. To illustrate the connection between (1.7) and (1.9), we rewrite (1.7) in Hamiltonian form:

$$i\frac{\partial}{\partial t}\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \frac{1}{i}\begin{pmatrix} 0 & -1 \\ H & 0 \end{pmatrix}\begin{pmatrix} u \\ \partial_t u \end{pmatrix}. \tag{1.11}$$

This was in fact the approach taken in [5], [6], [1], [10], [4]. By diagonalizing the matrix in (1.11) one may reduce the problem to studying the unitary group $\exp(-it\sqrt{H})$, cf. [1], [4]. It was further demonstrated in [4] that certain results of this type may be deduced from their analogues for the Schrödinger unitary group e^{-itH}. We hope to use similar methods to make progress on the scattering theory for a nonlinear variant of (1.7).

The paper is organized as follows. Section 2 takes care of preliminaries such as the conservation of the L^2 and energy norms for the solutions of (1.9). Assuming (1.10), the problem becomes effectively one-dimensional. We simplify it further in Section 3 by applying a suitable unitary transformation. The point of this reduction is that the kinetic energy part becomes simply $D_{r_*}^2$; the price we pay is that we have to add a "potential" V. Our main estimates are proved in Sections 4–6. In Section 7 we combine them with the known L^1 estimates for a 1-dimensional linear Schrödinger equation (see [15], where such estimates

were proved and applied to a similar nonlinear scattering problem), and obtain the time decay of L^p norms of the solutions. The global existence of solutions, and the existence and completeness of the wave operators between the nonlinear equation and the corresponding *linear* Schrödinger equation, follow by a standard argument (cf. [13]).

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2 Preliminaries

Let $u = u(r, \omega, t)$ be a solution to (1.9), where H is given by (1.8); we will also use the notation $u_t(\cdot) \equiv u(\cdot, t)$. Recall that r_* is defined in (1.3). For future reference we note that r is an increasing function of r_* and:

- as $r_* \to \infty$, $r \to \infty$ and $1 - \frac{2M}{r} \to 1$;
- as $r_* \to -\infty$, $r \sim 2M + e^{-1+r_*/2M} \to 2M$ and:

$$1 - \frac{2M}{r} \sim \frac{1}{2M} e^{-1-|r_*|/2M} \tag{2.1}$$

vanishes exponentially in $|r_*|$.

We will assume that the initial data u_0 satisfies (1.10) and belongs to the energy space:

$$\mathcal{H} = \{u \in L^2(\mathbf{R} \times S^2; r^2 dr_* \, d\omega) : \; u(r, \omega) = u(r) \text{ and } \mathbf{E}(u) < \infty\},$$

where the *energy* $\mathbf{E}(u)$ is defined as:

$$\mathbf{E}(u) = \int \left(\bar{u} \, H u + \frac{2\lambda}{p+1} |u|^{p+1} \right) r^2 dr_* d\omega. \tag{2.2}$$

We denote by $\|u\|_{\mathcal{H}} = (\mathbf{E}(u))^{1/2}$ the energy norm of u.

Observe first that the L^2 norm of u_t is conserved: this follows from the fact that the operator H is symmetric and from the form of the nonlinearity in (1.9). The second basic fact we will need is the conservation of energy.

Proposition 2.1 *For any solution $u_t(r, \omega) = u(r, \omega, t)$ of (1.9) (not necessarily radially symmetric), $\mathbf{E}(u_t)$ is independent of t. In particular, if $u_0 \in \mathcal{H}$, then $u_t \in \mathcal{H}$ and $\|u_t\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}}$ for all t.*

Proof. We compute:

$$\frac{d}{dt}\int_{\mathbf{R}\times S^2} \bar{u}_t \cdot (H + \lambda|u_t|^{p-1})u_t\, r^2 dr_* d\omega = \int_{\mathbf{R}\times S^2} \bar{u}_t \cdot \frac{\partial}{\partial t}(\lambda|u_t|^{p-1})u_t\, r^2 dr_* d\omega$$
$$= \int \frac{\partial}{\partial t}(\lambda|u_t|^{p-1})|u_t|^2 r^2 dr_* d\omega = \frac{p-1}{p+1}\int \frac{\partial}{\partial t}(\lambda|u_t|^{p+1}) r^2 dr_* d\omega.$$

Hence:

$$\frac{d}{dt}\int \left(\bar{u}_t H u_t + \lambda|u_t|^{p+1}\right) r^2 dr_* d\omega = \frac{\lambda(p-1)}{p+1}\frac{\partial}{\partial t}\int (\lambda|u_t|^{p+1}) r^2 dr_* d\omega,$$

which proves the proposition. □

Corollary 2.2 *Suppose that u_t solves (1.9), $u_0 \in \mathcal{H}$. Then:*

(i) $\int \bar{u}_t H u_t\, r^2 dr_* d\omega < C\|u_0\|^2_{\mathcal{H}}$,

(ii) $\int |u_t|^{p+1}\, r^2 dr_* d\omega < C\|u_0\|^2_{\mathcal{H}}$,

uniformly in t.

Proof. By Proposition 2.1,

$$\mathbf{E}(u_t) = \int \bar{u}_t H u_t\, r^2 dr_* d\omega + \frac{2\lambda}{p+1}\int |u_t|^{p+1}\, r^2 dr_* d\omega \tag{2.3}$$

is constant in t. Since both terms on the right-hand side of (2.3) are positive, this implies the corollary. □

3 Reduction to a one-dimensional problem

¿From now on, we restrict our attention to the space $L^2_{radial}(\mathbf{R}\times S^2; r^2 dr_* d\omega)$ of radially symmetric functions in $L^2(\mathbf{R}\times S^2; r^2 dr_* d\omega)$. We define a unitary operator $U : L^2(\mathbf{R}, dr_*) \to L^2_{radial}(\mathbf{R}\times S^2; r^2 dr_* d\omega)$ by:

$$U : \psi(r) \to u(r,\omega) := r^{-1}\psi(r),$$

and the symmetric operator $\tilde{H}$ on $L^2(\mathbf{R}, dr_*)$ by:

$$\tilde{H} = U^{-1} H U = r H r^{-1}.$$

Using (1.8) and (1.10), we find that:

$$\tilde{H} = D^2_{r_*} + V(r_*), \tag{3.1}$$

where $D = -i\partial$ and:

$$V(r_*) = \frac{2M}{r^3}\left(1 - \frac{2M}{r}\right). \tag{3.2}$$

Substituting $\psi = U^{-1}u = ru$ in (1.9), we obtain that ψ satisfies::

$$i\frac{\partial}{\partial t}\psi = \tilde{H}_\psi \psi, \tag{3.3}$$

where $\tilde{H}_\psi$ is the nonlinear operator:

$$\tilde{H}_\psi = \tilde{H} + \lambda r^{-p+1}|\psi|^{p-1}. \tag{3.4}$$

The energy space $\mathcal{H}$ is mapped by U^{-1} to:

$$\tilde{\mathcal{H}} = \{\psi \in L^2(\mathbf{R}; dr_*) : \ \|\psi\|_{\tilde{\mathcal{H}}}^2 := \int \bar{\psi} \cdot \tilde{H}_\psi \psi dr_* < \infty\}.$$

The remark before Proposition 2.1, and the unitarity of U, imply the conservation of the L^2 norm for solutions of (3.3). Moreover, from Corollary 2.2 we obtain the following.

Proposition 3.1 *U is a unitary operator from $\tilde{\mathcal{H}}$ to $\mathcal{H}$. Moreover, if $\psi_t(r) = \psi(r,t)$ solves (3.3) and $\psi_0 \in \tilde{\mathcal{H}}$, then:*

(i) $\int |\frac{\partial}{\partial r_*}\psi_t|^2 \, dr_* < C\|\psi_0\|_{\tilde{\mathcal{H}}}^2$;

(ii) $\int r^{-p+1}|\psi_t|^{p+1} \, dr_* < C\|\psi_0\|_{\tilde{\mathcal{H}}}^2$,

uniformly in t.

