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Consistent Histories and POV Measures

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Abstract This review is devoted to the history formulation of standard Hilbert space quantum mechanics. We will give an overview over the basic ideas and concepts of the history approach. The consistent histories approach is usually formulated using the standard notions of observable and state. We will argue in the second part of this review that the natural notion of an observable in quantum mechanics is that of a *positive-operator-valued measure* (POV measure) and will show that the consistent history formalism can be generalized to incorporate POV measures in a natural and simple way.

1 Introduction

This article is about the foundations of quantum mechanics. Ever since the invention of quantum mechanics many scientists – physicists, mathematicians and philosophers – have thought about the problem of how the physical world could possibly be how quantum mechanics says it is. This is clearly (at least in part) a metaphysical problem. In quantum physics, it is the problem of *interpreting* quantum mechanics. From a point of view of a philosopher, the problem of interpreting quantum mechanics is an interesting problem in its own right. In contrast, a physicist is mainly interested in the question whether a given theory is empirically adequate and successful. Quantum mechanics enjoys an overwhelming amount of empirical success and it has shown an ever increasing range of applicability. The spectacular empirical success of quantum mechanics as a physical theory is based on the interpretative rule known as *Born's rule.* [1] Born's rule in its simplest form states that

[1] For a historical account of Born's rule we refer to the book by Jammer [1].

the probability to find a system initially in the state φ in the state ψ is given by $p_\varphi(\psi) = |(\varphi, \psi)|^2$. Thus, Born's rule translates the abstract assertions of the theory (in terms of state vectors etc.) to empirically testable statements in terms of probabilities. The probabilistic predictions of quantum mechanics based on Born's rule (or at least some of them) can (in principle) be tested by comparing them with the relative frequencies of the various data in a long series of trials. [2] Born's rule (or some appropriate generalized formulation thereof) is contained or reproduced in any more extensive interpretation of quantum mechanics known to the author. Quantum mechanics admits many different interpretations, e.g., the Copenhagen type interpretations, [3] the many worlds interpretation, the modal interpretations, the quantum event interpretation [8] and the statistical interpretation [2] to name only a few of them. However, all these interpretations contain or reproduce Born's rule in some way or other and thus the empirical content of quantum mechanics is remarkably independent of the interpretation adopted. Different interpretations do not change the empirical content of quantum mechanics.

So, why another work about the foundations of quantum mechanics ?

From a point of view of a physicist new work on the foundations of quantum mechanics will be worth while only if it offers something new in content or perspective.

I will argue briefly in the sequel that the consistent histories approach to quantum mechanics indeed offers something new as well in content as in perspective.

The investigation of how quantum mechanics is to be interpreted and what quantum theory is really telling us about the deeper nature of physical reality is a slippery business. This is so because interpretations add no empirical content to the theory interpreted and accordingly we have no empirical criterion to make a decision between competing interpretations. Due to the authority of the founders of quantum mechanics and due to the lack of a sensible alternative interpretation, the Copenhagen interpretation of quantum mechanics has for many years acquired the status of a dogma. This goes as far as that in some quantum mechanics textbooks the Copenhagen interpretation is treated as if it were the only conceivable interpretation of quantum mechanics. Since the interpretation is undoubtedly part of quantum mechanics it is striking to see how little space is devoted to it and how uncritically the assertions of the Copenhagen interpretation are adopted in many of the standard textbooks (with notable exceptions, including the book by David Bohm [3]. The recommendable lecture notes by Chris Isham [9] may also serve as a supplementary text to the standard textbooks of quantum mechanics). It is well-known that the Copenhagen interpretation is plagued

[2] This does neither mean that probabilities should or must be interpreted only as approximate relative frequencies nor that probabilistic statements always (and exclusively) refer to ensembles of (similarly prepared) systems (as in the so-called *statistical ensemble interpretation*, see Ballentine [2]).

[3] I use the phrase *Copenhagen type interpretation* to denote collectively the different variants of the Copenhagen or orthodox interpretation, to wit, the body of ideas which is with some justice also often called "orthodoxy" and which is usually associated with the names of Bohr, Heisenberg, von Neumann, Pauli, Born, Jordan and others. More or less rudimentary accounts of the Copenhagen interpretation can be found in almost all textbooks on quantum mechanics. The best account of the ideas of the *Copenhagen school* ever written is perhaps the highly recommendable book by David Bohm [3] (see in particular Chapters 6-8,22-23). Bohm's book will serve as our general reference of the Copenhagen interpretation. Different variants of the Copenhagen interpretation may also be found in the recommendable books by von Neumann [4] and Scheibe [5] and also in [6]. Our cumulative nomenclature ignores the differences in the various versions of the Copenhagen type interpretations given by different authors; for an overview the reader is referred to the monographs by Jammer [1, 7].

with problems, mysteries and anthropomorphisms. The main problem of the Copenhagen interpretation is the understanding and description of the measuring process. Usually, the time evolution of the wavefunction of two interacting physical systems is unitary and governed by the Schrödinger equation. Consider a macroscopic measuring apparatus designed to measure the value of some observable A. Accordingly, if a suitable quantum system interacts with this measuring apparatus, then the (unitarily timedeveloped) state of the apparatus-object system after the interaction will in general (e.g., if the quantum system is initially *not* in an Eigenstate of the observable A) be a superposition of macroscopically distinguishable states. Even if some kind of decoherence mechanism is assumed to be present, the state of the apparatus after the interaction will in general be a mixture of macroscopically distinguishable states [10]. In contrast, the time evolution during a measurement process contains a nonunitary collapse. This is due to the fact that every measurement has a definite outcome, the measuring result. The problem is that the measurement process is also a physical interaction between the measuring apparatus and the measuring object and that we have no intrinsic criterion to distinguish ordinary interactions from measurement situations. In its most general form this problem is called the *objectification problem* of quantum mechanics [11]. This problem assumes different faces within different variants of the Copenhagen interpretation. We now describe briefly the ubiquitous solution of the Copenhagen type interpretation to the objectification problem. According to this solution, one has to abandon the idea that macroscopic objects – at least when they serve as measuring apparatuses – can be described by quantum mechanics. Instead one has to introduce the following two assumptions (cited from [3], Chapter 23)

1. Quantum theory presupposes a classical level and the correctness of classical concepts in describing this level.

2. The classically definite aspects of large-scale systems cannot be deduced from the quantum-mechanical relationships of assumed small-scale elements. Instead, classical definiteness and quantum potentialities complement each other in providing a complete description of the system as a whole.

These two assumptions clearly contradict the widespread belief that quantum concepts are more fundamental than classical concepts and that the classical theory is a limiting case of the quantum theory. If one wants to maintain the latter view, then clearly the Copenhagen interpretation must be altered or abandoned.

It is beyond the scope of the present work to discuss and criticize the Copenhagen interpretation and its problems in detail. However, to the best of my knowledge at present there is no interpretation of quantum mechanics totally free of 'problems' and a satisfactory solution of the objectification problem is not known. It seems as if metaphysical problems of the kind just mentioned can never all be solved by solely changing one's metaphysical presuppositions. That said, I do believe, though, that it is important for a physicist to be aware of the multitude of metaphysical presuppositions and beliefs compatible with quantum mechanics. Metaphysical dogmas in theoretical physics and 'wrong' metaphysical research programmes may be an obstacle for further progress and may hinder thus new developments. This is one of the main reasons why it is important to question the pervasive

Copenhagen type interpretation of quantum mechanics and to investigate alternative interpretations of quantum mechanics.

The *consistent* or *decoherent histories approach* to quantum theory is a fresh, novel attempt to formulate a generalization of standard quantum mechanics. The consistent histories approach introduces new concepts into quantum mechanics and is structurally different from all other approaches to quantum mechanics. Nonrelativistic quantum mechanics in its standard formulation is not a theory which describes the dynamical evolution pattern of events in time, but it is a theory which gives probabilities to the various possibilities or events. The history approach to quantum mechanics can be looked upon as an attempt to remedy this situation by introducing time sequences of possibilities (or events) as a rough substitute for dynamical processes.

Among others the consistent histories scheme provides a novel framework for the interpretation of standard Hilbert space quantum mechanics. This new interpretation has been developed mainly by Robert Griffiths and Roland Omnès [12] - [19]. In this interpretation quantum mechanics is asserted to be a theory describing individual (microscopic and macroscopic) systems and their real properties regardless of whether the systems are open or closed and regardless of whether there is an external observer or not. That is, it is asserted that quantum mechanics provides an objective description of physical phenomena. There is no fundamental observer-system split and no reduction of the wave-packet induced by external measurements. The concept of measurement does not retain the central and fundamental status it possesses in the Copenhagen type interpretations of quantum mechanics. This philosophy is particularly interesting for quantum cosmology. In the philosophy of science the consistent histories interpretation of quantum mechanics would be called a *realistic* (but indeterministic) interpretation. According to the point of view of philosophical realism a physical theory is a *description* of entities, structures and processes which really exist or occur in the real physical world regardless of whether there are observers present or not. In contrast, an *instrumentalist* would say that all physical theories are in a sense phenomenological and that the purpose of a physical theory is solely to provide an economical tool for making predictions. According to this point of view the concepts and structures in a physical theory do not correspond to some real entities or structures behind the phenomena. It should be stressed, however, that a *realistic* interpretation does not necessarily interpret the theory in naive classical terms. Formulated differently: philosophical realism does neither entail determinism nor the realism of classical physics. A realist would not claim that all of reality can be described using classical terminology and classical concepts. Indeed, for instance the Kochen-Dieks interpretation [20] - [23] of quantum mechanics is a realistic but indeterministic interpretation not using any classical pictures. In the interpretation based on the consistent histories approach to quantum mechanics the wave function (or more generally the state) is not interpreted as physical wave which really exists materially in space time, but as comprising all propensities and tendencies inherent in the system in question. It is in this sense that the state can be considered real in the consistent histories approach. In contrast, the Copenhagen interpretation in its pure form is not a realistic interpretation of quantum mechanics. However, it is also not a purely instrumentalistic interpretation of quantum mechanics. In the Copenhagen type interpretation realistic and instrumentalistic point of views are mixed to a certain extend (depending on the author). An example for a purely instrumentalistic interpretation of quantum mechanics is the statistical ensemble interpretation [2].

