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Representation Theory for Q-Algebras

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Abstract. Q-algebras provide a non-boolean logical model and have been used to study quantum measurement, interference phenomena in quantum physics and the nature of the quantum probabilities. Each Q-algebra can be represented on a pre-Hilbert space, thus resulting in the standard model of quantum theory, but the representations considered by the author in recent papers involve unnecessarily large pre-Hilbert spaces (with an infinite dimension in all non-commutative cases even if the Q-algebra itself has a finite dimension).

In the present paper, an "optimal" representation is constructed. It uses a pre-Hilbert space of minimum dimension, and is unique in a certain sense. A Q-algebra of finite dimension becomes isomorphic to a finite direct sum of matrix algebras.

1 Introduction

Q-algebras have been introduced in [2]. They provide a non-boolean logical model that can be used to investigate quantum measurement, interference phenomena in quantum physics and the nature of the quantum probabilities.

An element E in a complex (associative) algebra $\mathcal{A}$ with an involution * is called an (orthogonal) *projection* if $E=E^*$ and $E^2=E$, and an *atom* is a projection E with $E\mathcal{A}E=\mathbb{C}\,E$.

A *Q-algebra* is a complex (associative) algebra $\mathcal{A}$ with unit element $\mathbb{I}$ and an involution * satisfying the following two conditions:

(a) $X\in\mathcal{A}, X^*X=0 \Rightarrow X=0$
(b) For every $0\neq X\in\mathcal{A}$ there exists an atom E with $XE\neq0$.[1]

If $\mathcal{A}$ is a Q-algebra, $X,Y\in\mathcal{A}$ and $YXY^*=\lambda YY^*$ with $\lambda\in\mathbb{C}$, X is said to be *statistically predictable under* Y; λ is called the *expectation value of* X *under* Y and is denoted by $\mathbb{E}\,(X|Y)$.

[1] This definition is equivalent to the one presented in [2], which has been shown in [3].

This is a new type of expectation value very different from the one familiar from Kolmogorovian mathematical probability theory. The interpretation of λ as an expectation value, however, is justified from quantum physics at least if Y is a projection or the product of a finite number of projections. For a projection E, $\mathbb{E}(X|E)$ is the expectation value of X after a measurement with the outcome E. For a finite number of projections $E_1,...,E_n$, $\mathbb{E}(X|E_1E_2\cdot...\cdot E_n)$ is the expectation value of X after a series of measurements with the outcomes $E_1,E_2,...,E_n$ and depends on the order of the different measurements if $E_1,E_2,...,E_n$ do not commute among each other[2].

If E is an atom in the Q-algebra $\mathcal{A}$, then every $X\in\mathcal{A}$ is statistically predictable under E and the expectation value $\mathbb{E}(X|E)$ exists for all $X\in\mathcal{A}$. Then:

- $\mathbb{E}(X|E)\in\mathbb{R}$ for $X\in\mathcal{A}$ with $X=X^*$, and
- $\mathbb{E}(X^*X|E)\geq 0$ for all $X\in\mathcal{A}$.

As shown in detail in [2], this simple purely algebraic model is already sufficient to study the quantum physical measurement process and typical quantum phenomena like indeterminism and interference, revealing the non-Boolean character of these phenomena.

Each Q-algebra has a representation on a pre-Hilbert space bringing us to the standard model of quantum physics, but the representations considered in [2,3] involve pre-Hilbert spaces that seem to be unnecessarily large. With the representation constructed in [2] the dimension of the pre-Hilbert space can not be lower than the cardinality of the set of atoms in the Q-algebra which is infinite for each non-commutative Q-algebra. Therefore the algebra of all 2×2-matrices would be represented on an infinite-dimensional space although, of course, a representation on a 2-dimensional space is possible as well.

In the present paper, we search for "optimal" representations taking the representation theory of C*-algebras as a model[1] which is possible only to a certain extent. The representation of C*-algebras are constructed using states while atoms are used with Q-algebras. The representation theory of Q-algebras turns out to be much simpler and more basic than the one of C*-algebras, and we do not need to make any use of C*-algebra theory.

2 General representations

Let $\mathcal{H}$ be a pre-Hilbert space. Then let $L(\mathcal{H})$ denote the space of those linear operators X from $\mathcal{H}$ to $\mathcal{H}$ for which an opertor X^* from $\mathcal{H}$ to $\mathcal{H}$ exists with

$$\langle\eta|X\xi\rangle=\langle X^*\eta|\xi\rangle \text{ for all } \eta,\xi\in\mathcal{H}.$$

The mapping $X\to X^*$ provides an involution on $L(\mathcal{H})$, and $L(\mathcal{H})$ is a Q-algebra with the atoms being the orthogonal projections on the one-dimensional linear subspaces of $\mathcal{H}$. If $\mathcal{H}$ is complete (i.e. a Hilbert space), every element from $L(\mathcal{H})$ is a bounded operator and $L(\mathcal{H})$ coincides with the algebra of bounded linear operators on $\mathcal{H}$. If $dim\mathcal{H}=n<\infty$, $L(\mathcal{H})$ coincides with the algebra of $n\times n$-matrices.