Proof. Substituting $u_t = r^{-1}\psi_t$ in Corollary 2.2*(i)*, we obtain that:

$$\begin{aligned}
&\int \overline{r^{-1}\psi_t} \cdot H(r^{-1}\psi_t) r^2 dr_* d\omega = \int \bar{\psi}_t \cdot rHr^{-1}\psi_t \, dr_* d\omega \\
&= \int \bar{\bar{\psi}}_t \cdot \tilde{H}\psi_t \, dr_* d\omega = \int \bar{\bar{\psi}}_t \cdot (D_{r_*}^2 + V(r_*))\psi_t \, dr_* d\omega \\
&= \int \bar{\psi}_t \cdot D_{r_*}^2 \psi_t \, dr_* d\omega + \int \bar{\psi}_t \cdot V(r_*)\psi_t \, dr_* d\omega
\end{aligned}$$

is bounded by $C\|u_0\|_{\mathcal{H}}^2 = C\|\psi_0\|_{\tilde{\mathcal{H}}}^2$ for all t. Moreover, since both terms in the last line are positive, we find that:

$$\int \bar{\psi}_t \cdot D_{r_*}^2 \psi_t \, dr_* d\omega < C\|\psi_0\|_{\tilde{\mathcal{H}}}^2$$

uniformly in t, which after integration by parts yields *(i)*. Part *(ii)* follows by substituting $u_t = r^{-1}\psi_t$ in Corollary 2.2*(ii)*. □

4 The dilation identity

The starting point for our analysis of (3.3) is the observation that both the "potential" V given by (3.2) and the nonlinear term in (3.4) are repulsive interactions. Hence the long-time behaviour of the solutions is largely determined by the dispersive term $D_{r_*}^2$. In particular,

we obtain the *local decay* estimates (Proposition 5.1); these in turn will be needed in the proof of our results on scattering.

Throughout the rest of this paper we will denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbf{R}; dr_*)$: $\langle \psi, \phi \rangle = \int_{-\infty}^{\infty} \bar{\psi} \phi dr_*$.

Proposition 4.1 *There is an $\alpha \in \mathbf{R}$ (given explicitly by (4.1)) such that:*

$$\langle \psi, [\tilde{H}, iA_\alpha]\psi \rangle > 0,$$
$$\langle \psi, [\lambda r^{-p+1}|\psi|^{p-1}, iA_\alpha]\psi \rangle > 0$$

for all $\psi \in \tilde{\mathcal{H}}$, where $A_\alpha = \frac{1}{2}((r_* - \alpha)D_{r_*} + D_{r_*}(r_* - \alpha))$.

Proof. We have:

$$i[\tilde{H}, A_\alpha] = 2D_{r_*}^2 - (r_* - \alpha)\frac{dV(r_*)}{dr_*},$$

where

$$\frac{dV(r_*)}{dr_*} = \frac{2M}{r^4}\Big(1 - \frac{2M}{r}\Big)\Big(\frac{8M}{r} - 3\Big)$$

is positive for $2M < r < 8M/3$ and negative for $r > 8M/3$. Let

$$\alpha = \frac{8M}{3} + 2M \log \frac{2M}{3} \tag{4.1}$$

be the value of r_* corresponding to $r = 8M/3$. Then $-(r_* - \alpha)\frac{dV}{dr_*} > 0$ for all $r_* \in \mathbf{R}$, $r_* \neq \alpha$; hence:

$$\int_{-\infty}^{\infty} \bar{\psi} \cdot [\tilde{H}, iA_\alpha]\psi \, dr_* > 0$$

for all $\psi \in \tilde{\mathcal{H}}$, $\psi \not\equiv 0$.

Next, we consider the commutator with the nonlinear term:

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} \bar{\psi} \cdot [\lambda r^{-p+1}|\psi|^{p-1}, iA_\alpha]\psi \, dr_* \\ &= -\lambda \textstyle\int_{-\infty}^{\infty} |\psi|^2 (r_* - \alpha)\frac{\partial}{\partial r_*}(r^{-p+1}|\psi|^{p-1}) dr_*. \end{aligned} \tag{4.2}$$

We have

$$|\psi|^2 \frac{\partial}{\partial r_*}(r^{-p+1}|\psi|^{p-1}) = \frac{p-1}{p+1} r^2 \frac{\partial}{\partial r_*}(r^{-p-1}|\psi|^{p+1}). \tag{4.3}$$

Using (4.3) and integrating the right-hand side of (4.2) by parts, we obtain that it is equal to:

$$\frac{\lambda(p-1)}{p+1} \int_{-\infty}^{\infty} \frac{\partial}{\partial r_*}(r^2(r_* - \alpha)) \cdot (r^{-p-1}|\psi|^{p+1}) dr_*. \tag{4.4}$$

Since r is a positive and increasing function of r_*, so is $r^2(r_* - \alpha)$, hence $\partial_{r_*}(r^2(r_* - \alpha)) \geq 0$ and the integrand in (4.4) is nonnegative. Since $\lambda > 0$, (4.4) is nonnegative. This completes the proof of the proposition. □

5 Local decay

The purpose of this section is to prove the following local decay estimates.

Proposition 5.1 *Suppose that ψ_t solves (3.3), $\psi_0 \in \tilde{\mathcal{H}}$, and let $\beta > 3/2$, $0 \le R < \infty$. Then*

$$\int_{-\infty}^{\infty} \|(1+r_*^2)^{-\beta/2}\psi_t\|^2 dt \le C_\beta \|\psi_0\|_{L^2(\mathbf{R};dr_*)} \|\psi_0\|_{\tilde{\mathcal{H}}}, \tag{5.1}$$

$$\int_{-\infty}^{\infty} dt \int_{-R}^{R} r^{-p-1}|\psi_t|^{p+1} dr_* \le C_R \|\psi_0\|_{L^2(\mathbf{R};dr_*)} \|\psi_0\|_{\tilde{\mathcal{H}}}. \tag{5.2}$$

Proposition 5.1 will be obtained as a consequence of Proposition 5.2 below.

Proposition 5.2 *Let $\tilde{g}(r_*) = g(r_* - \alpha)$, where $g(s) = \int_0^s (1+t^2)^{-\sigma} dt$ for some $\sigma \in (1/2, 3/2)$ and α is as in Proposition 4.1. Define:*

$$\gamma = \frac{1}{2}(\tilde{g}(r_*)D_{r_*} + D_{r_*}\tilde{g}(r_*)). \tag{5.3}$$

(i) Let ψ_t solve (3.3), $\psi_0 \in \tilde{\mathcal{H}}$. Then $\|\gamma\psi_t\|_{L^2(\mathbf{R};dr_*)} \le C\|\psi_0\|_{\tilde{\mathcal{H}}}$ uniformly in t.