In the consistent histories approach probabilities are thought of as *propensities* reflecting the tendency that certain events will take place or that certain properties will be realized (upon repetition).

So, one important point in favour of the consistent histories approach is that it can be used as a framework for a *realistic* interpretation of quantum mechanics. This kind of interpretation was clearly not envisaged by the fathers of quantum mechanics. [4] But there is more to consistent histories.

The consistent histories approach incorporates radically new concepts whose introduction is, however, well motivated by standard nonrelativistic quantum mechanics. Although the consistent histories approach is accompanied by a radical change in the basic concepts of quantum theory, the mathematical framework of standard quantum mechanics and quantum field theory can be to a large extent retained. The new concepts of the consistent histories approach allow for a reassessment of several conceptual problems of quantum physics in the framework of Isham's general quantum history theories [24]. Isham's formulation of nonrelativistic quantum mechanics in terms of the concepts of the histories approach provide an attractive framework for a quantum theory in which space and time appear in a more symmetric way than in the usual formalism of standard quantum mechanics - moreover, in Isham's general quantum history theories [24], which are not discussed in the present review, time plays a subsidiary role.

In standard Hamiltonian quantum mechanics the time variable is fixed from the outset as the variable conjugate to the Hamiltonian. One important new ingredient in the consistent histories scheme is the notion of *history*. In nonrelativistic quantum mechanics a history in its simplest form is simply a time sequence of events. However, in standard quantum mechanics probabilities are solely associated with events at some fixed time. In contrast, in the consistent histories approach probabilities are associated with complete histories. [5] Moreover, in general a history is a more general object than simply a sequence of single-time events. The idea to investigate general quantum histories and general quantum history theories has first been put forward by Chris Isham in Ref. [24]. Isham characterizes general quantum history theories by a list of axioms abstracted from the mathematical structure of the standard consistent histories formalism.

It is not the purpose of this review to discuss all issues touched upon in this introduction in full detail. The reader is referred to the references for a fuller account. Further standard references for the consistent histories scheme are [25] - [52].

This paper is organized as follows: Section 2 is devoted to a general discussion of the notion of state and observable in quantum mechanics. The general notion of observable discussed there is perhaps not too well known. In the last two sections of Section 2 we discuss some issues which motivate and illustrate the consistent histories formalism which is presented in Section 3. We discuss the most general formulation of the consistent histories approach to quantum mechanics compati-

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[4] It should be stressed that the ideas and principles of the consistent histories interpretation are by no means necessary logical consequences of the mathematical formalism. On the contrary, the formalism of the consistent histories approach is quite independent from the details of the interpretation adopted.

[5] This is the reason why it is meaningless to talk about reductions of the state at the times of measurements in the realm of histories.

ble with the notion of observable as self-adjoint operator. In Section 4 we present the generalized effect history approach which incorporates the generalized notion of observable put forward in Section 2. It should be stressed that the approach presented in Section 4 of this work is equivalent to the approach in Ref. [43] and more general than the approach in Ref. [42]. However, making use of a theorem due to Foulis and Bennett considerably improves and simplifies the presentation (and sets right a mistake in Lemma 4 in [42]). Section 5 presents the summary. Some background material has been collected in Appendices A and B. These appendices recall some well-known facts to establish terminology, and also present some less standard results. The material presented in the Appendix A is used at many places in this paper (often without further notice). Before proceeding with Section 2, the reader is invited to go rapidly through the Appendix A.

Notations and Conventions

Throughout this work we will make use of Dirac's well-known ket and bra notation to denote vectors in Hilbert space and dual vectors in the dual Hilbert space respectively.

Throughout this work $\mathfrak{H}$ denotes some Hilbert space, $\mathcal{P}(\mathfrak{H})$ denotes the lattice of all projection operators on $\mathfrak{H}$, $\mathcal{B}(\mathfrak{H})$ denotes the set of all bounded operators on $\mathfrak{H}$.

2 Operational Quantum Physics

2.1 States in Quantum Mechanics

In standard Hilbert space quantum mechanics the *state* of some quantum mechanical system at time t comprises all probabilistic predictions of quantum mechanics at time t for the system in question.

Let S denote a quantum mechanical system with associated Hilbert space $\mathfrak{H}$. In standard quantum mechanics the set of all possible states of a quantum mechanical system is given by the set of all density operators on $\mathfrak{H}$. [6] It is often stated that the *pure states*, i.e., states of the form $\varrho = |\psi\rangle\langle\psi|$, provide the most detailed possible description of the system in question. Accordingly, such states are often referred to as states of maximum information. In contrast, *mixed* or *nonpure states*, i.e., states which are not one-dimensional projection operators, are often said to provide incomplete descriptions or to be states of less than maximal information. This is due to the fact that every density operator ϱ has a decomposition as $\varrho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ where $\{|\psi_i\rangle\}$ denotes an orthonormal system in $\mathfrak{H}$. It is at first sight tempting to interpret ϱ as a mixture of pure states $|\psi_i\rangle\langle\psi_i|$ with weights $p_i \geq 0$. However, the decomposition of ϱ is unique if and only if ϱ has no degenerate Eigenvalue. Moreover, any density operator ϱ admits infinitely many convex decompositions into (possibly nonorthogonal) pure states. Accordingly, mixed states **do not** admit an *ignorance interpretation* contrary to what is suggested by the nomenclature. According to the ignorance interpretation of mixed states, an individual system S prepared in the state ϱ is actually in one of the component states $|\psi_i\rangle$ with probability p_i. That the ignorance interpretation of mixed states is problematic was already recog-

[6] For simplicity we consider only systems without superselection rules.

nized by Fano [53]. [7] Hence, in any interpretation which asserts that quantum mechanics describes individual systems it is natural to treat pure and nonpure states on the same footing and not to regard pure states to be more fundamental than nonpure states. This is in accordance with Gleason's Theorem (cf. [54, 55]) which can be used as another more formal argument in favour of this assertion.

Consequently, there is a widespread agreement that states should be identified with density operators on Hilbert space. However, another possibility would be to adopt a more restricted notion of state according to which states are identified with rays in Hilbert space. I.e., the notion of state is restricted to what has above been called 'pure state' and the notion of 'mixed state' is completely discarded. The price to be paid for this is that in certain situations there may be systems with which no pure state can be associated. Such situations can be easily imagined. Consider for instance a system which is part of a compound system. Assume that the compound system is described by some pure state. Then in general there will be no pure state associated with the subsystem.

The notion of *isolated system* is clearly an idealization. Real physical systems (for instance in the laboratory) are never totally isolated from their environments. However, rejecting the idea of mixed state ultimately leads to the conclusion that quantum mechanics is in general only applicable to isolated systems not interacting with their environments. If some real physical system is initially in a pure state, then due to the interaction with its environment this pure state will in general develop to a mixed state. In general only the overall time development of the system plus its environment is unitary.

The identification of states with density operators is also forced upon us by Gleason's theorem (cf. [55]) and by the requirement that self-adjoint operators represent quantum mechanical observables [56]. The notion of density operator plays also a central role in the theory of the decoherence process [10].

We see that the restriction of the notion of state to rays in Hilbert space is extremely unnatural. It is deeply rooted in the formalism and the structure of Hilbert space quantum mechanics that the states should be identified with general density operators. Nevertheless, it is a possible – although arguably artificial – point of view that only one-dimensional projection operators should be identified with states. We will not, however, adopt this latter point of view in this work. The reason why we dwell on this point is that the situation for states is analogous to the situation for observables. However, in contrast to the situation for states in the case of observables the natural notion of observable inherent in the quantum mechanical formalism is not generally used.

2.2 Observables in Quantum Mechanics

The term "observable" already suggests that an observable is something which can be observed (this terminology is appropriate in the Copenhagen interpretation of quantum mechanics, whereas in realistic interpretations the term "beable" or the term "speakable" would be more appropriate). Every observable $\mathcal{O}$ has a certain range $\Omega_{\mathcal{O}}$ of possible values. We will not impose any restriction, whatsoever, on $\Omega_{\mathcal{O}}$. Quantum mechanics is a probabilistic theory. Hence, an observable $\mathcal{O}$ is fully

[7] In practical experimental situations, the "ignorance interpretation" of mixed states does provide, however, an intuitive and often useful way to think about *ensembles* of systems.

determined by the specification of a σ-algebra $\mathcal{F}_{\varrho,\mathcal{O}}$ of subsets of $\Omega_{\mathcal{O}}$ and a probability measure $p_{\varrho,\mathcal{O}} : \mathcal{F}_{\varrho,\mathcal{O}} \to [0,1]$ for *every* state ϱ. It is natural to consider only the case that $\mathcal{F}_{\varrho,\mathcal{O}}$ is independent of the state ϱ and to write $\mathcal{F}_{\mathcal{O}}$.

The positive and bounded operators E on $\mathfrak{H}$, satisfying

$$0 \leq E \leq 1,$$

are called EFFECT OPERATORS or briefly EFFECTS and the set of all effects on the Hilbert space $\mathfrak{H}$ will be denoted by $\mathfrak{E}(\mathfrak{H})$.

A GENERALIZED OBSERVABLE $\mathcal{O}$ is now a positive-operator-valued (POV) measure on some measurable space $(\Omega_{\mathcal{O}}, \mathcal{F}_{\mathcal{O}})$, i.e., a map $\mathcal{O} : \mathcal{F}_{\mathcal{O}} \to \mathfrak{E}(\mathfrak{H})$, with the properties:

- $\mathcal{O}(A) \geq \mathcal{O}(\emptyset)$, for all $A \in \mathcal{F}_{\mathcal{O}}$;
- Let $\{A_i\}$ be a countable set of disjoint sets in $\mathcal{F}_{\mathcal{O}}$, then $\mathcal{O}(\cup_i A_i) = \sum_i \mathcal{O}(A_i)$, the series converging ultraweakly;
- $\mathcal{O}(\Omega_{\mathcal{O}}) = 1$.