Definition 1: *Let $\mathcal{A}$ be a Q-algebra.*

(i) *A representation Π of $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$ is a homomorphism $\Pi:\mathcal{A}\rightarrow L(\mathcal{H})$ with $\Pi(X^*)=(\Pi(X))^*$ for all $X\in\mathcal{A}$.*

(ii) *A representation Π of $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$ is called irreducible if each linear subspace of $\mathcal{H}$ that is invariant for $\Pi(\mathcal{A})$ equals either $\mathcal{H}$ or $\{0\}$.*

(iii) *Two representations Π_1 and Π_2 of $\mathcal{A}$ on the pre-Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are called equivalent if there is a linear bijection $T:\mathcal{H}_1\rightarrow\mathcal{H}_2$ with*

$$T\Pi_1(X)=\Pi_2(X)T \text{ for all } X\in\mathcal{A}.$$

They are called unitarily equivalent if, in addition, $\langle T\eta|T\xi\rangle=\langle\eta|\xi\rangle$ *for all* $\eta,\xi\in\mathcal{H}_1$.

Lemma 1: *If a representation Π of a Q-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$ is irreducible, then the dimension of $\Pi(E)\mathcal{H}$ equals 1 or 0 for every atom E in $\mathcal{A}$.*

Proof: We assume that there is an atom E in $\mathcal{A}$ such that $\Pi(E)\mathcal{H}$ contains two orthogonal non-zero vectors η,ξ. The linear subspace $\Pi(\mathcal{A})\eta$ is invariant for $\Pi(\mathcal{A})$ and contains η, but does not contain ξ since

$$\begin{aligned}\langle\Pi(X)\eta|\xi\rangle&=\langle\Pi(X)\Pi(E)\eta|\Pi(E)\xi\rangle=\langle\Pi(EXE)\eta|\xi\rangle\\&=\boldsymbol{E}(X|E)\langle\Pi(E)\eta|\xi\rangle=\boldsymbol{E}(X|E)\langle\eta|\xi\rangle=0\end{aligned}$$

for all $X\in\mathcal{A}$. Thus Π is reducible. □

3 Representations resulting from atoms

With each atom E in a Q-algebra $\mathcal{A}$, a representation Π_E of $\mathcal{A}$ can be associated in the following way. The pre-Hilbert space $\mathcal{H}_E$ is the left ideal $\mathcal{A}E$ with the scalar product

$$\langle Y|Z\rangle:=\boldsymbol{E}\left(Y^*Z|E\right) \text{ for } Y,Z\in\mathcal{H}_E=\mathcal{A}E$$

and, for $X\in\mathcal{A}$,

$$\Pi_E(X)Y=XY \text{ for } Y\in\mathcal{H}_E=\mathcal{A}E.$$

Lemma 2: (i) *Π_E is irreducible for each atom E in a Q-algebra $\mathcal{A}$, and $\Pi_E(\mathcal{A})$ is a Q-algebra.*

(ii) *If Π is any irreducible representation of a Q-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$ with $\Pi(E)\neq 0$ for an atom E in $\mathcal{A}$, then Π is unitarily equivalent to Π_E.*

(iii) *If $\dim(\mathcal{A}E)=n<\infty$ for an atom E in a Q-algebra $\mathcal{A}$, then $\Pi_E(\mathcal{A})$ is isomorphic to the algebra of $n\times n$-matrices.*

Proof: (i) Let $\mathcal{Q}$ be an invariant subspace of $\mathcal{H}_E=\mathcal{A}E$ with $0\neq Y\in\mathcal{Q}$. Then $XY\in\mathcal{Q}$ for each $X\in\mathcal{A}$ Now let Z be any element from $\mathcal{H}_E=\mathcal{A}E$. For

$$V:=\frac{1}{\boldsymbol{E}\left(Y^*Y|E\right)}ZY^*$$

we get:

$$\mathcal{Q} \ni VY = \frac{1}{\boldsymbol{E}(Y^*Y|E)} ZY^*Y = \frac{1}{\boldsymbol{E}(Y^*Y|E)} ZEY^*YE = ZE = Z\,.$$