(ii) For any $0 < R < \infty$ there are $c_1, c_{2,R} > 0$, such that for all $\psi \in \tilde{\mathcal{H}}$:

$$\langle \psi, i[\tilde{H}_\psi, \gamma]\psi\rangle \ge c_1 \int_{-\infty}^{\infty} (1+r_*^2)^{-\sigma-1}|\psi|^2 dr_* + c_{2,R}\int_{-R}^{R} r^{-p-1}|\psi|^{p+1} dr_*. \tag{5.4}$$

Proof of Proposition 5.1, given Proposition 5.2. Clearly, it suffices to prove the proposition for $3/2 < \beta < 5/2$. By Proposition 5.2*(ii)*, we have:

$$\begin{aligned} &\tfrac{d}{dt}\langle\psi_t, \gamma\psi_t\rangle = \langle\psi_t, i[\tilde{H}_\psi, \gamma]\psi_t\rangle \\ &\ge c_1\langle\psi_t, (1+r_*^2)^{-\beta}\psi_t\rangle + c_{2,R}\textstyle\int_{-R}^{R} r^{-p-1}|\psi_t|^{p+1} dr_*, \end{aligned}$$

where $\beta = \sigma + 1$. Integrating this inequality from $-\infty$ to ∞ in t, we obtain:

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty}\langle\psi_t, (1+r_*^2)^{-\beta}\psi_t\rangle + \int_{-\infty}^{\infty} dt \int_{-R}^{R} r^{-p-1}|\psi_t|^{p+1} dr_* \\ &\le \overline{\lim}_{t\to\infty}\langle\psi_t, \gamma\psi_t\rangle - \underline{\lim}_{t\to-\infty}\langle\psi_t, \gamma\psi_t\rangle \\ &\le 2\sup_{t\in\mathbf{R}}|\langle\psi_t, \gamma\psi_t\rangle| \\ &\le 2\|\psi_t\|_{L^2}\|\psi_t\|_{\tilde{\mathcal{H}}} \le 2\|\psi_0\|_{L^2}\|\psi_0\|_{\tilde{\mathcal{H}}}, \end{aligned}$$

by Proposition 5.2*(i)*. □

Proof of Proposition 5.2. We first note that $\tilde{g}$ is bounded if $\sigma > 1/2$. Hence:

$$\|\gamma\psi_t\|_{L^2(\mathbf{R};dr_*)} \leq C\|\psi_t\|_{H^1(\mathbf{R};dr_*)},$$

which implies *(i)*.

To prove *(ii)*, it suffices to verify that:

$$\langle\psi, i[\tilde{H},\gamma]\psi\rangle \geq c_1 \int_{-\infty}^{\infty} (1+{r_*}^2)^{-\sigma-1}|\psi|^2\, dr_*, \tag{5.5}$$

$$\langle\psi, i[\lambda r^{-p+1}|\psi|^{p-1},\gamma]\psi\rangle \geq c_{2,R}\int_{-R}^{R} r^{-p-1}|\psi|^{p+1}dr_*. \tag{5.6}$$

The proof of (5.6) is similar to that of Proposition 4.1. We have:

$$i[\Psi(r_*),\gamma] = -\tilde{g}\frac{\partial}{\partial r_*}\Psi(r_*).$$

Putting $\Psi(r_*) = \lambda r^{-p+1}|\psi|^{p-1}$, we obtain:

$$\begin{aligned}
&\textstyle\int_{-\infty}^{\infty} \bar{\psi}\cdot[\lambda r^{-p+1}|\psi|^{p-1}, i\gamma]\psi\, dr_* \\
&= -\textstyle\int_{-\infty}^{\infty} |\psi|^2\tilde{g}(r_*)\frac{\partial}{\partial r_*}(\lambda r^{-p+1}|\psi|^{p-1})dr_* \\
&= -\textstyle\int_{-\infty}^{\infty} \tilde{g}(r_*)\frac{\lambda(p-1)}{p+1}r^2\frac{\partial}{\partial r_*}(r^{-p-1}|\psi|^{p+1})dr_* \\
&= \textstyle\frac{\lambda(p-1)}{p+1}\int_{-\infty}^{\infty}\frac{\partial}{\partial r_*}(r^2\tilde{g}(r_*))\cdot(r^{-p-1}|\psi|^{p+1})dr_*,
\end{aligned}$$

where we used (4.3) and, at the last step, integrated by parts. We now use that for any $R > 0$ there is an $\epsilon > 0$ such that

$$\frac{\partial}{\partial r_*}\Big(r^2\tilde{g}(r_*)\Big) \geq \epsilon$$

for $-R \leq r_* \leq R$, and conclude that the last integral is

$$\geq \epsilon\int_{-R}^{R} r^{-p-1}|\psi|^{p+1}dr_*.$$

The proof of (5.5) essentially follows [9]. We compute:

$$i[D_{r_*}^2,\gamma] = 2D_{r_*}\tilde{g}D_{r_*} - \frac{1}{2}\tilde{g}''', \tag{5.7}$$

$$i[V,\gamma] = -\tilde{g}\frac{d}{dr_*}V(r_*). \tag{5.8}$$

Since $\tilde{g}(r_*) > 0$ for $r_* > \alpha$ and $\tilde{g}(r_*) < 0$ for $r_* < \alpha$, and the opposite inequalities hold for $V'(r_*)$ (see Section 4), the term (5.8) is nonnegative. It remains to prove that:

$$\int \bar{\psi} \cdot i[D_{r_*}^2, \gamma]\psi dr_* \geq \int (1 + {r_*}^2)^{-\sigma-1} |\psi|^2 dr_*. \tag{5.9}$$

We first define the unitary transformation:

$$S: \ L^2(\mathbf{R}; dr_*) \to L^2(\mathbf{R}; s^2 ds),$$
$$\psi(r_*) \to s^{-1}\psi(s + \alpha) =: \phi(s).$$

Then:

$$\begin{aligned} Si[D_{r_*}^2, \gamma]S^* &= -2s^{-1}\tfrac{d}{ds}\, g'(s)\tfrac{d}{ds}\, s - \tfrac{1}{2}g'''(s) \\ &= -2\tfrac{d}{ds}\, g'\tfrac{d}{ds} - \tfrac{4}{s}g'\tfrac{d}{ds} - \tfrac{2}{s}g'' - \tfrac{1}{2}g'''. \end{aligned} \tag{5.10}$$

Since

$$\int \bar{\psi} \cdot i[D_{r_*}^2, \gamma]\psi \, dr_* = \int \bar{\phi} \cdot Si[D_{r_*}^2, \gamma]S^*\phi \, s^2 ds,$$

(5.9) is equivalent to

$$\int \bar{\phi} \cdot Si[D_{r_*}^2, \gamma]S^*\phi \, s^2 ds \geq c_1 \int (1 + {r_*}^2)^{-\sigma-1} |\phi|^2 s^2 ds. \tag{5.11}$$

It therefore suffices to prove (5.11) for $\phi \in L^2(\mathbf{R}; s^2 ds)$.

We first prove that the operator

$$L = -\frac{d}{ds}g'\frac{d}{ds} - \frac{2}{s}g'\frac{d}{ds} \tag{5.12}$$

is nonnegative on $L^2(\mathbf{R}; s^2 ds)$. Writing $\int_{-\infty}^{\infty} \bar{\phi} L\phi \, s^2 ds = \int_{-\infty}^{0} + \int_0^{\infty}$, and changing variables $s \to -s$ in the integral $\int_{-\infty}^{0}$, we see that it suffices to prove that

$$\int_0^{\infty} \bar{\phi} L\phi \, s^2 ds > 0 \tag{5.13}$$

for $\phi \in \mathcal{H}_1 := L^2([0, \infty); s^2 ds)$. (We denote here by L the operator defined in (5.12) acting on $\mathcal{H}_1$.) To this end, we observe that $\mathcal{H}_1$ can be identified with the subspace $\mathcal{H}_2$ of $L^2(\mathbf{R}^3; d^3x)$ consisting of spherically symmetric functions. Namely, we introduce spherical coordinates (s, ω) in $\mathbf{R}^3$ so that $s^2 = x_1^2 + x_2^2 + x_3^2$, $d^3x = s^2 ds$. Then the operator

$$T: \mathcal{H}_1 \to \mathcal{H}_2,$$
$$\phi(s) \to \tilde{\phi}(s, \omega) = \phi(s),$$

is unitary. Under this identification, L becomes an operator on $\mathcal{H}_2$ equal to TLT^*. However, an explicit computation shows that

$$TLT^* = \sum_{i=1}^{3} D_{x_i} \frac{x_i}{s} g'(s) \frac{x_i}{s} D_{x_i},$$

which is obviously nonnegative since $g' = (1+s^2)^{-\sigma} > 0$. Hence L is nonnegative.