Generalized observables are also called EFFECT-VALUED MEASURES.

Given any state ϱ of $\mathcal{S}$, then every generalized observable $\mathcal{O}$ induces a probability measure $p_{\varrho,\mathcal{O}}$ on the measurable space $(\Omega_{\mathcal{O}}, \mathcal{F}_{\mathcal{O}})$ by

$$p_{\varrho,\mathcal{O}} : \mathcal{F}_{\mathcal{O}} \to [0,1], p_{\varrho,\mathcal{O}}(A) := \mathrm{tr}_{\mathfrak{H}}(\mathcal{O}(A)\varrho).$$

The number $p_{\varrho,\mathcal{O}}(A)$ is interpreted as the probability that the observable $\mathcal{O}$ assumes a value in the set $A \subset \Omega_{\mathcal{O}}$ in the state ϱ.

Conversely, let $p_\varrho : \mathcal{F} \to [0,1]$ be a probability measure on some measurable space $(\Omega, \mathcal{F})$. It is natural to assume that the map $\varrho \mapsto p_\varrho$ preserves the convex structure of the space of all states, i.e., $p_{\sum_i w_i \varrho_i} = \sum_i w_i p_{\varrho_i}$ whenever $0 \leq w_i \leq 1$ with $\sum_i w_i = 1$. Then $p_{|\psi\rangle\langle\psi|}(A)$ induces a unique bounded symmetric sesquilinear form t_A on $\mathfrak{H}$ for every $A \in \mathcal{F}$. From Riesz's theorem it follows that there exists a unique bounded operator $T(A)$ such that $p_{|\psi\rangle\langle\psi|}(A) = \mathrm{tr}_{\mathfrak{H}}(T(A)|\psi\rangle\langle\psi|)$. It is easy to check that the map $A \mapsto T(A)$ is indeed a positive-operator-valued measure on $(\Omega, \mathcal{F})$. Thus, we have seen that POV measures are the most general notion of observable compatible with the probabilistic structure of quantum mechanics. The argument which has led us to this conclusion is also valid if we restrict the notion of state to one-dimensional subspaces of Hilbert space (pure states).

In standard texts on quantum mechanics observables are identified with self-adjoint operators on $\mathfrak{H}$. As a consequence of the spectral theorem, these ordinary observables (associated with self-adjoint operators on $\mathfrak{H}$) are then in one-to-one correspondence with projection-operator-valued Borel measures on the real line $\mathbb{R}$, to wit, with maps $\mathcal{O}_s : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathfrak{H})$, such that $\mathcal{O}_s(\mathbb{R}) = 1$ and $\mathcal{O}_s(\cup_i K_i) = \sum_i \mathcal{O}_s(K_i)$ for every pairwise disjoint sequence $\{K_i\}_i$ in $\mathcal{B}(\mathbb{R})$ (the series converging in the ultraweak topology). $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$ and $\mathcal{P}(\mathfrak{H})$ denotes the set of projection operators on $\mathfrak{H}$. This notion of observable is usually motivated by the requirement that the *expectation value* of every observable should be real. This argument per se does not exclude more general concepts of observables as considered here. Our above discussion shows however that the notion of generalized observable is deeply rooted within Hilbert space quantum mechanics and is naturally and almost automatically forced upon us by the mathematical formalism of quantum mechanics.

Quantum mechanics is totally consistent without POV measures. Quantum mechanics is also consistent without the notion of mixed state. As discussed in the last section, the price to be paid for the abandonment of the notion of mixed state is that in certain situations there are quantum mechanical systems with which no state can be associated. Analogously, abandoning POV measures would have the consequence that in certain measurement situations there would be no observable which actually is measured.

An example for such a measurement situation is easily imagined. This example is essentially due to Ludwig [57]. Consider a measuring device $\mathcal{M}$ consisting of a detector $\mathcal{D}$ and some test particle $\mathcal{T}$ and assume that the detector $\mathcal{D}$ is designed to measure the value of some ordinary observable $\mathcal{O}_\mathcal{T} : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathfrak{H}_\mathcal{T})$ associated with the test particle $\mathcal{T}$. An appropriately prepared incident physical system $\mathcal{Z}$ (e.g., a particle) interacts first with $\mathcal{T}$ and then the detector $\mathcal{D}$ measures the value of the observable $\mathcal{O}_\mathcal{T}$ of the particle $\mathcal{T}$. To obtain the observable $\mathcal{O}_\mathcal{Z}$ measured by the device $\mathcal{M}$ one has to apply the unitary transformation given by the S-matrix S of the interaction between $\mathcal{Z}$ and $\mathcal{T}$ in an appropriate way to the observable measured by $\mathcal{D}$. Let $\varrho_\mathcal{Z}$ denote the initial state of $\mathcal{Z}$ and $\varrho_\mathcal{T}$ denote the initial state of $\mathcal{T}$. Then $\mathcal{O}_\mathcal{Z}$ can be expressed through S, $\varrho_\mathcal{T}$ and $\mathcal{O}_\mathcal{T}$ as a partial trace

$$\mathcal{O}_\mathcal{Z} : \mathcal{B}(\mathbb{R}) \to \mathfrak{E}(\mathfrak{H}_\mathcal{Z}), \mathcal{O}_\mathcal{Z}(A) := \mathrm{tr}_{\mathfrak{H}_\mathcal{T}} \left((1 \otimes \varrho_\mathcal{T}) S^\dagger (1 \otimes \mathcal{O}_\mathcal{T}(A)) S \right),$$

where $\mathfrak{H}_\mathcal{T}$ denotes the Hilbert space of the test particle $\mathcal{T}$ and $\mathfrak{H}_\mathcal{Z}$ denotes the Hilbert space of $\mathcal{Z}$. For realistic physical S-matrices S the expression $\mathrm{tr}_{\mathfrak{H}_\mathcal{T}} \left((1 \otimes \varrho_\mathcal{T}) S^\dagger (1 \otimes \mathcal{O}_\mathcal{T}(A)) S \right)$ is in general not a projection operator but an effect operator on $\mathfrak{H}_\mathcal{Z}$. The observable $\mathcal{O}_\mathcal{Z}$ is so chosen as to conform with the following requirement

$$\mathrm{tr}_{\mathfrak{H}_\mathcal{Z} \otimes \mathfrak{H}_\mathcal{T}} \left(S (\varrho_\mathcal{Z} \otimes \varrho_\mathcal{T}) S^\dagger (1 \otimes \mathcal{O}_\mathcal{T}(A)) \right) = \mathrm{tr}_{\mathfrak{H}_\mathcal{Z}} \left(\varrho_\mathcal{Z} \mathcal{O}_\mathcal{Z}(A) \right),$$

for all $A \subset \mathcal{B}(\mathbb{R})$.

Therefore whether a measuring device measures the value of a generalized observable or of an ordinary observable associated with some self-adjoint operator may depend on an arbitrary cut between the system and the apparatus. This argument can be formalized, see Ref. [58]. In summary, we see that if we abandon the idea that POV measures represent the observables and stick only to self-adjoint operators, then it cannot be said that in the above example of a measuring process the measuring apparatus $\mathcal{M}$ measures the value of some observable of $\mathcal{Z}$.

In the last decades many examples for POV measures have been discussed in the literature, which show that POV measures are also useful. In the sequel I will only briefly mention a few particularly interesting examples

- The problem of finding the 'phase' observable canonically conjugate to the number operator for a harmonic oscillator has a long history. A satisfactory self-adjoint operator representing 'phase' has not been found and possibly does not exist [59]. However, in terms of POV measures the problem of defining a 'phase' observable has a simple solution unifying various approaches to the problem. This can be found in Ref. [60, 61]. In this reference Busch, Grabowski and Lahti consider the following POV measure M_o on the Borel sets of $[0, 2\pi[$:

$$M_o : \mathcal{B}([0, 2\pi[) \to \mathfrak{E}(\mathfrak{H}), I \mapsto M_o(I) := \sum_{m,n} \frac{|n\rangle\langle m|}{2\pi} \int_I \exp(i(n-m)\varphi)d\varphi,$$

where $\mathfrak{H}$ denotes the Fock space of the harmonic oscillator. Busch et al. put forward the suggestion that M_o represents the appropriate quantum phase observable for the harmonic oscillator. The POV measure M_o is canonically conjugate to the number operator N in the following sense:

$$e^{iN\varphi} M_o(I) e^{-iN\varphi} = M_o(I + \varphi),$$

where $I + \varphi := \{\varphi' + \varphi \bmod 2\pi \mid \varphi' \in I\}$. The relation of M_o and various previous proposals to the quantum phase problem is discussed in detail in Ref. [60].

- The simultaneous unsharp specification of several noncommuting observables can conveniently be described with the aid of POV measures. An unsharp position-momentum observable is described in the book by Davies [62].

Many further examples and arguments in favour of POV measures can be found in the monographs by Davies [62] and Busch, Grabowski and Lahti [61] and references therein.

Summarizing our discussion in the last two sections, we have seen that the most general and natural notions of state and observable in quantum mechanics are density operators on Hilbert space and POV measures, respectively. These notions are particularly reasonable in the context of realistic interpretations of quantum mechanics. I have further briefly argued that these notions are not only compatible with quantum mechanics but are in fact useful.

2.3 The Projection Postulate

In this subsection we make a brief digression to the intricacies of the orthodoxy.