Thus $\mathcal{H}_E = \mathcal{Q}$, and we have shown that Π_E is irreducible. Now we prove that $\Pi_E(\mathcal{A})$ owns sufficiently many atoms. $\Pi_E(F)$ is an atom in $\Pi_E(\mathcal{A})$ for each atom F in $\mathcal{A}$ with $\Pi_E(F) \neq 0$. If $X \in \mathcal{A}$ such that $\Pi_E(X)\,\Pi_E(F) = 0$ for all atoms F in $\mathcal{A}$, then

$$0 = (\Pi_E(X)\Pi_E(F))^* = \Pi_E(FX^*) \text{ for all atoms } F \text{ in } \mathcal{A}$$

$$\Rightarrow \quad FX^*Y = 0 \text{ for all atoms } F \text{ in } \mathcal{A} \text{ and } Y \in \mathcal{A}E$$

$$\Rightarrow \quad Y^*XF = 0 \text{ for all atoms } F \text{ in } \mathcal{A} \text{ and } Y \in \mathcal{A}E$$

$$\Rightarrow \quad Y^*X = 0 \text{ for all } Y \in \mathcal{A}E \Rightarrow X^*Y = 0 \text{ for all } Y \in \mathcal{A}E$$

$$\Rightarrow \quad 0 = \Pi_E(X^*) = (\Pi_E(X))^* \Rightarrow 0 = \Pi_E(X).$$

(ii) We select $\eta \in \Pi(E)\mathcal{H}$ with $\|\eta\| = 1$ and define $T: \mathcal{A}E \to \mathcal{H}$ via $TY := \Pi(Y)\eta$ for $Y \in \mathcal{A}E$. T is linear. Since $\Pi(\mathcal{A})\eta$ is invariant for Π and $0 \neq \eta \in \Pi(\mathcal{A})\eta$, $\Pi(\mathcal{A})\eta$ must equal $\mathcal{H}$ because of the irreducibility and thus T is surjective. For $X, Y \in \mathcal{A}E$, we get:

$$X^*Y = EX^*YE = \boldsymbol{E}(X^*Y|E)E$$

and then

$$\langle TX | TY\rangle = \langle \Pi(X)\eta | \Pi(Y)\eta\rangle = \langle \eta | \Pi(X^*Y)\eta\rangle$$
$$= \boldsymbol{E}(X^*Y|E)\langle \eta | \Pi(E)\eta\rangle = \boldsymbol{E}(X^*Y|E) = \langle X|Y\rangle.$$

Furthermore for all $X \in \mathcal{A}$ and $Y \in \mathcal{A}E$: $T\Pi_E(X)Y = TXY = \Pi(XY)\eta = \Pi(X)\Pi(Y)\eta = \Pi(X)TY$.

(iii) $\Pi_E(\mathcal{A})$ is isomorphic to a subalgebra of the $n \times n$-matrices, and it is therefore sufficient to prove that $dim\, \Pi_E(\mathcal{A}) \geq n^2$. Let $Y_1, \ldots, Y_n$ be an orthonormal basis of $\mathcal{A}E$. Then

$$Y_k^*Y_l = EY_k^*Y_lE = \boldsymbol{E}(Y_k^*Y_l|E)E = \langle Y_k | Y_l\rangle E = \begin{cases} E & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}$$

and hence

$$Y_jY_k^*Y_l = \begin{cases} Y_j & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}$$

such that $\Pi_E(Y_jY_k^*)$, $j = 1, \ldots, n,\ k = 1, \ldots, n$, are linearly independent in $\Pi_E(\mathcal{A})$. □

Now we introduce an equivalence relation ~ on the system of atoms in a Q-algebra, which will later help us to construct the "optimal" representation.

Definition 2: *Two atoms E and F in a Q-algebra $\mathcal{A}$ are said to be equivalent ($E \sim F$) if the representations Π_E and Π_F are unitarily equivalent.*

Lemma 3: *The following conditions are equivalent for two atoms E and F in a Q-algebra $\mathcal{A}$:*
(i) *$E \sim F$ (i.e. the representations Π_E and Π_F are unitarily equivalent).*
(ii) *The representations Π_E and Π_F are equivalent.*
(iii) $\Pi_F(E) \neq 0$.
(iv) *There exists an X in $\mathcal{A}$ with $EXF \neq 0$.*
(v) *There exists an Y in $\mathcal{A}$ with $E = Y^*FY$.*
(vi) *There are X, Y in $\mathcal{A}$ with $E = XFY$.*

Proof: The implications (i)⇒(ii) and (v)⇒(vi) are trivial.
(ii)⇒(iii): Let $T: \mathcal{A}E \to \mathcal{A}F$ be the linear bijection with $T\Pi_E(X) = \Pi_F(X)T$ for all $X \in \mathcal{A}$. Then $\Pi_F(E) \neq 0$ since $\Pi_E(E) \neq 0$.
(iii)⇒(iv): $\Pi_F(E) \neq 0$ means that there is $X \in \mathcal{A}F$ with $0 \neq \Pi_F(E)X = EX = EXF$.
(iv)⇒(v): $EXF \neq 0 \Rightarrow 0 \neq (EXF)^* \Rightarrow 0 \neq EXF(EXF)^* = EXFX^*E = \boldsymbol{E}\left(XFX^*|E\right)E$. With

$$\lambda := \boldsymbol{E}\left(XFX^*|E\right) > 0 \text{ and } Y := \frac{1}{\sqrt{\lambda}} X^*E,$$

we then get:

$$Y^*FY = \frac{1}{\lambda} EXFX^*E = E.$$

(vi)⇒(i): $E = XFY$. Then

$$E = XFYE \Rightarrow FYE \neq 0 \Rightarrow EY^*F \neq 0 \Rightarrow 0 \neq FYEY^*F = \mathbb{E}\,(YEY^*|F)F \Rightarrow \lambda := \mathbb{E}\,(YEY^*|F) > 0.$$