To finish the proof of (5.11), it remains to check that

$$-\frac{2}{s}g'' - \frac{1}{2}g''' \geq c_0(1+s^2)^{-\sigma-1} \tag{5.14}$$

for some $c_0 > 0$. However, by an explicit computation the left-hand side of (5.14) is equal to

$$\sigma(1+s^2)^{-\sigma-2}(5+(3-2\sigma)s^2),$$

so that (5.14) holds if $0 < \sigma < 3/2$. □

6 Pseudoconformal identity

Let $\mathcal{X} = \{\phi \in \tilde{\mathcal{H}} : \ \|\phi\|_{\mathcal{X}} < \infty\}$, where:

$$\|\phi\|_{\mathcal{X}} = \|\phi\|_{\tilde{\mathcal{H}}} + \|r_*\phi\|_{L^2(\mathbf{R};dr_*)}.$$

We continue to denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbf{R};dr_*)$.

Proposition 6.1 *Assume that $p > 3$, and let $\epsilon > 0$. Let ψ_t be a solution of (3.3) such that $\psi_1 \in \mathcal{X}$. Then:*

$$\int_1^T \langle \psi_t, \Big(\frac{r_*}{2t} - D_{r_*}\Big)^2 \psi_t\rangle dt < CT^{\epsilon}(\|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}); \tag{6.1}$$

$$\int_1^T \int_{-\infty}^{\infty} r^{-p+1}|\psi_t|^{p+1} dr_* dt < CT^{\epsilon}(\|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}); \tag{6.2}$$

and, for $t > 1$,

$$\langle \psi_t, \Big(\frac{r_*}{2t} - D_{r_*}\Big)^2 \psi_t\rangle < Ct^{-1+\epsilon}(\|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}); \tag{6.3}$$

$$\int_{-\infty}^{\infty} r^{-p+1}|\psi_t|^{p+1} dr_* < Ct^{-1+\epsilon}(\|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}); \tag{6.4}$$

$$\langle \psi_t, V(r_*)\psi_t\rangle < Ct^{-1+\epsilon}(\|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}); \tag{6.5}$$

the constants in (6.1)-(6.5) may depend on λ, p, ϵ but are independent of T, t.

Proof. Throughout this proof we will denote by C a constant which may depend on p, λ, ϵ and may change from line to line, but is always independent of ψ, t, T.

Let

$$\Phi(t) = \Phi_0(t) + t\Psi(t),$$

where

$$\Phi_0(t) = t\Big(\Big(\frac{r_*}{2t} - D_{r_*}\Big)^2 + V\Big), \ \Psi(t) = \lambda r^{-p+1}|\psi_t|^{p-1}.$$

Observe that:

$$0 \leq \langle \psi_1, \Phi_0(1)\psi_1 \rangle \leq C\|\psi_1\|_{\mathcal{X}}^2, \tag{6.6}$$

$$0 \leq \langle \psi_1, \Psi(1)\psi_1 \rangle \leq C\|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}. \tag{6.7}$$

Indeed, (6.6) is obvious from the definition of Φ_0 and $\mathcal{X}$. To prove (6.7), we will use that in dimension 1:

$$\|\psi\|_\infty \leq C\|\psi\|_2^{1/2}\|D_{r_*}\psi\|_2^{1/2}, \tag{6.8}$$

and that r^{-1} is bounded, hence:

$$\langle \psi_1, \Psi_1\psi_1 \rangle \leq C\int_{-\infty}^{\infty} |\psi_1|^{p+1} dr_* \leq C\|\psi_1\|_\infty^{p-1} \int_{-\infty}^{\infty} |\psi_1|^2 dr_* \leq C\|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}.$$

The main idea of the proof of Proposition 6.1 is to estimate

$$\langle \psi_T, \Phi(T)\psi_T \rangle - \langle \psi_0, \Phi(0)\psi_0 \rangle = \int_1^T \frac{d}{dt}\langle \psi_t, \Phi(t)\psi_t \rangle dt$$

from below. We have $\frac{d}{dt}\langle \psi_t, \Phi(t)\psi_t \rangle = \langle \psi_t, D\Phi(t)\psi_t \rangle$, where

$$D\Phi(t) = \frac{\partial}{\partial t}\Phi(t) + i[\tilde{H}_\psi, \Phi(t)].$$

We will also write

$$D_0\Phi_0(t) = \frac{\partial}{\partial t}\Phi_0(t) + i[\tilde{H}, \Phi_0(t)]$$

We compute:

$$\begin{aligned}
D\Phi(t) &= \tfrac{\partial}{\partial t}(\Phi_0(t) + t\Psi) + i[\tilde{H} + \Psi, \Phi_0(t) + t\Psi] \\
&= D_0\Phi_0(t) + \tfrac{\partial}{\partial t}(t\Psi) + i[\tilde{H}, t\Psi] + i[\Psi, \Phi_0] \\
&= D_0\Phi_0(t) + \tfrac{\partial}{\partial t}(t\Psi) + i[D_{r_*}^2, t\Psi] + i[\Psi, t\left(\tfrac{r_*}{2t} - D_{r_*}\right)^2] \\
&= D_0\Phi_0(t) + \tfrac{\partial}{\partial t}(t\Psi) + i[\Psi, -\tfrac{1}{2}(r_* D_{r_*} + D_{r_*} r_*)] \\
&= D_0\Phi_0(t) + \Psi + t\tfrac{\partial}{\partial t}\Psi + r_*\tfrac{\partial}{\partial r_*}\Psi(r_*).
\end{aligned} \tag{6.9}$$

We first estimate the nonlinear terms, beginning with $\langle \psi_t, r_*\frac{\partial}{\partial r_*}\Psi(r_*)\psi_t \rangle$:

$$\begin{aligned}
&\textstyle\int_{-\infty}^{\infty} |\psi_t|^2 r_*\frac{\partial}{\partial r_*}\Psi dr_* = \int_{-\infty}^{\infty} \frac{\lambda(p-1)}{p+1} r^2 r_*\frac{\partial}{\partial r_*}(r^{-p-1}|\psi_t|^{p+1}) dr_* \\
&= -\tfrac{\lambda(p-1)}{p+1}\textstyle\int_{-\infty}^{\infty} \frac{\partial}{\partial r_*}(r^2 r_*)\, r^{-p-1}|\psi_t|^{p+1} dr_* \\
&= -\tfrac{\lambda(p-1)}{p+1}\textstyle\int_{-\infty}^{\infty} (\frac{\partial}{\partial r_*}(r^2) r_* + r^2) r^{-p-1}|\psi_t|^{p+1} dr_*. \\
&= -\tfrac{\lambda(p-1)}{p+1}\Big(\textstyle\int_{-\infty}^{-R} + \int_{-R}^{R} + \int_R^{\infty}\Big)\frac{\partial}{\partial r_*}(r^2) r_* r^{-p-1}|\psi_t|^{p+1} dr_* \\
&-\tfrac{\lambda(p-1)}{p+1}\textstyle\int_{-\infty}^{\infty} r^{-p+1}|\psi_t|^{p+1} dr_*.
\end{aligned} \tag{6.10}$$

By (2.1), for any $\delta > 0$ there is $R > 0$ such that $|\frac{\partial}{\partial r_*}(r^2)r_* r^{-2}| < \lambda\delta$ for $r_* < -R$. Hence

$$\int_{-\infty}^{-R} |\frac{\partial}{\partial r_*}(r^2)r_*| r^{-p-1}|\psi_t|^{p+1} dr_* \leq \lambda\delta \int_{-\infty}^{\infty} r^{-p+1}|\psi_t|^{p+1} dr_*, \tag{6.11}$$

provided that R is large enough. Next, $\frac{\partial}{\partial r_*}(r^2)r_*$ is bounded for $-R \leq r_* \leq R$, so that:

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} dt \int_{-R}^{R} |\frac{\partial}{\partial r_*}(r^2)r_*| r^{-p-1}|\psi_t|^{p+1} dr_* \\ &\leq C \textstyle\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} r^{-p-1}|\psi_t|^{p+1} dr_* \leq C\|\psi_1\|_{\mathcal{X}}^2, \end{aligned} \tag{6.12}$$

by (5.1); the constant C depends on R and hence on δ, but not on T. Finally,

$$\int_R^{\infty} \frac{\partial}{\partial r_*}(r^2)r_* r^{-p-1}|\psi_t|^{p+1} dr_* \geq 0. \tag{6.13}$$

Plugging (6.11)–(6.13) into (6.10), we obtain:

$$\begin{aligned} &\textstyle\int_1^T \langle \psi_t, r_* \frac{\partial}{\partial r_*}\Psi(t)\psi_t\rangle dt \\ &\leq -\lambda\Big(\frac{p-1}{p+1} - \delta\Big) \textstyle\int_1^T dt \int r^{-p+1}|\psi_t|^{p+1} dr_* + C\|\psi_1\|_{\mathcal{X}}^2 \\ &= -(\frac{p-1}{p+1} - \delta) \textstyle\int_1^T \langle \psi_t, \Psi(t)\psi_t\rangle + C\|\psi_1\|_{\mathcal{X}}^2. \end{aligned} \tag{6.14}$$

Next, we have

$$\begin{aligned} &\textstyle\int_1^T t\langle \psi_t, \frac{\partial\Psi}{\partial t}\psi_t\rangle dt = \lambda \int_1^T t \int_{-\infty}^{\infty} |\psi_t|^2 \frac{\partial}{\partial t}(r^{-p+1}|\psi_t|^{p-1}) dr_* dt \\ &= \textstyle\frac{\lambda(p-1)}{p+1} \int_1^T t \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(r^{-p+1}|\psi_t|^{p+1}) dr_* dt \\ &= \textstyle\frac{p-1}{p+1} \int_1^T t \frac{\partial}{\partial t}\langle \psi_t, \Psi(t)\psi_t\rangle dt \\ &= \textstyle\frac{p-1}{p+1}\Big(T\langle \psi_T, \Psi(T)\psi_T\rangle - \langle \psi_1, \Psi(1)\psi_1\rangle - \int_1^T \langle \psi_t, \Psi(t)\psi_t\rangle\Big) dt. \end{aligned} \tag{6.15}$$

Combining (6.14) and (6.15), we obtain:

$$\begin{aligned} &\textstyle\int_1^T \langle \psi_t, \Big(\Psi(t) + t\frac{\partial}{\partial t}\Psi(t) + r_* \frac{\partial}{\partial r_*}\Psi(t)\Big)\psi_t\rangle \, dt \\ &\leq \textstyle\frac{p-1}{p+1}\Big(T\langle \psi_T, \Psi(T)\psi_T\rangle - \langle \psi_1, \Psi(1)\psi_1\rangle\Big) - c_1 \int_1^T \langle \psi_t, \Psi(t)\psi_t\rangle dt + C\|\psi_1\|_{\mathcal{X}}^2, \end{aligned} \tag{6.16}$$

where $c_1 = \frac{p-3}{p+1} - \delta$ satisfies $c_1 > 0$ if $\delta > 0$ was chosen small enough.

It remains to estimate:

$$D_0\Phi_0 = -\Big(\frac{r_*}{2t} - D_{r_*}\Big)^2 + W(r_*),$$

where

$$W(r_*) = V(r_*) + r_* \frac{\partial}{\partial r_*}V(r_*) = \frac{2M}{r^3}\Big(1 - \frac{2M}{r}\Big) + \frac{2Mr_*}{r^4}\Big(1 - \frac{2M}{r}\Big)\Big(\frac{8M}{r} - 3\Big).$$

We have:

$$|W(r_*)| \leq C(1+r_*^2)^{-3/2},$$

which is almost – but not quite – sufficient for the local decay estimate (5.1) to be applicable. To remedy this, we write:

$$\int_1^T \langle \psi_t, W(r_*)\psi_t\rangle dt \leq C\int_1^T \langle \psi_t, (1+r_*^2)^{-3/2}\,\psi_t\rangle dt \leq I_1 + I_2,$$

where:

$$I_1 = C\int_1^T \langle \psi_t, \chi(1+r_*^2)^{-3/2}\,\psi_t\rangle dt,$$

$$I_2 = C\int_1^T \langle \psi_t, (1-\chi)(1+r_*^2)^{-3/2}\,\psi_t\rangle dt,$$

and $\chi(r_*,t)$ is a bounded C^∞ function such that $\chi \equiv 1$ for $1+r_*^2 \leq t$, $t \geq 1$, and $\chi \equiv 0$ for $1+r_*^2 \geq 2t$, $t \geq 1$. To estimate I_2, we use that $(1+r_*^2)^{-3/2} \leq t^{-3/2}$ on $\operatorname{supp}\chi$, hence:

$$I_2 \leq C\int_1^T t^{-3/2}\|\psi_t\|_2^2\, dt \leq C\|\psi_1\|_{\mathcal{X}}^2,$$

where we also used that the L^2 norm of ψ_t is constant. It remains to estimate I_1. We have for any $\epsilon > 0$:

$$\chi(1+r_*^2)^{-\frac{3}{2}} = \chi(1+r_*^2)^{\epsilon}\,(1+r_*^2)^{-\frac{3}{2}-\epsilon} \leq Ct^{\epsilon}(1+r_*^2)^{-\frac{3}{2}-\epsilon} \leq CT^{\epsilon}(1+r_*^2)^{-\frac{3}{2}-\epsilon}$$

for $1 \leq t \leq T$. Hence using (5.1) we obtain:

$$I_2 \leq CT^{\epsilon}\int_1^T \langle \psi_t, (1+r_*^2)^{-3/2-\epsilon}\psi_t\rangle dt \leq CT^{\epsilon}\|\psi_1\|_{\mathcal{X}}^2.$$

Thus:

$$\int_1^T \langle \psi_t, D_0\Phi_0\psi_t\rangle dt \leq -\int_1^T \langle \psi_t, \Big(\frac{r_*}{2t} - D_{r_*}\Big)^2\psi_t\rangle dt + CT^{\epsilon}\|\psi_1\|_{\mathcal{X}}^2. \tag{6.17}$$

Combining (6.17) and (6.16), we find that:

$$\begin{aligned}
&\langle \psi_T, (\Phi_0(T) + T\Psi(T))\psi_T\rangle - \langle \psi_1, (\Phi_0(1)+\Psi(1))\psi_1\rangle \\
&= \textstyle\int_1^T \langle \psi_t, D\Phi(t)\psi_t\rangle dt \\
&\leq -\textstyle\int_1^T \langle \psi_t, \Big(\frac{r_*}{2t} - D_{r_*}\Big)^2\psi_t\rangle dt - c_1\int_1^T \langle \psi_t, \Psi(t)\psi_t\rangle dt \\
&+\tfrac{p-1}{p+1}\Big(T\langle \psi_T, \Psi(T)\psi_T\rangle - \langle \psi_1, \Psi(1)\psi_1\rangle\Big) + CT^{\epsilon}\|\psi_1\|_{\mathcal{X}}^2.
\end{aligned} \tag{6.18}$$

Rearranging (6.18) and using (6.6)–(6.7), we finally obtain:

$$\begin{aligned}
&\langle \psi_T, \Phi_0(T)\psi_T\rangle + \tfrac{2}{p+1}T\langle \psi_T, \Psi(T)\psi_T\rangle \\
&\leq -\textstyle\int_1^T \langle \psi_t, \Big(\frac{r_*}{2t} - D_{r_*}\Big)^2\psi_t\rangle dt - c_1\int_1^T \int r^{-p+1}|\psi_t|^{p+1}\, dr_* dt \\
&+C\Big(T^{\epsilon}\|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}\Big).
\end{aligned} \tag{6.19}$$