One of the central concepts in the usual formulations of the orthodox interpretation of quantum mechanics is the concept of *reduction of the wave packet* as a reaction to measurements performed on the system in question. This concept is closely related to the idea of *ideal* measurement. A measurement is said to be ideal if it is repeatable (loosely speaking an ideal measurement minimizes the disturbance of the state due to the measurement). The projection postulate can be formulated as follows: consider a quantum system in the state ϱ. Suppose an ideal measurement of the ordinary observable $\mathcal{A}$ is performed on the system and suppose that the Eigenvalue a is found upon this measurement, then the state of the system undergoes the nonunitary transformation as a response to the measurement

$$\varrho \mapsto \frac{P(a)\varrho P(a)}{\mathrm{tr}(P(a)\varrho)},$$

where $P(a)$ denotes the projection operator onto the Eigenspace corresponding to the Eigenvalue a in the spectral decomposition of $\mathcal{A}$. This is the well-known *Lüders-von Neumann projection postulate.* It can essentially be traced back to von Neumann [4] but the formulation given here is due to Lüders [63]. The objectification problem of quantum mechanics can now reformulated as the problem of how the reduction of the state can be understood as a *physical process.*

The projection postulate can be generalized to sequential measurements. The resulting formula has seemingly been first given by Wigner [64]. Suppose that a quantum system initially in the state ϱ is exposed to a succession of ideal measurements and suppose that at time t_1 the observable $\mathcal{A}_1$ is measured with the result a_1, at time t_2 the observable $\mathcal{A}_2$ is measured with the result a_2 and so on, then the final state after the last measurement is

$$\frac{P(a_n)\cdots P(a_2)P(a_1)\varrho P(a_1)P(a_2)\cdots P(a_n)}{\mathrm{tr}(P(a_n)\cdots P(a_2)P(a_1)\varrho P(a_1)P(a_2)\cdots P(a_n))}.$$

Notice that we are using the Heisenberg picture and that accordingly the projection operators are time dependent. For notational simplicity the time dependence is suppressed.

In the consistent histories approach the projection postulate and its underlying philosophy is rejected. However, in its construction the following heuristic principle is adopted: *The results of measurement theory – in particular the Lüders-von Neumann rule for the wave function collapse caused by measurements and its generalization for sequential measurements (the Wigner formula) – have to be a* **consequence** *of the formalism of the consistent histories approach.*

2.4 Quantum Properties

Consider propositions of the form "The value of the observable $\mathcal{O}$ is in the set $A \in \mathcal{F}_{\mathcal{O}}$." If $\mathcal{O}$ is an ordinary observable we can associate the projection operator $\mathcal{O}(A)$ with this proposition. If $\mathrm{tr}(\mathcal{O}(A)\varrho) = 1$ for some state ϱ, then we say that the proposition corresponding to $\mathcal{O}(A)$ is *true* in the state ϱ or briefly that $\mathcal{O}(A)$ is true in the state ϱ. [8] Clearly, there may be another observable $\mathcal{O}'$ such that $\mathcal{O}(A) = \mathcal{O}'(A')$ for some $A' \in \mathcal{F}_{\mathcal{O}'}$. Hence, the propositions associated with $\mathcal{O}(A)$ and $\mathcal{O}'(A')$ can only simultaneously be true. This observation has led von Neumann to the notion of *property* of a quantum mechanical system. According to von Neumann, there is a one-to-one correspondence between properties of quantum mechanical systems and projection operators on Hilbert space. Birkhoff and von Neumann have studied in their classic work [65] the structure of the space of all projection operators of a quantum system. [9] They proved that this space carries the structure of a nondistributive and hence non-Boolean lattice; for more details see Appendix A.5.

In classical physics meaningful propositions about a system are typically of a similar form: "The value of the observable $\mathcal{A}$ is in the set A," for some appropriately chosen subset A of $\mathbb{R}$. Since observables in classical physics are real-valued functions on phase space $\mathfrak{P}$, we can associate with every meaningful proposition the subset

$$\mathfrak{P}[\mathcal{A} \in A] := \{\mathfrak{p} \in \mathfrak{P} \mid \mathcal{A}(\mathfrak{p}) \in A\}$$

[8] We will discuss the meaning of the notion of truth briefly but in more depth in the next section.

[9] Notice, that the original terminology in von Neumann [4] and Birkhoff and von Neumann [65] is in the spirit of the Copenhagen interpretation.

of phase space. In contrast to the quantum case, the set of all propositions about a classical physical system is isomorphic to the space of all subsets of the classical phase space, specifically it constitutes a Boolean lattice. [10] Birkhoff and von Neumann argued that the characteristic feature which distinguishes the quantum mechanical propositional calculus from the classical propositional calculus is the breakdown of the distributive law in the quantum case.

The Hilbert lattice of all projection operators does not satisfy all requirements which might intuitively be associated with a space of properties. The situation is even worse, when we consider the set of propositions about a quantum mechanical system and take into account the full set of generalized observables as introduced in the last section. In this case the space of all "properties" of a quantum system is isomorphic to the space of all effect operators on Hilbert space. The latter space carries the structure of a D-poset, see Appendix A. The use of the term "property" with regard to effect operators is counterintuitive. I shall thus refrain from using such a terminology. Instead, I will use a different terminology better suited to the structure of the space of effect operators and a realistic attitude.

Two propositions are said to be *equivalent* if they correspond to the same effect operator. Hence, there is a one-to-one correspondence between effect operators on Hilbert space and equivalence classes of propositions. I shall say that the effect operators represent the possible *events* which may occur in the physical system in question. The idealized notion of *real event* as proposed here describes irreversible transitions from the possible to the actual. The notion of event is clearly an idealization, but hopefully a useful one. The notion of event is not assumed to be restricted to the macroscopic realm. Quantum events are assumed to take place also at a microscopic scale. Accordingly, quantum mechanics is viewed as a fundamentally stochastic theory describing the evolving pattern of events taking place in the real world. This evolutionary picture of physics has been recently put forward by Haag [66, 67].

3 Standard Consistent Histories

3.1 Homogeneous Histories

We consider a quantum mechanical system $\mathcal{S}$ without superselection rules represented by a separable complex Hilbert space $\mathfrak{H}$ and a Hamiltonian operator H. Every physical state of the considered system is mathematically represented by a density operator on $\mathfrak{H}$, i.e., a linear, positive, trace-class operator on $\mathfrak{H}$ with trace 1. We denote the set of all trace-class operators on $\mathfrak{H}$ by $\mathcal{T}(\mathfrak{H})$ and the set of all density operators on $\mathfrak{H}$ by $\mathcal{T}(\mathfrak{H})_1^+$. The time evolution is governed by the unitary operator $U(t',t) = \exp(-i(t'-t)H/\hbar)$ which maps states at time t into states at time t' and satisfies $U(t'',t')U(t',t) = U(t'',t)$ and $U(t,t) = 1$.

In standard nonrelativistic quantum mechanics the observables are identified with PV Borel measures on the real line and, accordingly, to every sequence of measurement outcomes there corresponds a sequence of projection operators. Led by the heuristic principle stated in Section 2.3,

[10] I do neither discuss here the question of whether some equivalence relation on the space of propositions should be taken into account nor how such equivalence relation should be chosen.

the basic idea in the consistent histories approach is to abstract such *histories* from their concrete realization as a sequence of measurement outcomes and to think of histories as independent entities in their own right. Histories are then, loosely speaking, sequences of projection operators on $\mathfrak{H}$. This idea can be formalized as follows:

Definition 3.1 *A* HOMOGENEOUS HISTORY *is a map $h : \mathbb{R} \to \mathcal{P}(\mathfrak{H}), t \mapsto h_t$. We call $t_i(h) := \inf(t \in \mathbb{R} \cup \{-\infty, \infty\} \mid h_t \neq 1)$ the* INITIAL *and $t_f(h) := \sup(t \in \mathbb{R} \cup \{-\infty, \infty\} \mid h_t \neq 1)$ the* FINAL TIME *of h, respectively. Furthermore, the* SUPPORT OF *h is given by $\mathfrak{s}(h) := \{t \in \mathbb{R} \mid h_t \neq 1\}$. If $\mathfrak{s}(h)$ is finite, countable or uncountable, then we say that h is a* FINITE, COUNTABLE *or* UNCOUNTABLE HISTORY *respectively. The space of all homogeneous histories will be denoted by $\mathcal{H}(\mathfrak{H})$, the space of all finite homogeneous histories by $\mathcal{H}_{fin}(\mathfrak{H})$ and the space of all finite homogeneous histories with support S by $\mathcal{H}_S(\mathfrak{H})$.*

In this work we focus attention on finite histories. Infinite or even continuous histories are much more difficult to handle, see [37]. In the following we will identify every homogeneous history h with the string of its nontrivial projection operators, i.e., we write $h \simeq \{h_{t_k}\}_{t_k \in \mathfrak{s}(h)}$.

Furthermore, to every finite homogeneous history $h \in \mathcal{H}_{fin}(\mathfrak{H})$ we associate its CLASS OPERATOR WITH RESPECT TO THE FIDUCIAL TIME t_0

$$
\begin{aligned}
C_{t_0}(h) &:= U(t_0, t_n)h_{t_n}U(t_n, t_{n-1})h_{t_{n-1}}...U(t_2, t_1)h_{t_1}U(t_1, t_0) && (1) \\
&= U(t_0, t_i(h))h_{t_n}(t_n)h_{t_{n-1}}(t_{n-1})...h_{t_1}(t_1)U(t_i(h), t_0), && (2)
\end{aligned}
$$

where we have defined the Heisenberg picture operators

$$h_{t_k}(t_k) := U(t_k, t_i(h))^\dagger h_{t_k} U(t_k, t_i(h))$$

with respect to the initial time $t_i(h)$ of h.

The following definition is motivated by Wigner's formula and the heuristic principle mentioned in Section 2.3.

Definition 3.2 *Let the state of a quantum mechanical system at time t_0 be given by the density operator $\varrho(t_0)$. For every pair h and k of finite homogeneous histories we define the* DECOHERENCE WEIGHT OF h AND k *by*

$$\mathrm{d}_\varrho(h, k) := \mathrm{tr}\left(C_{t_0}(h)\varrho(t_0)C_{t_0}(k)^\dagger\right). \qquad (3)$$

The functional $\mathrm{d}_\varrho : \mathcal{H}_{fin}(\mathfrak{H}) \times \mathcal{H}_{fin}(\mathfrak{H}) \to \mathbb{C}, (h, k) \mapsto \mathrm{d}_\varrho(h, k)$ will be called the HOMOGENEOUS DECOHERENCE FUNCTIONAL ASSOCIATED WITH THE STATE ϱ.