We now define $T: \mathcal{A}E \to \mathcal{A}F$ via

$$T(Z) = \frac{1}{\sqrt{\lambda}} ZY^*F.$$

We get for $V, Z \in \mathcal{A}E$:

$$\langle T(V)|T(Z)\rangle = \frac{1}{\lambda}\, \boldsymbol{E}\left(FYV^*ZY^*F|F\right) = \frac{1}{\lambda}\, \boldsymbol{E}\left(YEV^*ZEY^*|F\right)$$
$$= \frac{1}{\lambda}\, \boldsymbol{E}\left(V^*Z|E\right) \boldsymbol{E}\left(YEY^*|F\right) = \boldsymbol{E}\left(V^*Z|E\right) = \langle V|Z\rangle.$$

This implies that T is injective. We still have to prove that it is surjective. For $V \in \mathcal{A}F$ we define $Z \in \mathcal{A}E$ as

$$Z := \frac{1}{\sqrt{\lambda}} VYE = \frac{1}{\sqrt{\lambda}} VFYE.$$

Then:

$$T(Z) = \frac{1}{\lambda} VFYEY^*F = FV = V. \ \square$$

Two atoms that are not orthogonal are equivalent (this follows fromLemma 3 (iv)), or two atoms that are not equivalent are orthogonal (but, of course, orthogonality does not imply non-equivalence).

Two atoms in a commutative Q-algebra are equivalent if and only if they are identical (since two commuting atoms are either equal or orthogonal and, in the second case, Lemma 3 (iv) excludes that they are equivalent).

In the Q-algebra $L(\mathcal{H})$ with a pre-Hilbert space $\mathcal{H}$, all atoms are equivalent among each other since, if the two atoms E and F are the orthogonal projections on the one-dimensional linear subspaces generated by the normalized vectors $\eta,\xi\in\mathcal{H}$, then the operator T

$$T:\mathcal{H}\rightarrow\mathcal{H},\quad T\psi:=\langle\xi|\psi\rangle\eta$$

belongs to $L(\mathcal{H})$ and

$$T^{*}\psi=\langle\eta|\psi\rangle\xi \text{ and } T^{*}ET\psi=\langle\xi|\psi\rangle T^{*}\eta=\langle\xi|\psi\rangle\xi=F\psi$$

for all $\psi\in\mathcal{H}$. Particularly in the algebra of all bounded linear operators on a Hilbert space or in the algebra of $n\times n$-matrices, all atoms are equivalent among each other.

Theorem 1: *For a Q-algebra $\mathcal{A}$ with finite dimension are equivalent:*
(i) *$\mathcal{A}$ is isomorphic to a matrix algebra.*
(ii) *$E\sim F$ for each pair E,F of atoms in $\mathcal{A}$.*

Proof: (i) $\Rightarrow$ (ii): see above. (ii) $\Rightarrow$ (i): Applying Lemma 2 (iii) it is sufficient to show the injectivity of Π_E for an atom E in $\mathcal{A}$. For $0\neq X\in\mathcal{A}$ there is an atom F with $XF\neq 0$, and since $E\sim F$, there is $Y\in\mathcal{A}$ with $F=Y^*EY$. Then: $0\neq XF=XY^*EY\Rightarrow 0\neq XY^*E=\Pi_E(X)Y^*E$, i.e. $\Pi_E(X)\neq 0$. □

Theorem 2: *Each finite-dimensional Q-algebra $\mathcal{A}$ is a finite direct sum of matrix algebras.*

Proof: Since orthogonal atoms are linearly independent, the number of equivalence classes of atoms is $n<\infty$, and a maximal set of pairwise orthogonal atoms in each equivalence class is finite. Let E_{jk}, $k=1,...,m_j$, be a maximal set of pairwise orthogonal atoms in the j^{th} equivalence class ($1\leq j\leq n$) and define

$$D_j:=\sum_{k=1}^{m_j}E_{jk}\,.$$

Then, D_j ($1\leq j\leq n$) are projections with $D_jD_{j'}=0$ and $D_jXD_{j'}=0$ for $j\neq j'$ and $X\in\mathcal{A}$. This follows from $E_{jk}XE_{j'k'}=0$ for all k,k' (Lemma 3 (iv)). Since furthermore

$$\sum_{j=1}^{n}D_j=\mathbb{I}\,,$$

we get:

$$\mathcal{A}=\bigoplus_{j=1}^{n}D_j\mathcal{A}D_j\,.$$

Now apply Theorem 1 to each $D_j\mathcal{A}D_j$. Each atom F in $D_j\mathcal{A}D_j$ is equivalent to E_{j1} since, otherwise, F would not be equivalent to any E_{jk}, $k=1,...,m_j$, F would thus be orthogonal to each E_{jk}, $k=1,...,m_j$, and therefore orthogonal to D_j, contradicting $FD_j=F\neq 0$. □

The *dimension* of a pre-Hilbert space $\mathcal{H}$ is the cardinal number of a maximal orthonormal family of vectors in $\mathcal{H}$.