Note first that the integrands in the integrals

$$\begin{aligned} &\int_1^T \langle \psi_t, \left(\tfrac{r_*}{2t} - D_{r_*}\right)^2 \psi_t \rangle dt, \\ &\int_1^T \int r^{-p+1} |\psi_t|^{p+1} dr_* dt \end{aligned} \tag{6.20}$$

are positive. Hence the left-hand side of (6.19) is bounded from above, uniformly in T, by

$$C\left(T^\epsilon \|\psi_1\|_{\mathcal{X}}^2 + \|\psi_1\|_{\tilde{\mathcal{H}}}^{p+1}\right).$$

However, it is also trivially bounded from below by 0, since $\Phi_0(T)$ and $\Psi(T)$ are nonnegative for all T. This yields (6.3)–(6.5). Finally, to get (6.1)–(6.2) we plug (6.3)–(6.5) back into (6.19). This ends the proof of the proposition. □

7 Global existence and scattering theory

In this section we prove our main L^∞ estimate on ψ_t (Proposition 7.1). Interpolating this estimate with the conservation of the L^2 norm yields L^p estimates, which will be an essential part of our proof of asymptotic completeness. Recall that the space $\mathcal{X}$ was defined at the beginning of Section 6.

Proposition 7.1 *Let ψ_t solve (3.3), $\psi_0 \in \mathcal{X}$. Assume that $\lambda > 0$ and $p > 3$. Then for any $\epsilon > 0$:*

$$\|\psi_t\|_\infty \le C_\epsilon t^{-\frac{1}{4}+\epsilon} \|\psi_1\|_2^{1/2} \left(\|\psi\|_{\mathcal{X}}^2 + \|\psi\|_{\tilde{\mathcal{H}}}^{p+1}\right)^{1/4}. \tag{7.1}$$

(This in particular implies global existence in L^∞.)

Proof. ¿From (6.8) we have:

$$\begin{aligned} \|\psi_t\|_\infty &= \|e^{ix^2/4t}\psi_t\|_\infty \le C\|\psi_t\|_2^{1/2} \|D_{r_*} e^{ix^2/4t}\psi_t\|_2^{1/2} \\ &= C\|\psi_t\|_2^{1/2} \|(\tfrac{x}{2t} - D_{r_*})\psi_t\|_2^{1/2} \\ &\le Ct^{-\frac{1}{4}+\epsilon} \|\psi_0\|_2^{1/2} \left(\|\psi\|_{\mathcal{X}}^2 + \|\psi\|_{\tilde{\mathcal{H}}}^{p+1}\right)^{1/4}. \end{aligned}$$

where we have used that, by (6.3),

$$\|(\frac{x}{2t} - D_{r_*})\psi_t\|_2 \le Ct^{-\frac{1}{2}+\epsilon} \left(\|\psi\|_{\mathcal{X}}^2 + \|\psi\|_{\tilde{\mathcal{H}}}^{p+1}\right)^{1/2}.$$

This proves (7.1). □

The main result of this section is the following theorem, in which we compare the nonlinear dynamics associated with the equation (3.3) to the linear evolution e^{-itH}. We will state all of our results on scattering for the case $t \to \infty$; for $t \to -\infty$ analogous results obviously hold.

Theorem 7.2 *(i) (Existence of wave operators) Assume that* $p > \frac{1}{2}(3+\sqrt{17})(\approx 3.56)$. *Then for any* $\psi_+ \in H^1(\mathbf{R}) \cap L^{q'}(\mathbf{R})$, *where* $q' = 1 + \frac{1}{p}$, *there is a* $\psi_0 \in H^1(\mathbf{R})$ *such that if* ψ_t *is the solution of (3.3) with initial condition* ψ_0 *at* $t = 0$, *then:*

$$\|e^{-it\tilde{H}}\psi_+ - \psi_t\|_{L^2(\mathbf{R})} \to 0 \text{ as } t \to \infty. \tag{7.2}$$

(ii) (Asymptotic completeness) Assume that $p > 4$, *and let* $\psi_0 \in \mathcal{X}$. *Let* ψ_t *be the solution of (3.3) with initial condition* ψ_0 *at* $t = 0$. *Then there is a* $\psi_+ \in L^2(\mathbf{R})$ *such that (7.2) is satisfied.*

Observe that by Hölder's inequality:

$$\int |\psi|^{q'} dr_* \le \Big(\int |\psi|^2 (r_*{}^2 + 1)\, dr_*\Big)^{q'/2} \Big(\int (r_*{}^2+1)^{-\frac{q'}{2-q'}} dr_*\Big)^{1-\frac{q'}{2}};$$

the last integral is convergent since $\frac{2q'}{2-q'} > 1$. Hence $\mathcal{X} \subset L^{q'}(\mathbf{R}) \cap H^1(\mathbf{R})$, and *(i)* holds in particular for $\psi_+ \in \mathcal{X}$.

Proof of Theorem 7.2. To prove the existence of wave operators (part *(i)*), we need to solve the integral equation

$$\psi_t = e^{-it\tilde{H}}\psi_+ - i\lambda \int_t^\infty e^{-i(t-s)\tilde{H}} r^{-p+1} |\psi_s|^{p-1}\psi_s\, ds \tag{7.3}$$

for a given $\psi_+ \in H^1(\mathbf{R}) \cap L^2(\mathbf{R}; r_* dr_*)$. Let

$$\mathcal{F}(\phi) = \int_t^\infty e^{-i(t-s)\tilde{H}} r^{-p+1} |\phi_s|^{p-1}\phi_s\, ds. \tag{7.4}$$

Let $X_T = L^k([T,\infty); L^q(\mathbf{R}))$ for suitable k, q which will be chosen later. We first prove that

$$\|\mathcal{F}(\phi)\|_{X_T} \le C_0 \|\phi\|_{X_T}^p, \tag{7.5}$$

with C_0 independent of ϕ and T.

We will use the L^q estimates for the Schrödinger unitary group e^{-itH} in dimension 1, proved recently by Weder [15]: if $H = D^2 + V(x)$ is a Schrödinger operator on $L^2(\mathbf{R})$ with the potential $V(x)$ satisfying $\int |V(x)|(1+|x|)^\gamma dx < \infty$ for some $\gamma > 5/2$ (which clearly holds for V given by (3.2), then:

$$\|e^{-itH} P_c\|_{B(L^{q'}, L^q)} \le C t^{-(\frac{1}{2}-\frac{1}{q})} \tag{7.6}$$

for $1 \le q' \le 2$, $\frac{1}{q} + \frac{1}{q'} = 1$. P_c is the projection on the continuous spectral subspace of H; for $H = \tilde{H}$, it follows *e.g.*, from Proposition 4.1 that $\tilde{H}$ has no point spectrum and therefore $P_c = \mathbf{1}$.