In view of the heuristic principle stated in Section 2.3 it is natural to attempt to interpret the value $d_\varrho(h, h)$ as the "probability" of the homogeneous history h. The problem to be addressed in this section is the problem of whether this interpretation makes sense, to wit, the problem of how and in what sense d_ϱ can be extended to a probability functional.

Following Isham [24], we will proceed in several steps. The first question to be addressed is the problem of what the appropriate mathematical representations of the grammatical connectives "and", "or" and "not" in the realm of histories are, so that under appropriate circumstances we can talk about propositions like "the history h is realized or (and) the history k is realized" etc.

In the second step (see Theorem 3.15), the functional d_ϱ will be extended to the space of generalized ("composed") history propositions, since as it stands, say, the expression $\mathsf{d}_\varrho(h \text{ or } k, h \text{ or } k)$ is not defined. The resulting extension d_ϱ is not additive, i.e., in general one has $d_\varrho(h \text{ or } k, h \text{ or } k) \neq \mathsf{d}_\varrho(h, h) + \mathsf{d}_\varrho(k, k)$.

Thus, in the last step we formulate the necessary and sufficient condition a set of histories has to satisfy in order that d_ϱ induces a probability measure on it, see Section 3.1.3. Only for those sets $\mathcal{C}$ of histories satisfying the *consistency condition* the interpretation of the number $d_\varrho(h, h)$ as probability of the history $h \in \mathcal{C}$ is justified.

3.2 Inhomogeneous Histories

For every finite subset S of $\mathbb{R}$ we can consider the Hilbert tensor product $\otimes_{t \in S}\mathfrak{H}$ and the algebra $\mathcal{B}_S^{\otimes}(\mathfrak{H})$ of bounded linear operators on $\otimes_{t \in S}\mathfrak{H}$. It was pointed out by Isham [24] that for any fixed S there is an injective (but not surjective) correspondence σ_S between finite histories with support S and elements of $\mathcal{B}_S^{\otimes}(\mathfrak{H})$ given by

$$\sigma_S : \mathcal{H}_S(\mathfrak{H}) \to \mathcal{B}_S^{\otimes}(\mathfrak{H}), h \simeq \{h_{t_k}\}_{t_k \in S} \mapsto \otimes_{t_k \in S} h_{t_k}. \tag{4}$$

The finite homogeneous histories with support S can therefore be identified with projection operators on $\otimes_{t \in S}\mathfrak{H}$. Hence, without the risk of confusion, we will almost everywhere in this work briefly write h instead of $\sigma_S(h)$. The set of all projection operators on $\otimes_{t \in S}\mathfrak{H}$ will in the sequel be denoted by $\mathcal{P}_S^{\otimes}(\mathfrak{H})$. Obviously, not all projection operators in $\mathcal{P}_S^{\otimes}(\mathfrak{H})$ have the form $\sigma_S(h)$ for some $h \in \mathcal{H}_S(\mathfrak{H})$.

When a homogeneous history vanishes for some $t_0 \in \mathbb{R}$, i.e., $h_{t_0} = 0$, then we say that h is a ZERO HISTORY. All zero histories are collectively denoted by 0, slightly abusing the notation.

Definition 3.3 *Let $h, k \in \mathcal{H}(\mathfrak{H})$. We say that k is* COARSER THAN *h if $h_t \leq k_t$ for all $t \in \mathbb{R}$ and write $h \leq k$. If furthermore $h \neq k$, then we write $h < k$. The set $\mathcal{H}(\mathfrak{H})$ equipped with the relation $\leq$ is a partially ordered set.*

Definition 3.4 *Two homogeneous histories h and k are said to be* DISJOINT *if there is some $t \in \mathbb{R}$ such that $h_t k_t = 0$.*

The identification of finite homogeneous histories with support S with projection operators on $\otimes_{t \in S}\mathfrak{H}$ allows for the introduction of a much broader class of histories. To this end we recall the well-known fact that the set $\mathcal{P}(\mathfrak{H})$ of projection operators on a Hilbert space $\mathfrak{H}$ carries the structure of an orthocomplemented complete lattice, see Appendix A.5.

Definition 3.5 *Let S be a finite subset of $\mathbb{R}$, then we call the space $\mathcal{P}_S^{\otimes}(\mathfrak{H})$ of projection operators on $\otimes_{t \in S}\mathfrak{H}$ the* SPACE OF FINITE INHOMOGENEOUS HISTORIES WITH SUPPORT S. *The direct limit of the directed system $\{\mathcal{P}_S^{\otimes}(\mathfrak{H}) \mid S \subset \mathbb{R} \text{ finite}\}$ will be called* THE SPACE OF ALL FINITE INHOMOGENEOUS HISTORIES WITH ARBITRARY SUPPORT *and will be denoted by $\mathcal{P}_{fin}^{\otimes}(\mathfrak{H})$.*

The definition of the direct limit of a directed system of D-posets can be found in Appendix A.2.3. The lattice operations on $\mathcal{P}^{\otimes}_{S}(\mathfrak{H})$ induce corresponding operations on the finite homogeneous histories in $\mathcal{H}_{fin}(\mathfrak{H})$, which are explicitly described in the following remarks. These operations are the sought mathematical representations of the grammatical connectives "and", "or" and "not" in the history formalism.

Remark 3.6 *Let $h, k \in \mathcal{H}_{fin}(\mathfrak{H})$ be two finite homogeneous histories, then the* JOIN *$h \vee k$ of h and k is defined to be the unique finite history with support $\mathfrak{s}(h) \cup \mathfrak{s}(k)$ which is represented in $\mathcal{P}^{\otimes}_{\mathfrak{s}(h)\cup\mathfrak{s}(k)}(\mathfrak{H})$ by $\left(\otimes_{t_i\in\mathfrak{s}(h)}h_{t_i}\right) \bigvee \left(\otimes_{s_j\in\mathfrak{s}(k)}k_{s_j}\right)$. The history $h \vee k$ is in general not a homogeneous but an inhomogeneous history. The* JOIN *$\bigvee_j h_j$ of any finite sequence $\{h_j\}$ of pairwise disjoint homogeneous histories is analogously defined to be the unique finite history with support $\bigcup_j \mathfrak{s}(h_j)$ which is represented in $\mathcal{P}^{\otimes}_{\bigcup_j\mathfrak{s}(h_j)}(\mathfrak{H})$ by $\bigvee_j \left(\otimes_{t_i\in\mathfrak{s}(h_j)}h_{t_i}\right)$.*

Remark 3.7 *Let $h, k \in \mathcal{H}_{fin}(\mathfrak{H})$ be two finite homogeneous histories, then the* MEET *$h \wedge k$ of h and k satisfies that $(h\wedge k)_t := h_t \wedge k_t$ is the projection operator on the intersection of the ranges of h_t and k_t for all $t \in \mathbb{R}$. The meet operation maps pairs of finite homogeneous histories to a finite homogeneous history.*

Remark 3.8 *Let h be a finite homogeneous history with support $\mathfrak{s}(h)$, then $\neg h$ is the unique history with support $\mathfrak{s}(h)$ which in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ is represented by $1 - \bigotimes_{t\in\mathfrak{s}(h)} h_t$. We call $\neg h$ the* NEGATION *of h. The negation $\neg h$ of a finite homogeneous history h will in general be inhomogeneous. Obviously the negation satisfies $h \vee \neg h = 1$ and $h \wedge \neg h = 0$. It is clear that $\neg h$ is uniquely determined by these two conditions.*

Lemma 3.9 *Let S be a finite subset of $\mathbb{R}$, then the set $\mathcal{P}^{\otimes}_{S}(\mathfrak{H})$ is an orthocomplemented complete lattice.*

Remark 3.10 *The join, meet and orthocomplementation operations on $\mathcal{P}^{\otimes}_{S}(\mathfrak{H})$ (where S is a finite subset of $\mathbb{R}$) and on $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ are denoted by the same symbols (slightly abusing the notation).*

In [24, 68] it is explained how to imbed $\mathcal{P}^{\otimes}_{S}(\mathfrak{H})$ into an infinite tensor product of operator algebras and how to furnish the latter with a Hilbert lattice structure.

Definition 3.11 *Two (possibly inhomogeneous) finite histories h and k are said to be* DISJOINT *if $h \leq \neg k$, where $\leq$ is the partial order on $\mathcal{P}^{\otimes}_{\mathfrak{s}(h)\cup\mathfrak{s}(k)}(\mathfrak{H})$. We write $h \perp k$.*

Lemma 3.12 *Let h and k denote two disjoint finite histories, then $h \wedge k = 0$.*

Remark 3.13 *For every finite $S \subset \mathbb{R}$ the meet, join and orthocomplementation operations on $\mathcal{P}^{\otimes}_{S}(\mathfrak{H})$ induce a meet, join and an orthocomplementation operation on $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ respectively which will be denoted by the same symbols.*

Definition 3.14 *Let $\mathcal{A}$ denote a finite collection $\{h_k\}$ of histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. Then $\mathcal{A}$ is said to be* DISJOINT *if each pair of histories in $\mathcal{A}$ is disjoint. $\mathcal{A}$ is said to be* COMPLETE *if $\bigvee_k h_k = 1$.*

Let $S \subset \mathbb{R}$. A history $h \in \mathcal{H}_S(\mathfrak{H})$ is called a SIMPLE HISTORY if h_t is a projection operator on a one dimensional subspace of $\mathfrak{H}$ for every $t \in S$. Denote the set of finite inhomogeneous histories which can be generated from the set of simple homogeneous histories in $\mathcal{H}_S(\mathfrak{H})$ by the application of a finite number of $\vee$ operations by $\mathcal{F}_S(\mathfrak{H})$. The class operators can be extended to $\mathcal{F}_S(\mathfrak{H})$ by requiring that C_{t_0} is additive in the following sense

$$\begin{aligned} C_{t_0}(h \vee k) &:= C_{t_0}(h) + C_{t_0}(k) \text{ whenever } h \perp k & (5) \\ C_{t_0}(\neg h) &:= 1 - C_{t_0}(h). & (6) \end{aligned}$$

These definitions are compatible with the lattice theoretical identities $\neg(h \vee k) = (\neg h) \wedge (\neg k)$ and $\neg(h \wedge k) = (\neg h) \vee (\neg k)$. Notice, that Equation 6 is a consequence of Equation 5. The fiducial time t_0 can be chosen completely arbitrary.