Lemma 4: *Let $\mathcal{A}$ be a Q-algebra and E an atom in $\mathcal{A}$. The cardinal number of a maximal set of mutually orthogonal atoms in the equivalence class of E equals dim($\mathcal{A}E$).*

Proof: Let Ω denote a maximal set of mutually orthogonal atoms in the equivalence class of E. For each $F\in\Omega$ there is $X_F\in\mathcal{A}$ with $FX_FE\neq 0$ (Lemma 3 (iv)). Then $EX_F^*FX_FE\neq 0$ and $\mathbb{E}(X_F^*FX_F|E)>0$. We will now prove that

$$\left\{\alpha_F FX_F E \middle| F\in\Omega\right\} \quad \text{with} \quad \alpha_F := \frac{1}{\sqrt{\mathbf{E}\left(X_F^* FX_F\middle|E\right)}}$$

is a maximal orthonormal family of vectors in $\mathcal{A}E$. We get for $F_1,F_2\in\Omega$:

$$\left\langle \alpha_{F_1} F_1 X_{F_1} E \middle| \alpha_{F_2} F_2 X_{F_2} E\right\rangle = \alpha_{F_1}\alpha_{F_2}\, \mathbf{E}\left(X_{F_1}^* F_1 F_2 X_{F_2}\middle|E\right)$$
$$= \begin{cases} 0 & \text{for } F_1 \neq F_2 \\ 1 & \text{for } F_1 = F_2 \end{cases}$$

Now we assume that $0\neq Y\in\mathcal{A}E$ exists with $Y\perp\left\{FX_F E\middle|F\in\Omega\right\}$. Then

$$D := \frac{1}{\mathbf{E}\left(Y^*Y\middle|E\right)} YEY^*$$

is an atom (cf. [3]) with $D\sim E$ (Lemma 3 (v)) and for all $F\in\Omega$:

$$0 = \left\langle Y\middle|FX_F E\right\rangle = \mathbb{E}\left(Y^* FX_F E\middle|E\right) \Rightarrow 0 = EY^* FX_F E \Rightarrow DFX_F E = 0 .$$

Since $F\sim E$, there is an element Z in $\mathcal{A}$ with $F=Z^*EZ$ (Lemma 3 (v)). Hence

$$0=DZ^*EZX_FE= \mathbb{E}(ZX_F|E)\, DZ^*E.$$

From $0\neq FX_FE=Z^*EZX_FE= \mathbb{E}(ZX_F|E)\, Z^*E$ we get $\mathbb{E}(ZX_F|E)\neq 0$. Therefore $DZ^*E=0$ and finally $DF=DZ^*EZ=0$. Thus, D is an atom that is equivalent to E and orthogonal to all $F\in\Omega$, which contradicts the maximality of Ω. □

4 Automorphisms

An *automorphism* of a Q-algebra $\mathcal{A}$ is a linear bijection $\theta:\mathcal{A}\to\mathcal{A}$ with $\theta(XY)=\theta(X)\theta(Y)$ and $\theta(X^*)=\theta(X)^*$ for all $X,Y\in\mathcal{A}$. The group of automorphisms of a Q-algebra $\mathcal{A}$ is denoted by $Aut(\mathcal{A})$.

An automorphism θ maps atoms to atoms with $\mathbb{E}(\theta(X)|\theta(E))=\mathbb{E}(X|E)$ for each atom E and $X\in\mathcal{A}$.

Lemma 5: (i) *If* θ *is an automorphism of a Q-algebra* $\mathcal{A}$ *and* E,F *are atoms in* $\mathcal{A}$ *with* $\theta(E)\sim F$, *then* $\Pi_F\circ\theta$, *which is a representation on* $\mathcal{H}_F$ *as well, is unitarily equivalent to* Π_E.
(ii) *If* θ *is an automorphism of a Q-algebra* $\mathcal{A}$ *and if* E *is an atom in* $\mathcal{A}$ *with* $\theta(E)\sim E$, *then there exists a unitary element* U *in* $L(\mathcal{H}_E)$ *with* $\Pi_E(\theta(X))=U\Pi_E(X)U^*$ *for all* $X\in\mathcal{A}$.