Using (7.6) and the fact that r^{-1} is bounded, we estimate:

$$\begin{aligned}
\|\mathcal{F}(\phi)\|_{X_T} &= \Big\| \| \textstyle\int_t^\infty e^{-i(t-s)H_T} r^{-p+1} |\phi_s|^{p-1}\psi_s \, ds \|_{L^q(dr_*)} \Big\|_{L^k(dt)} \\
&\le \Big\| \textstyle\int_t^\infty |t-s|^{-(\frac{1}{2}-\frac{1}{q})} \|r^{-p+1}\phi_s^p\|_{L^{q'}(dr_*)} ds \Big\|_{L^k(dt)} \\
&\le \Big\| \textstyle\int_t^\infty |t-s|^{-(\frac{1}{2}-\frac{1}{q})} \|\phi_s\|^p_{L^{pq'}(dr_*)} ds \Big\|_{L^k(dt)} \\
&\le C_0 \Big\| \|\phi_s\|_{L^{pq'}(dr_*)} \Big\|^p_{L^{p\eta}(dt)},
\end{aligned} \tag{7.7}$$

where:

$$1 + \frac{1}{k} = \frac{1}{\kappa} + \frac{1}{\eta}, \ \frac{1}{q} + \frac{1}{q'} = 1, \ \Big(\frac{1}{2} - \frac{1}{q}\Big)\kappa = 1 \tag{7.8}$$

(at the last step we used the generalized Young's inequality). The double norm in the last line of (7.7) is equal to $\|\phi\|^p_{X_T}$ if

$$p\eta = k, \ pq' = q. \tag{7.9}$$

Solving (7.8)–(7.9), we obtain:

$$q = p+1, \ \kappa = \frac{2(p+1)}{p-1}, \ k = \frac{2(p-1)(p+1)}{p+3}. \tag{7.10}$$

Hence, if q, k are chosen as in (7.10), the mapping $\mathcal{F}$ is a contraction on the ball $B_{\epsilon,T} = \{\|\phi\|_{X_T} \le \epsilon\}$, where ϵ depends on p but not on T.

Using (7.6), we obtain that for $\psi_+ \in L^2(\mathbf{R}) \cap L^{q'}(\mathbf{R})$,

$$\| \| e^{-it\tilde{H}} \psi_+ \|_{L^q(dr_*)} \|_{L^k([T,\infty);dt)} \to 0 \text{ as } T \to \infty,$$

provided that $(\frac{1}{2} - \frac{1}{q})k > 1$. For q and k as in (7.10), this condition is satisfied when:

$$p > \frac{3+\sqrt{17}}{2} \approx 3.56.$$

Therefore, given a $\psi_+ \in H^1(\mathbf{R}) \cap L^{q'}(\mathbf{R})$, we may choose T large enough so that $e^{-it\tilde{H}}\psi_+ \in B_{\epsilon/3,T}$. Using a standard contraction argument, we can now solve (by iteration) the equation (7.3) for $t \ge T$. The solution $\psi_t(r_*), t \ge T$, belongs to X_T and solves (in the weak sense) the differential equation (3.3) with the initial condition $\psi_T = e^{-iT\tilde{H}}\psi_+ \in H^1(\mathbf{R})$. By conservation of energy, $\psi_t \in H^1(\mathbf{R})$ for $t \ge T$. Finally, we extend ψ_t to all $t \in \mathbf{R}$ by solving (3.3) backwards (*i.e.*, for $t < T$) with the initial data as above at $t = T$. We obtain a solution ψ_t with

$$\|\psi_t\|_{H^1(\mathbf{R})} \le C \|e^{-it_0\tilde{H}}\psi_+\|_{H^1(\mathbf{R})} \le C\|\psi_+\|_{H^1(\mathbf{R})}$$

for all $t \in \mathbf{R}$.

Next, we claim that

$$\|e^{it\tilde{H}}\psi_t - \psi_+\|_{L^q(dr_*)} \to 0 \text{ as } t \to \infty. \tag{7.11}$$

Indeed, multiplying both sides of (7.3) by $e^{it\tilde{H}}$ and then proceeding as in the proof of (7.5), we obtain for $t > T$:

$$\begin{aligned} &\|e^{it\tilde{H}}\psi_t - \psi_+\|_{L^q(dr_*)} = \lambda \| \textstyle\int_t^\infty e^{-is\tilde{H}} r^{-p+1} |\psi_s|^{p-1}\psi_s ds\|_{L^q(dr_*)} \\ &\le \lambda \textstyle\int_t^\infty s^{-(\frac{1}{2}-\frac{1}{q})} \|\psi_s\|^p_{L^{pq'}(dr_*)} \\ &\le \lambda \Big(\textstyle\int_t^\infty s^{-(\frac{1}{2}-\frac{1}{q})\xi} ds\Big)^{1/\xi} \Big(\textstyle\int_t^\infty \|\psi_s\|^{p\eta}_{L^{pq'}(dr_*)} ds\Big)^{1/\eta}, \end{aligned} \tag{7.12}$$

for $\frac{1}{\eta} + \frac{1}{\xi} = 1$. The last line in (7.12) is bounded by $C_1(t)\|\psi\|^p_{X_{t_0}}$, with $C_1(t) \to 0$ as $t \to \infty$, if

$$(\frac{1}{2} - \frac{1}{q})\xi > 1, \ p\eta = k, \ pq' = q.$$

Again, it is easy to verify that one can choose ξ, η so that these conditions are satisfied.

By (7.11), $e^{it\tilde{H}}\psi_t$ converges to ψ_+ strongly in $L^q(\mathbf{R})$; moreover, we saw earlier that the H^1 norms of $e^{it\tilde{H}}\psi_t$ are bounded uniformly for all t. Hence $e^{it\tilde{H}}\psi_t$ has a weak limit in L^2. Since the L^2 norm of $e^{it\tilde{H}}\psi_t$ is constant in t, the L^2 convergence is strong. The L^q limit, ψ_+, belongs to L^2, hence the L^2 limit must also be equal to ψ_+.

We obtain that:

$$\|\psi_t - e^{-it\tilde{H}}\psi_+\|_{L^2(dr_*)} = \|e^{it\tilde{H}}\psi_t - \psi_+\|_{L^2(dr_*)} \to 0,$$

which proves Theorem 7.2*(i)*.

To prove *(ii)*, *i.e.*, the completeness of wave operators, we consider the integral equation

$$e^{it\tilde{H}}\psi_t = \psi_0 - i\lambda \int_0^t e^{is\tilde{H}} r^{-p+1} |\psi_s|^{p-1}\psi_s \, ds, \tag{7.13}$$

for $\psi_0 \in \mathcal{X}$. The equation (7.13) is equivalent to (3.3) with initial condition ψ_0 at $t = 0$.

We will prove that the integral

$$\int_0^\infty e^{is\tilde{H}} r^{-p+1} |\psi_s|^{p-1}\psi_s \, ds \tag{7.14}$$

is norm convergent in $L^q(\mathbf{R})$ for some $2 < q < \infty$ (depending on p). Indeed, by (7.6) we have:

$$\int_0^t \|e^{is\tilde{H}} r^{-p+1} |\psi_s|^{p-1}\psi_s\|_{L^2(\mathbf{R})} ds \le \int_0^t s^{-(\frac{1}{2}-\frac{1}{q})} \|\psi_s\|^p_{L^{q'p}(\mathbf{R})} ds,$$

for $\frac{1}{q} + \frac{1}{q'} = 1$, $q > 2$. The last integral can be broken up into $\int_0^1 + \int_1^\infty$. Since $p > 4$ and $q' \ge 1$, $q'p > 4$, so that by Sobolev's inequality:

$$\|\psi_s\|_{L^{q'p}} < C\|\psi_s\|_{H^1(\mathbf{R})} = C\|\psi_0\|_{H^1(\mathbf{R})}.$$

Therefore the integral $\int_0^1$ is bounded by:

$$C\|\psi_0\|_{H^1(\mathbf{R})} \int_0^t s^{-(\frac{1}{2}-\frac{1}{q})} ds \le C'\|\psi_0\|_{H^1(\mathbf{R})},$$

since $\frac{1}{2} - \frac{1}{q} < 1$.