Using Equation 5, the homogeneous decoherence functional can in an obvious way be extended to the set $\mathcal{F}_S(\mathfrak{H})$.

However, a much stronger result is true. The homogeneous decoherence functional d_ϱ from Definition 3.2 can be extended to a functional defined for arbitrary pairs of inhomogeneous histories.

Let S be a finite subset of $\mathbb{R}$ and define $\mathfrak{H}(S) := \otimes_{t \in S} \mathfrak{H}$. For a finite dimensional Hilbert space $\mathfrak{H}$ Isham, Linden and Schreckenberg [39] have proven that for every ϱ there exists a unique trace class operator $\mathfrak{X}_\varrho$ on $\mathfrak{H}(S) \otimes \mathfrak{H}(S)$ such that d_ϱ can be written as

$$\mathrm{d}_\varrho(h, k) = \mathrm{tr}_{\mathfrak{H}(S) \otimes \mathfrak{H}(S)}(h \otimes k \mathfrak{X}_\varrho),$$

provided $\mathfrak{s}(h) \subset S$ and $\mathfrak{s}(k) \subset S$. A proof for this assertion can be found in Appendix B. For an infinite dimensional Hilbert space $\mathfrak{H}$ the author and Wright [45] have shown that $\mathfrak{X}_\varrho$ is in general only bounded but not of trace class. Accordingly the homogeneous decoherence functional d_ϱ cannot be extended to a finitely valued decoherence functional on the space of all histories but we have to allow for infinite values of the extension. The main ideas are informally outlined in Appendix B, the details can be found in [45]. Thus we have

Theorem 3.15 *The homogeneous decoherence functional* $\mathrm{d}_\varrho : \mathcal{H}_{fin}(\mathfrak{H}) \times \mathcal{H}_{fin}(\mathfrak{H}) \to \mathbb{C}$, $(h, k) \mapsto \mathrm{d}_\varrho(h, k)$ *can be uniquely extended to a* DECOHERENCE FUNCTIONAL $d_\varrho : \mathcal{P}^{\otimes}_{fin}(\mathfrak{H}) \times \mathcal{P}^{\otimes}_{fin}(\mathfrak{H}) \to \mathbb{C} \cup \{\infty\}$. *Let h, h' and k denote finite histories. The decoherence functional d_ϱ satisfies for all* $h, h', k \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$

- $d_\varrho(h, h) \in \mathbb{R}$ *and* $d_\varrho(h, h) \geq 0$.
- $d_\varrho(h, k) = d_\varrho(k, h)^*$, *whenever* $d_\rho(h, k) \in \mathbb{C}$.
- $d_\varrho(1, 1) = 1$.
- $d_\varrho(h \vee h', k) = d_\varrho(h, k) + d_\varrho(h', k)$, *whenever* $h \perp h'$ *and* $d_\varrho(h \vee h', k), d_\varrho(h, k)$ *and* $d_\varrho(h', k) \in \mathbb{C}$.
- $d_\varrho(0, h) = 0$, *for all* h.

- $d_\varrho(h,k) \in \mathbb{C}$ *for all* $h,k \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *iff* $\mathfrak{H}$ *is finite dimensional.*

The proof for the first item $d_\varrho(h,h) \geq 0$ and for the second item $d_\varrho(h,k) = d_\varrho(k,h)^*$ can be found in Appendix B. It is clear that d_ϱ is also σ-orthoadditive on $\mathcal{P}^{\otimes}_S(\mathfrak{H})$ for every finite $S \subset \mathbb{R}$ in the finite sector of d_ϱ.

3.3 Consistent Sets of Histories

Definition 3.16 *Let h and k be two disjoint histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. Two histories h and k are said to be* PRECONSISTENT WITH RESPECT TO THE STATE ϱ *if $d_\varrho(h,k) \in \mathbb{C}$ and if* Re $d_\varrho(h,k) = 0$. *Any collection C of histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ is said to be* PRECONSISTENT WITH RESPECT TO THE STATE ϱ *if every pair of disjoint histories in C is preconsistent with respect to the state ϱ.*
Let C' be a Boolean lattice of histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ with respect to the meet, join and orthocomplementation induced from $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ (see Remark 7) such that $0_{C'} = 0 \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. The unit in C' will be denoted by $1_{C'}$. Such a Boolean lattice C' of histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ is said to be CONSISTENT WITH RESPECT TO THE STATE ϱ *if (i) $d_\varrho(h,k) \in \mathbb{C}$ for all $h,k \in C'$, (ii) C' is preconsistent with respect to the state ϱ and (iii) $d_\varrho(1_{C'}, 1_{C'}) \leq 1$.*

The reader may wonder what is meant by the "zero" and the "unit" in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. Technically, $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ may be thought of as the direct limit of the directed set $\{\mathcal{P}^{\otimes}_S(\mathfrak{H}) | S \subset \mathbb{R}, S \text{ finite }\}$. According to Appendix A.2.3 this direct limit carries the structure of a D-poset (actually, it also carries the structure of a lattice) and it is understood that the zero history and the unit history of $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ are the zero and the unit in this D-poset structure, respectively.

The condition Re $d_\varrho(h,k) = 0$ is often expressed in physical terms by saying that the events h and k have vanishing interference in the state ϱ.

The notion of consistency is important because it is the key to a probability interpretation of the numbers $d_\varrho(h,h)$ for some (pre-)consistent sets of histories. The Definition 3.16 of a consistent set of histories is the minimal necessary requirement that d_ϱ induces a probability measure on the consistent set in question as we will discuss more fully below.
Note that our above terminology in Definition 3.16 differs somewhat from the terminology used by other authors. Further, some authors discuss more severe conditions. These authors call a pair h, k of histories *weakly decoherent* if it satisfies Re $d_\varrho(h,k) = 0$ and *mediumly decoherent* if it satisfies $d_\varrho(h,k) = 0$.

There are other related notions of decoherence and consistency in the literature, see, e.g. [10]. However, since Definition 3.16 represents the mathematically minimal requirement we shall stick to it and will not consider the stronger conditions. Let us recall that usually a probability space is defined to be a triple $(\Omega, \mathcal{A}, p)$, where Ω is an arbitrary set, $\mathcal{A}$ is a Boolean σ-algebra of subsets of Ω and p is a probability measure on $\mathcal{A}$. This can be generalized as follows

Definition 3.17 *Let $\mathcal{L}$ be a partially ordered set and $\mathcal{B} \subset \mathcal{L}$ be a Boolean lattice. A nonnegative valuation $m : \mathcal{B} \longrightarrow \mathbb{R}^+$ on $\mathcal{B}$ which is additive*

$$m\left[\bigvee_{k=1}^{N} \alpha_k\right] = \sum_{k=1}^{N} m[\alpha_k], \text{ if } \alpha_k \wedge \left(\bigvee_{i=1}^{k-1} \alpha_i\right) = 0, \text{ for every } k < N,$$

is called a FINITE MEASURE ON $\mathcal{B}$. *If* $\mathcal{B}$ *is a Borel lattice, then* N *may be taken to be* ∞. *In this case* m *is* σ*-additive. If* $\mathcal{B}$ *is not a Borel lattice, then* N *is always finite. If furthermore* $m[1_{\mathcal{B}}] = 1$, *then* m *is called a* PROBABILITY MEASURE ON $\mathcal{B}$ *and the triple* $(\mathcal{L}, \mathcal{B}, m)$ *is called a* PROBABILITY LATTICE.

A Borel lattice is a Boolean σ-lattice [71].

Theorem 3.18 *Let* $C \subset \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *be a Boolean lattice. If* C *is consistent with respect to the state* ϱ, *then the triple* $(\mathcal{P}^{\otimes}_{fin}(\mathfrak{H}), C, p_\varrho)$ *is a probability lattice, where* p_ϱ *is defined by*

$$p_\varrho : C \to \mathbb{R}^+, p_\varrho(h) := \frac{d_\varrho(h,h)}{d_\varrho(1_C, 1_C)}. \tag{7}$$

The proof is straightforward.

In the literature it is often tacitly assumed that the preconsistent set of histories under consideration forms (or generates) a Boolean lattice so that a probability interpretation of the diagonal values of the decoherence functional makes sense. The probability defined by Equation 7 can for finite homogeneous histories be interpreted as conditional probability, namely as the probability of the sequence of the propositions $h_{t_f} = h_{t_k}, ..., h_{t_{k-j}}$ given that the sequence of propositions $h_{t_{k-j-1}}, ..., h_{t_0}$ is realized.

Lemma 3.19 *Let* $(\mathcal{P}^{\otimes}_{fin}(\mathfrak{H}), C, p_\varrho)$ *be the probability lattice from Theorem* 3.18, *where* p_ϱ *is defined by Equation* 7, *then for all* $h, k \in C$

- $0 \leq p_\varrho(h) \leq 1$.
- $p_\varrho(h \vee k) + p_\varrho(h \wedge k) = p_\varrho(h) + p_\varrho(k)$.
- $p_\varrho(h) \leq p_\varrho(k)$ *whenever* $h \leq k$.