Proof: (i) Since $\theta(E)\sim F$, Π_F and $\Pi_{\theta(E)}$ are unitarily equivalent. Then $\Pi_F\circ\theta$ and $\Pi_{\theta(E)}\circ\theta$, which are representations on $\mathcal{A}F$ and $\mathcal{A}\theta(E)$, respectively, are unitarily equivalent as well. On the other hand, with

$$U:\mathcal{A}E\to\mathcal{A}\theta(E),\ UY:=\theta(Y),$$

we get

$$U\Pi_E(X)Y=\theta(XY)=\theta(X)\theta(Y)=\Pi_{\theta(E)}(\theta(X))U(Y)$$

for $Y\in\mathcal{A}E$ and $X\in\mathcal{A}$ such that $\Pi_{\theta(E)}\circ\theta$ is unitarily equivalent to Π_E. Therefore $\Pi_F\circ\theta$ is unitarily equivalent to Π_E.
(ii) Apply (i) with $F=E$. □

Each unitary element U of a Q-algebra $\mathcal{A}$ (i.e. $U^*U=UU^*=\mathbb{I}$) defines an automorphism $\theta_U\in Aut(\mathcal{A})$ via $\theta_U(X):=UXU^*$ for $X\in\mathcal{A}$, and $\theta\in Aut(\mathcal{A})$ is called an *inner* automorphism if a unitary element $U\in\mathcal{A}$ exists with $\theta=\theta_U$.

Corollary 1: *Every automorphism of the Q-algebra* $L(\mathcal{H})$ *with a pre-Hilbert space* $\mathcal{H}$ *is inner.*

Proof: Applying Lemma 5 (ii) and having in mind that all atoms are equivalent in this case, it is sufficient to show that the trivial representation of $L(\mathcal{H})$ on $\mathcal{H}$ is unitarily equivalent to Π_E for some atom E, and due to Lemma 2 (ii) it is sufficient to prove its irreducibility. The irreducibility immediately follows from the fact that there is an $X\in L(\mathcal{H})$ with $X\xi=\eta$ for each pair of vectors $\xi,\eta\in\mathcal{H}$ with $\xi\neq0$. X can be defined as $X(\psi):=\langle\xi|\psi\rangle\eta$ for $\psi\in\mathcal{H}$. □

5 The "optimal" representation

Different representations of a Q-algebra have been considered in [2,3], but the pre-Hilbert spaces involved with them seem to be unnecessarily large. The representation of an atom is an orthogonal projection, but the dimension of the range of this projection is higher than one (indeed infinite in many cases).

We are now interested in a representation that maps atoms to orthogonal projections with one-dimensional range. We first note that all such representations are injective.

Lemma 6: *A representation Π of a Q-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$ such that $\Pi(E)\neq 0$ for each atom E in $\mathcal{A}$ is injective.*

Proof: For each $0\neq X\in\mathcal{A}$ there is an atom E with $XE\neq 0$. Then

$$0\neq EX^*XE=\mathbb{E}\,(X^*X|E)E$$
$$\Rightarrow 0\neq \mathbb{E}\,(X^*X|E) \text{ and}$$
$$0\neq \mathbb{E}\,(X^*X|E)\,\Pi(E)=\Pi(\,\mathbb{E}\,(X^*X|E)\,E)=\Pi(EX^*XE)=\Pi(E)\,\Pi(X^*)\,\Pi(X)\,\Pi(E)$$
$$\Rightarrow 0\neq\Pi(X).\ \Box$$

For a family of representations Π_α of a Q-algebra $\mathcal{A}$ on pre-Hilbert spaces $\mathcal{H}_\alpha$ $(\alpha\in J)$ the *direct sum* is defined as follows: The linear space

$$\underset{\alpha\in J}{\oplus}\mathcal{H}_\alpha:=\left\{\xi:J\to\bigcup_{\alpha\in J}\mathcal{H}_\alpha\,\middle|\,\xi(\alpha)\in\mathcal{H}_\alpha \text{ for all } \alpha\in J \text{ and } \xi(\alpha)\neq 0 \text{ only for a finite number of } \alpha\in J\right\}$$

becomes a pre-Hilbert space with the scalar product:

$$\langle\xi_1|\xi_2\rangle:=\sum_{\alpha\in J}\langle\xi_1(\alpha)|\xi_2(\alpha)\rangle.$$

The direct sum $\underset{\alpha\in J}{\oplus}\Pi_\alpha$ is the representation of $\mathcal{A}$ on $\underset{\alpha\in J}{\oplus}\mathcal{H}_\alpha$ defined via

$$\left(\underset{\alpha\in J}{\oplus}\Pi_\alpha\right)(X):=\underset{\alpha\in J}{\oplus}\Pi_\alpha(X) \text{ for } X\in\mathcal{A}.$$

$\underset{\alpha\in J}{\oplus}\Pi_\alpha(X)$ maps ξ to η with $\eta(\alpha):=\Pi_\alpha(X)\xi(\alpha)$.