To prove that $\int_1^t$ is finite, it suffices to verify that

$$\|\psi_s\|_{L^{q'p}(\mathbf{R})} \leq C(\psi_0)\, s^{-\alpha} \tag{7.15}$$

for some α satisfying $\frac{1}{2} - \frac{1}{q} + \alpha p > 1$, *i.e.*,

$$\frac{1}{q'} + \alpha p > \frac{3}{2}. \tag{7.16}$$

We obtain (7.15) by interpolating between (7.1) and the conservation of the L^2 norm: $\|\psi_s\|_{L^2} = \|\psi_0\|_{L^2}$. Such interpolation yields (7.15) for

$$\alpha < \frac{1}{4}(1-\theta) = \frac{q'p-2}{4q'p}. \tag{7.17}$$

We may find an α satisfying both (7.17) and (7.16) if $q'p + 2 > 6q'$. It therefore suffices to take $1 < q' < \frac{2}{6-p}$ if $4 < p < 5$ and $1 < q' < 2$ if $p \geq 5$.

Let $\phi_+ \in L^q(\mathbf{R})$ be the L^q limit of (7.14). Then (7.13) implies that:

$$\|e^{it\tilde{H}}\psi_t - \psi_0 + i\lambda\phi_+\|_{L^q(\mathbf{R})} \to 0 \text{ as } t \to \infty.$$

The same argument as in the proof of *(i)* proves now that the convergence takes place also in L^2. Let $\psi_+ = \psi_0 + i\lambda\phi_+$, then:

$$\lim_{t\to\infty} \|e^{it\tilde{H}}\psi_t - \psi_+\|_{L^2(\mathbf{R})} = \lim_{t\to\infty} \|\psi_t - e^{-it\tilde{H}}\psi_+\|_{L^2(\mathbf{R})} = 0.$$

This ends the proof of the theorem. □

Recall that $\tilde{H} = D_{r_*}^2 + V(r_*)$, where $V(r_*)$ is a short-range C^∞ potential given by (3.2). We can therefore apply the well known results on short range scattering (see *e.g.*, [3]) to the evolution $e^{-it\tilde{H}}$. As noted before, $\tilde{H}$ has no point spectrum, hence the wave operators

$$W_+ = s - \lim_{t\to\infty} e^{it\tilde{H}} e^{-itD_{r_*}^2}$$

exist and are complete (*i.e.*, are unitary on L^2). Thus for any initial condition $\psi_+ \in L^2(\mathbf{R})$ there is a $\phi_+ \in L^2(\mathbf{R})$ ($\phi_+ = W_+^*\psi_+$) such that:

$$\|e^{-it\tilde{H}}\psi_+ - e^{-itD_{r_*}^2}\phi_+\|_{L^2(\mathbf{R})} \to 0 \text{ as } t \to \infty.$$

Combining this with Theorem 7.2*(ii)*, we obtain the following corollary.

Corollary 7.3 *Assume that $p > 4$, and let ψ_t be the solution of (3.3) with the initial condition $\psi_0 \in H^1(\mathbf{R}) \cap L^2(\mathbf{R}; r_* dr_*)$ at $t = 0$. Then there is a $\phi_+ \in L^2(\mathbf{R})$ such that:*

$$\|\psi_t - e^{-itD_{r_*}^2}\phi_+\|_{L^2(\mathbf{R})} \to 0 \text{ as } t \to \infty. \tag{7.18}$$

Similarly, by Theorem 7.2*(i)* there is a subspace S of $L^2(\mathbf{R})$ (equal to $W_+^*(H^1 \cap L^{1+1/p})$) such that for any $\phi_+ \in S$ there is a $\psi_0 \in H^1(\mathbf{R})$ for which (7.18) holds.

Finally, let us reformulate Corollary 7.3 in terms of the original equation (2.1), to which (3.3) is unitarily equivalent. Recall from Section 3 that ψ_t solves (3.3) if and only if

$$u_t(r,\omega) = U\psi_t(r) = r^{-1}\psi_t(r)$$

solves (2.1). Moreover, the condition $\psi_0 \in \mathcal{X}$ is equivalent to:

$$u \in \mathcal{H},\ r_*u \in L^2(\mathbf{R} \times S^2; r^2 dr_* d\omega). \tag{7.19}$$

Corollary 7.3 states that if u_t solves (2.1) and the initial condition u_0 at $t = 0$ satisfies (7.19), then there is a $u_+ \in L^2(\mathbf{R} \times S^2; r^2 dr_* d\omega)$ such that:

$$\|Ju_t - e^{-itD_{r_*}^2} Ju_+\|_{L^2(\mathbf{R};dr_*)} \to 0 \text{ as } t \to \infty, \tag{7.20}$$

where

$$J = U^{-1}:\ L^2(\mathbf{R} \times S^2; r^2 dr_* d\omega) \to L^2(\mathbf{R}; dr_*),$$
$$(Ju)(r) = (U^{-1}u)(r) = ru(r,\omega)$$

for radially symmetric u. (7.20) may be interpreted as follows (cf. [5], [6], [1]). All solutions $e^{-itD_{r_*}^2} Ju_+$ of the free Schrödinger equation on the cylindrical manifold $\mathbf{R} \times S^2$ (with the usual metric) split up into two parts, one of which escapes to the "spatial infinity" $r_* \to \infty$, the other approaches the horizon $r_* \to -\infty$. Hence Ju_t, where u_t solves (2.1), will have similar characteristics. However, if we return to the usual coordinates on the Schwarzschild manifold, the waves approaching the horizon and the spatial infinity will begin to look differently. As $r_* \to \infty$, $r \sim r_*$ and the asymptotic dynamics generated by $J^{-1}D_{r_*}^2 J$, is similar to that for a free Schrödinger equation in $\mathbf{R}^3$. On the other hand, when $r_* \to -\infty$, $r \to 2M$ and the identification operator J is essentially a multiplication by $2M$; hence the asymptotic evolution is given by a one-dimensional Schrödinger equation. This phenomenon seems to be typical for evolution equations on Schwarzschild manifolds, cf. [1], Section 1.

References

[1] A. Bachelot: *Asymptotic completeness for the Klein-Gordon equation on the Schwarzschild metric*, Ann. Inst. Henri Poincaré **61**(1994), 411–441.

[2] A. Bachelot, J.-P. Nicolas: *Equation non linéaire de Klein-Gordon dans des metriques de type Schwarzschild*, C. R. Acad. Sci. Paris, Série 1 **316**(1993), 1047–1050.

[3] H.L. Cycon, R. Froese, W. Kirsch, B. Simon: *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer 1987.

[4] S. DeBièvre, P. D. Hislop, I. M. Sigal: *Scattering theory for the wave equation on non-compact manifolds*, Rev. Math. Phys. **4**(1992), 575–608.

[5] J. Dimock: *Scattering for the wave equation on the Schwarzschild metric*, Gen. Rel. and Grav. **17**(1985), 353–369.

[6] J. Dimock, B.S. Kay: *Classical and quantum scattering theory for linear scalar fields on the Schwarzschild metric I*, Ann. Phys. **175**(1987), 366–426.

[7] R. Froese, P. D. Hislop: *Spectral analysis of second-order elliptic operators on noncompact manifolds*, Duke Math. J. **58**(1989), 103–129.

[8] T. Kato: *Nonlinear Schrödinger equations*, in: *Schrödinger operators*, Lect. Notes Phys. 345, Springer 1989, 218–263.

[9] R. Lavine: *Scattering theory for long range potentials*, J. Funct. Anal. 5(1970), 368–382.

[10] J.-P. Nicolas: *Non linear Klein-Gordon equation on Schwarzschild-like metrics*, J. Math. Pures Appl. **74**(1995), 35–58.

[11] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, III. Scattering theory*, Academic Press, 1979.

[12] A. Sá Barreto, M. Zworski: *Distribution of resonances for spherical black holes*, preprint, 1998.

[13] W. A. Strauss: *Nonlinear Wave Equations*, AMS Publ. 73 (1989)

[14] W. T. Shu: *Spin Field Equations and Yang-Mills Equations*, Ph.D. thesis, Princeton University, 1990.

[15] R. Weder: $L^p - L^{p'}$ *estimates for the Schrödinger equation on the line*, preprint, 1998.