Corollary 3.20 *Let* $C \subset \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *be a Boolean lattice. Then* C *is a preconsistent set of histories w.r.t. the state* ϱ *if and only if* $0_C = 0 \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$, $d_\varrho(1_C, 1_C) \leq 1$ *and if every pair* h, k *of histories in* C *satisfies*

$$d_\varrho(h \vee k, h \vee k) + d_\varrho(h \wedge k, h \wedge k) = d_\varrho(h,h) + d_\varrho(k,k). \tag{8}$$

Remark 3.21 *We notice that* d_ϱ *induces also probability functionals on sets of histories which are not Boolean lattices. Let* C *be a preconsistent set of pairwise disjoint histories, then* $m_\varrho : C \to \mathbb{R}^+, m_\varrho(h) := d_\varrho(h,h) / \left(\sum_{k \in C} d_\varrho(k,k)\right)$ *is an additive functional on* C *and* $m_\varrho(h)$ *can be interpreted as probability of* $h \in C$. *However, since* C *generates a Boolean sublattice of* $\mathcal{K}_{fin}(\mathfrak{H})$ *on which* d_ϱ *induces a probability measure extending* m_ϱ, *it is enough to consider Boolean algebras of histories.*

3.4 The Consistent Histories Interpretation

The interpretation of quantum mechanics based on the consistent histories approach has been developed by Robert Griffiths and Roland Omnès. In this subsection I briefly summarize the most important aspects of the version of this interpretation adopted in this review. The version presented here differs in some minor points from the original expositions by Omnès and by Griffiths (which differ themselves in some aspects). And it differs slightly from the version given in [42]. I do not claim that Omnès and Griffiths will necessarily fully agree with my presentation.

At the heart of the consistent histories interpretation lies the following philosophical maxim: *Whether or not some assertion or proposition is meaningful depends upon the context and the framework into which the assertion is placed.* Accordingly, one has always be careful not only to specify the assertion itself but also its context. Propositions are always contextual. In the consistent histories approach this principle is carefully obeyed. According to the consistent histories approach in quantum mechanics probabilistic predictions and state histories are only meaningful with respect to a consistent set of histories. *Per se* (i.e., without the specification of a consistent set of histories) state histories and probabilistic predictions have no meaning.

The consistent histories interpretation of quantum mechanics is a realistic interpretation of quantum mechanics. Quantum mechanics is asserted to be a theory describing individual systems regardless of whether they are open or closed and regardless of whether they are observed or not. The basic ingredients in the formalism of the consistent histories approach are the space of histories on the one hand and the space of decoherence functionals on the other hand.

In standard Hilbert space quantum mechanics the state of some quantum mechanical system comprises all probabilistic predictions of quantum mechanics for the system in question. In the words of Popper [69] *"the real state of a physical system, at any moment, may be conceived as the sum total of its dispositions—or its potentialities, or possibilities, or propensities."* This idea of the notion of state can be carried over to the history formulation of quantum mechanics: it is in this sense that decoherence functionals can be said to represent the *state* of a system described by the history version of quantum mechanics. This notion of state of a system has a peculiar transtemporal meaning.

In the last section we have seen that in standard quantum mechanics it is natural from a mathematical point of view and for aesthetic reasons to identify the space of histories with $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. Whereas the homogeneous histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ admit a direct physical interpretation in terms of sequences of single time "events," there is no such immediate interpretation available for general inhomogeneous histories. It is natural, however, to interpret the latter as representatives of *unsharp quantum events*, i.e, events which cannot be associated with some fixed time, but which are smeared out in time. This proposal is supported by the following example which is adapted (rather stolen) from [70]. In this reference Aharonov and Albert consider so-called *multiple-time observables*. For definiteness, consider a spin-$\frac{1}{2}$ particle and the following object

$$\sigma_{zx}(t_1, t_2) := \sigma_z(t_1) + \sigma_x(t_2),$$

where $\sigma_z(t_1)$ denotes the self-adjoint operator representing the single-time spin observable in z direction in the Heisenberg picture at time t_1 and where $\sigma_x(t_2)$ denotes the single-time spin observable in x direction in the Heisenberg picture at time t_2. (The argument in [70] does not presuppose that the particle is free and is for instance also valid if the particle interacts with an external magnetic field.) By considering an appropriate *Gedankenexperiment*, Aharonov and Albert argue that the value of this multiple-time observable can be measured without measuring $\sigma_z(t_1)$ or $\sigma_x(t_2)$ individually. In the consistent histories approach the notion of measurement has no fundamental status, but as a working hypothesis we take seriously the idea that objects like $\sigma_{zx}(t_1, t_2)$ are physically meaningful also in the consistent histories approach and that they represent a new kind of observable. Now, when we ask the question what kind of history corresponds to the proposition "the value of the observable $\sigma_{zx}(t_1, t_2)$ is 0" we see that the corresponding history is of the form $P_{z,\uparrow}(t_1) \otimes P_{x,\downarrow}(t_2) + P_{z,\downarrow}(t_1) \otimes P_{x,\uparrow}(t_2)$ in an obvious notation. Clearly, this is in general an *inhomogeneous history*. This example can obviously be generalized to much more general situations involving also n-time observables. It follows that the propositions associated with general Aharonov-Albert type multiple-time observables are in general inhomogeneous histories. This gives a physical meaning to (at least some of) the inhomogeneous histories introduced in Section 3.1. The discussion substantiates the proposal that inhomogeneous histories correspond to events which are spread out or unsharp in time.

In the formulation of the history approach given above the most general propositions about a quantum mechanical system which have a physical meaning are identified with finite (or at least countably infinite) history propositions. Other statements about a system which cannot be cast into the framework of history propositions are not considered to be meaningful and hence are excluded from consideration. Histories may be said to represent the possible *temporal events* which may occur. The probabilities associated with histories are considered to be objective entities in their own right and are interpreted as measures of the tendency or propensity of an individual system to realize certain *histories*. The probability measure on a consistent Boolean algebra of history propositions induced by the decoherence functional (according to Theorem 3.18) defines in this consistent Boolean algebra two logical relations, namely an implication and an equivalence relation between histories. A history proposition h is said to IMPLY a history proposition k if the conditional probability $p_\varrho(k|h) \equiv \frac{p_\varrho(h \wedge k)}{p_\varrho(h)}$ is well-defined and equal to one. Two history propositions h and k are said to be EQUIVALENT if h implies k and vice versa.

It is straightforward to show that the notion of implication is independent of the consistent Boolean lattice employed. If the history h implies the history k (w.r.t. ϱ) in some consistent Boolean lattice of histories, then h implies k (w.r.t. ϱ) in every consistent Boolean lattice containing h and k. Omnès' "universal rule of interpretation" of quantum mechanics can now be formulated as

Rule 1 (Omnès) *Propositions about quantum mechanical systems should solely be expressed in terms of history propositions. Every description of an isolated quantum mechanical system should be expressed in terms of finite history propositions belonging to a common consistent Boolean algebra of histories. Every reasoning relating several propositions should be expressed in terms of the logical relations induced by the probability measure from Theorem* 3.18 *in that Boolean algebra.*

We briefly discuss the significance of the different requirements in this rule. It is well-known that every Boolean lattice is isomorphic to the algebra of clopen subsets of some topological space, see Theorem A.12. Hence, by requiring that every description and every reasoning should be *within* a fixed Boolean lattice, all paradoxa and inconsistencies in quantum mechanics which are due to the nondistributivity of the lattice of projection operators have been expelled from the language of quantum mechanics by one single rule. The requirement that the Boolean lattice is consistent is technical and due to the fact that a physically sensible probability functional can only be defined on consistent Boolean lattices. There is a strong methodological argument supporting the Rule 1, i.e., the requirement of simplicity and economy of principles. There is a vast literature discussing examples illustrating the Rule 1. We refer in particular to Omnès' extensive book [19] and to Griffiths [12, 29] and Omnès [13]-[18].

In standard quantum mechanics texts, by and large, no attention is paid to the important issue under which conditions probability assertions and combinations of probability assertions are justified and meaningful. Consistent historians are more careful. According to Rule 1 the assignment of probabilities to certain histories is only admissible when these histories belong to a common Boolean lattice of histories which satisfies the consistency condition. The philosophy underlying this assertion is as follows (adapted from Griffiths [30]): We assert that for a quantum mechanical system there are several incompatible (or complementary) *frameworks* for its theoretical description in terms of possible events and for making logical inferences about possible events or about time sequences of possible events. All different frameworks (or in Griffiths' terminology *topics of conversation*) are similarly objective. That is, the symmetrical treatment of several incompatible frameworks in the mathematical formalism of quantum mechanics is not broken in the interpretation and (as is asserted in the interpretation) also not in the physical reality in the following sense: *it is the integral objective physical situation (for instance, but not necessarily, an experimental arrangement) which determines the framework that should be used for the description and reasoning.* The propensity that some particular event occurs depends upon the quantum system itself and upon the integral physical situation. Particularly, in measurement situations the result of the measurement depends upon the object under study and upon the mode of observation. However, for isolated individual systems, we are free to choose the framework we want. The multitude of alternative frameworks reflects the peculiar nature of quantum reality. It is in a sense as if with each alternative framework we inspect another degree of freedom of the system of which there are in principle an infinite supply.

It is easy to derive counterintuitive and even contradictory conclusions by using different incompatible frameworks for the description of a given system. Thus we are led to the conclusion that the nature of quantum reality cannot be easily imagined and can only in part be described by using ordinary language. The Rule 1 has to be understood as a 'semantical' rule which systematizes the language of quantum mechanics. It clearly states what the assertions and predictions of quantum mechanics are and it once and for all makes sure, whether a reasoning or an implication is allowed or not. The causal relationship between different histories is coded into a logical relationship.

The notions of *truth* and of *reality* are also framework dependent. We say that a history h is true or an *element of reality* in the state ϱ if $d_\varrho(h,h) = 1$. However, it is important to notice that the assertion that a history h is true (or real) is only meaningful with respect to a ϱ-consistent Boolean lattice of history propositions. This statement clearly unravels that the assertion that the consistent histories approach is a " realistic" interpretation must not be understood in a naive (to wit, classical) spirit. The alleged "quantum reality" is quite unusual and has rather strange features. Recently, Isham [35] has proposed that the possible truth values which can be associated with a history actually lie in a Heyting algebra. [11] The physical idea beyond this proposal is briefly as follows: consider a Boolean lattice of history propositions $\mathcal{B}$ which is not consistent with respect to d_ϱ. Accordingly, no history in $\mathcal{B}$ can be said to be true. Nevertheless there may be a coarse-graining of $\mathcal{B}$, i.e., a Boolean sublattice of $\mathcal{B}$, which is ϱ-consistent. This consistent sublattice may contain 'true' histories h. Isham argues that the truth value associated with such histories in the fine-grained description provided by $\mathcal{B}$ should account for the fact that in a coarse-grained description these histories are true. The formalization of this idea has led Isham to his proposal that the possible truth values associated with histories lie in a Heyting algebra.