Definition 3: *A Q-algebra $\mathcal{A}$ is said to be of type ω with a cardinal number ω if the cardinal number of every set of mutually orthogonal atoms in $\mathcal{A}$ does not exceed ω and if there exists a set of mutually orthogonal atoms in $\mathcal{A}$ with cardinal number ω.*

Theorem 3: *For each Q-algebra $\mathcal{A}$ there is a representation Π on a pre-Hilbert space $\mathcal{H}$ satisfying the following conditions:*

(i) *$\Pi(F)\mathcal{H}$ is one-dimensional for each atom F in $\mathcal{A}$, and $\mathcal{H}$ is the linear hull of*

$$\bigcup_{\substack{\text{all atoms}\\ F \text{ in } \mathcal{A}}} \Pi(F)\mathcal{H}.$$

(ii) *$\dim\mathcal{H}$ equals the type of $\mathcal{A}$.*

(iii) *For each $\theta\in Aut(\mathcal{A})$ there is a unitary element U_θ in $L(\mathcal{H})$ with*

$$\Pi(\theta(X))U_\theta = U_\theta\Pi(X)$$

for all $X\in\mathcal{A}$.

(iv) *Each representation Π' on a pre-Hilbert space $\mathcal{H}'$ such that $\Pi'(F)\mathcal{H}'$ is one-dimensional for each atom F in $\mathcal{A}$ and $\mathcal{H}'$ is the linear hull of*

$$\bigcup_{\substack{\text{all atoms}\\ F \text{ in } \mathcal{A}}} \Pi'(F)\mathcal{H}'$$

is unitarily equivalent to Π.

If Ω is a set of atoms containing one and only one atom out of each equivalence class, then Π can be defined as

$$\Pi := \bigoplus_{E\in\Omega} \Pi_E .$$

Proof: (i) For Ω and Π as above and for an atom F we have to show that $\Pi(F)$ has a 1-dimensional range. Since $\Pi_E(F)=0$ for all atoms $E\in\Omega$ that are not equivalent to F (Lemma 3 (iii)), it is sufficient to consider $\Pi_E(F)$ with $E\sim F$. The range of $\Pi_E(F)$ has dimension 0 or 1 (Lemma 1 and Lemma 2), and since $\Pi_E(F)\neq 0$ (Lemma 3), the dimension must equal 1.

For each $\xi\in\mathcal{H}$ there are finitely many $E_1,\ldots,E_n\in\Omega$ such that ξ has the shape

$$\xi = \sum_{k=1}^{n} \xi_k \quad \text{with } \xi_k(E) = \begin{cases} 0 & \text{for } \Omega \ni E \neq E_k \\ Y_k \in \mathcal{A}E_k & \text{for } \Omega \ni E = E_k . \end{cases}$$

For $\xi\neq 0$ we get $Y_kE_k=Y_k\neq 0$, and

$$F_k := \frac{1}{\boldsymbol{E}\left(Y_k^* Y_k \middle| E\right)} Y_k E Y_k^*$$

is an atom in $\mathcal{A}$ with $\xi_k \in \Pi(F_k)\mathcal{H}$ since $F_k\xi_k(E) = 0 = \xi_k(E)$ for $E\neq E_k$,

$$F_k\xi_k(E_k) = F_kY_k = F_kY_kE_k = \frac{1}{\boldsymbol{E}\left(Y_k^* Y_k \middle| E_k\right)} Y_kE_kY_k^*Y_kE_k = Y_kE_k = Y_k = \xi_k(E_k)$$

and hence

$$\left(\Pi(F_k)\xi_k\right)(E) = F_k\xi_k(E) = \xi_k(E) \text{ for } E\in\Omega, \text{ i.e. } \Pi(F_k)\xi_k = \xi_k .$$

(ii) This follows from Lemma 4, taking into account that atoms that are not equivalent are orthogonal.

(iii) We define $j:\Omega\to\Omega$ such that $j(E)\sim\theta(E)$, i.e. $j(E)$ is that representative out of the equivalence class of $\theta(E)$ that has been selected for Ω. Then we get from Lemma 5 (i) that $\Pi_{j(E)}\circ\theta$ and Π_E are unitarily equivalent representations, i.e. there is a linear bijection $T_E:\mathcal{H}_{j(E)}\to\mathcal{H}_E$ with

$$T_E\Pi_{j(E)}(\theta(X))=\Pi_E(X)T_E \quad \text{for all } X\in\mathcal{A} \text{ and}$$
$$\langle T_E\eta|T_E\xi\rangle=\langle\eta|\xi\rangle \quad \text{for all } \eta,\xi\in\mathcal{H}_{j(E)}$$

for each $E\in\Omega$. We now define V_θ as follows:

$$V_\theta:\bigoplus_{E\in\Omega}\mathcal{H}_E\to\bigoplus_{E\in\Omega}\mathcal{H}_E,\ V_\theta\xi(E):=T_E\xi(j(E)).$$

Since j and the T_E $(E\in\Omega)$ are bijections, V_θ is a bijection. Furthermore:

$$\langle V_\theta\eta|V_\theta\xi\rangle=\sum_{E\in\Omega}\langle T_E\eta(j(E))|T_E\xi(j(E))\rangle=\sum_{E\in\Omega}\langle\eta(j(E))|\xi(j(E))\rangle$$
$$=\sum_{E\in\Omega}\langle\eta(E)|\xi(E)\rangle=\langle\eta|\xi\rangle \quad \text{for } \eta,\xi\in\bigoplus_{E\in\Omega}\mathcal{H}_E$$

and for $X\in\mathcal{A}$:

$$(V_\theta\Pi(X)\xi)(E)=T_E\Pi_{j(E)}(X)\xi(j(E))=\Pi_E(\theta^{-1}(X))T_E\xi(j(E))$$
$$=(\Pi(\theta^{-1}(X))V_\theta\xi)(E) \quad \text{for each } E\in\Omega,\xi\in\bigoplus_{E\in\Omega}\mathcal{H}_E,$$

i.e. $V_\theta\Pi(X)=\Pi(\theta^{-1}(X))V_\theta$. Finally, we replace X by $\theta(X)$ and define $U_\theta:=V_\theta^{-1}$.

(iv) For each $E\in\Omega$ we select $\eta_E\in\Pi'(E)\mathcal{H}'$ with $\|\eta_E\|=1$, and we define for $\xi\in\mathcal{H}$:

$$T\xi:=\sum_{\substack{E\in\Omega,\\ \xi(E)\neq 0}}\Pi'(\xi(E))\eta_E\in\mathcal{H}'.$$

T is a linear mapping from $\mathcal{H}$ to $\mathcal{H}'$. Furthermore for $\xi_1,\xi_2\in\mathcal{H}$ and $E,F\in\Omega$:

$$\xi_1(E)^*\xi_2(E)=E\xi_1(E)^*\xi_2(E)E=\mathbb{E}(\xi_1(E)^*\xi_2(E)|E)E=\langle\xi_1(E)|\xi_2(E)\rangle E$$

and

$$\xi_1(E)^*\xi_2(F)=E\xi_1(E)^*\xi_2(F)F=0 \quad \text{for } E\neq F$$

(E and F are not equivalent, and use Lemma 3 (iv)), hence

$$\langle\Pi'(\xi_1(E))\eta_E|\Pi'(\xi_2(F))\eta_F\rangle=\langle\eta_E|\Pi'((\xi_1(E))^*\xi_2(F))\eta_F\rangle=\begin{cases}0 & \text{if } E\neq F\\ \langle\xi_1(E)|\xi_2(E)\rangle & \text{if } E=F\end{cases}$$

($\Pi'(E)\eta_E=\eta_E$ and $\|\eta_E\|=1$) and therefore $\langle T\xi_1|T\xi_2\rangle=\langle\xi_1|\xi_2\rangle$.

To prove that $T(\mathcal{H})=\mathcal{H}'$ it is sufficient to show that there is a $\xi\in\mathcal{H}$ with $T\xi=\eta$ for each $0\neq\eta\in\Pi'(F)\mathcal{H}'$ with an atom F in $\mathcal{A}$. There is one and only one atom D in Ω with $D\sim F$ and $Y\in\mathcal{A}$ with $FYD\neq 0$ (Lemma 3 (iv)). We now define $0\neq\xi\in\mathcal{H}$ via:

$$\xi(E):=\begin{cases} 0 & \text{for } E\neq D \\ FYD & \text{for } E=D. \end{cases}$$

Then

$$0\neq T\xi=\Pi'(FYD)\eta_D=\Pi'(F)\Pi'(YD)\eta_D=\alpha\eta$$

with $0\neq\alpha\in\mathbb{C}$ since $dim\Pi'(F)\mathcal{H}'=1$, and finally $T\left(\frac{1}{\alpha}\xi\right)=\eta$.

We will now prove that $T\Pi(X)=\Pi'(X)T$ for all $X\in\mathcal{A}$. Let $\xi\in\mathcal{H}$.

$$\Pi'(X)T\xi:=\Pi'(X)\sum_{\substack{E\in\Omega,\\ \xi(E)\neq 0}}\Pi'(\xi(E))\eta_E=\sum_{\substack{E\in\Omega,\\ \xi(E)\neq 0}}\Pi'(X\xi(E))\eta_E=T\Pi(X)\xi.\ \Box$$

With an injective representation of a Q-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$, the dimension of $\mathcal{H}$ can not be lower than ω if $\mathcal{A}$ is of type ω. Thus the representation of Theorem 3 uses a pre-Hilbert space of minimum size. Furthermore, part (iv) of Theorem 3 means that this representation is unique in a certain sense.

6 Conclusions

The above Theorem 3 yields a representation of a Q-algebra $\mathcal{A}$ on a pre-Hilbert space such that each automorphism of $\mathcal{A}$ can be expressed via unitary operators on the pre-Hilbert space, but the map from the automorphisms to the unitary operators need not be a group homomorphism. This is a major difference to the representation constructed in [2] where this map is a group homomorphism. Nevertheless, the representation of Theorem 3 is superior since it owns interesting properties (pre-Hilbert space with minimum dimension, uniqueness) that the one constructed in [2] does not provide.

References

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