In summary, we assert that for every physical system there are elements of physical reality which cannot be combined either in constructing a theoretical description or in making logical inferences about them. *Such complementary elements of reality are not independent.* The exact form of the framework for the theoretical description and for making logical inferences was specified above in Rule 1 for the standard 'logical' interpretation of quantum mechanics and in Rule 2 below for the generalized 'logical' interpretation developed in Section 4.

4 Consistent Effect Histories

4.1 Homogeneous Effect Histories

In Section 3.1 we have introduced the standard consistent histories formalism. One of the basic ingredients of this formalism is the proposal that the observables in quantum mechanics have to be identified with self-adjoint operators on Hilbert space.

We have argued in Section 2 that Hilbert space quantum mechanics allows for a much richer notion of observable. Therefore it is worthwhile to study whether the consistent histories approach can be generalized to incorporate POV measures. It is not claimed that the resulting extension can be used in any concrete application of quantum mechanics to make qualitatively new predictions different from the predictions of quantum mechanics. The problem investigated in this section is mainly a matter of principle in the foundations of quantum mechanics. It is the question of whether both the consistent histories approach and the concept of generalized observable can be uphold together or whether they are incompatible.

I will argue that the consistent histories approach can indeed be generalized.

Definition 4.1 *A* HOMOGENEOUS EFFECT HISTORY (OF THE FIRST KIND) *is a map* $u : \mathbb{R} \to \mathfrak{E}(\mathfrak{H}), t \mapsto u_t$. *The* SUPPORT OF u *is given by* $\mathfrak{s}(u) := \{t \in \mathbb{R} \mid u_t \neq 1\}$. *If* $\mathfrak{s}(u)$ *is finite, countable or uncountable, then we say that* u *is a* FINITE, COUNTABLE *or* UNCOUNTABLE EFFECT HISTORY *respectively. The space of all homogeneous effect histories (of the first kind) will be denoted by*

[11] A *Heyting algebra* H is a lattice with universal bounds with an additional binary operation $\Rightarrow$ with the property that for $x, y, z \in H, x \wedge y \leq z$ if and only if $x \leq (y \Rightarrow z)$.

$\mathbb{E}(\mathfrak{H})$, the space of all finite homogeneous effect histories (of the first kind) by $\mathbb{E}_{fin}(\mathfrak{H})$ and the space of all finite homogeneous effect histories (of the first kind) with support S by $\mathbb{E}_S(\mathfrak{H})$. All homogeneous effect histories for which there exists at least one $t \in \mathbb{R}$ such that $u_t = 0$ are collectively denoted by 0, slightly abusing the notation.

The choice of the decoherence weight in Section 3 was motivated by the heuristic principle from Section 2.3 and the form of the Lüders-von Neumann projection postulate. The state transformation formula of the Lüders-von Neumann projection postulate can be generalized to the measurement of POV measures. Assume that upon a measurement of the observable $\mathcal{O}$ the value of $\mathcal{O}$ is found to be in the set $A \in \mathcal{F}_{\mathcal{O}}$, then the generalized Lüders-von Neumann state transformation prescription is

$$\varrho \mapsto \frac{\sqrt{\mathcal{O}(A)}\varrho\sqrt{\mathcal{O}(A)}}{\mathrm{tr}(\mathcal{O}(A)\varrho)}.$$

In general, this prescription does not correspond to an ideal or repeatable measurement of $\mathcal{O}$ of course. Anyhow, in the consistent histories approach we are looking for something different. We are looking for a rule assigning probabilities to sequences of quantum events (represented by some homogeneous effect history) subject to the following conditions:

- If the effect history u is degenerate, i.e., if its support contains exactly one time point $\mathfrak{s}(u) = \{t\}$, then the probability assigned to the "history" u should be equal to the standard quantum mechanical probability $\mathrm{tr}(u_t\varrho)$ of the effect u_t in the state ϱ.

- The probability assigned to ordinary histories should coincide with the probability assigned in the standard consistent histories theory.

This shows that the following definitions are sensible.

The class operator C_{t_0} defined above for finite ordinary homogeneous histories can be extended to homogeneous finite effect histories $u \in \mathbb{E}_{fin}(\mathfrak{H})$

$$\begin{aligned} C_{t_0}(u) &:= U(t_0,t_n)\sqrt{u_{t_n}}U(t_n,t_{n-1})\sqrt{u_{t_{n-1}}}...U(t_2,t_1)\sqrt{u_{t_1}}U(t_1,t_0) && (9)\\ &= U(t_0,t_i(u))\sqrt{u_{t_n}}(t_n)\sqrt{u_{t_{n-1}}}(t_{n-1})...\sqrt{u_{t_1}}(t_1)U(t_i(u),t_0), && (10)\end{aligned}$$

where we have defined the Heisenberg picture operators

$$\sqrt{u_{t_k}}(t_k) := U(t_k,t_i(u))^\dagger\sqrt{u_{t_k}}U(t_k,t_i(u))$$

with respect to the initial time $t_i(u)$ of u.

Definition 4.2 *For every pair u and v of finite homogeneous effect histories (of the first kind) we define the* DECOHERENCE WEIGHT *of u and v by*

$$\mathsf{d}_\varrho(u,v) := \mathrm{tr}\left(C_{t_0}(u)\varrho(t_0)C_{t_0}(v)^\dagger\right).$$

The functional $\mathsf{d}_\varrho : \mathbb{E}_{fin}(\mathfrak{H}) \times \mathbb{E}_{fin}(\mathfrak{H}) \to \mathbb{C}$, $(u,v) \mapsto \mathsf{d}_\varrho(u,v)$ will be called the HOMOGENEOUS DECOHERENCE FUNCTIONAL ASSOCIATED WITH THE STATE ϱ.

As in the standard consistent histories approach we wish to interpret the value $\mathsf{d}_\varrho(u,u)$ as the probability of u. There immediately arises a serious difficulty with this decoherence functional. At first sight it seems difficult (if not impossible) to construct a natural mathematical structure on the space of effect histories such that the decoherence functional is additive in both arguments. Without this structure a consistency condition generalizing Equation 8 cannot even be formulated and an interpretation of $\mathsf{d}_\varrho(u,u)$ as probability seems to be impossible. These questions are investigated in the next section.

4.2 Inhomogeneous Effect Histories

The notion of inhomogeneous effect history is introduced in close analogy to the parallel concepts in the standard consistent histories approach. However, under which circumstances some inhomogeneous effect histories can be interpreted as composition of homogeneous histories by the grammatical connectives "and", "or" and "not" – which was a major motivation for the definition of the notion inhomogeneous history in Section 3 – will be studied only at the end of this section.

The map σ_S given by Equation 4 can be extended to a map

$$\sigma_{fin} : \mathbb{E}_{fin}(\mathfrak{H}) \to \mathcal{B}^{\otimes}_{fin}(\mathfrak{H}), u \simeq \{u_{t_k}\}_{t_k \in s(u)} \mapsto \otimes_{t_k \in s(u)} u_{t_k}, \tag{11}$$

where $\mathcal{B}^{\otimes}_{fin}(\mathfrak{H})$ denotes the disjoint union of all $\mathcal{B}^{\otimes}_{S}(\mathfrak{H}), S \subset \mathbb{R}$ finite. The map σ_{fin} is neither injective nor surjective. However, $d_\varrho(u,v)$ depends on u and v only through $\sigma_{fin}(u)$ and $\sigma_{fin}(v)$. From a mathematical point of view it thus is to be natural to define the notion of *inhomogeneous* effect history as follows:

Definition 4.3 *Let S be a finite subset of $\mathbb{R}$, then we call the space $\mathfrak{E}^{\otimes}_{S}(\mathfrak{H}) := \mathfrak{E}(\otimes_{t \in S}\mathfrak{H})$ of effect operators on $\otimes_{t\in S}\mathfrak{H}$ the* SPACE OF FINITE INHOMOGENEOUS EFFECT HISTORIES WITH SUPPORT S. *The direct limit of the directed system $\{\mathfrak{E}^{\otimes}_{S}(\mathfrak{H}) \mid S \subset \mathbb{R}$ finite$\}$ will be called* THE SPACE OF ALL FINITE INHOMOGENEOUS EFFECT HISTORIES WITH ARBITRARY SUPPORT *and will be denoted by $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$. The elements in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ will also be called* EFFECT HISTORY PROPOSITIONS.

The construction of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is described in detail in Appendix A.3. The homogeneous elements in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ represent equivalence classes of homogeneous effect histories. In this work we will carefully distinguish between homogeneous effect histories as defined in Definition 4.1 and homogeneous elements in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$. For clarity of exposition we will call the former *homogeneous effect history of the first kind* or (where no confusion can arise) simply homogeneous effect histories, whereas the latter will be called *homogeneous effect histories of the second kind.*
All the $\mathfrak{E}^{\otimes}_{S}(\mathfrak{H})$, $S \subset \mathbb{R}$ and $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ carry several isomorphic distinct D-poset structures, as discussed in Appendix A.3.

Recent results of the author and Wright [45] immediately imply that the decoherence functional d_ϱ as defined above on the space of homogeneous effect histories (of the second kind) can indeed be extended to a possibly infinitely valued functional on the space of inhomogeneous effect histories with the desired properties. In particular, the author and Wright have shown that the decoherence functional d_ϱ in Definition 3.2 can be canonically extended to a functional $\mathcal{D}_{\varrho,S}$ on $\mathfrak{E}^{\otimes}_{S}(\mathfrak{H}) \times \mathfrak{E}^{\otimes}_{S}(\mathfrak{H})$

(with values in the Riemann sphere) which is additive in both arguments with respect to the canonical D-poset structure on $\mathfrak{E}_S^{\otimes}(\mathfrak{H})$ in the finite sector. Notice, however, that $\mathcal{D}_{\varrho,S}$ does not extend the homogeneous decoherence functional from Definition 4.2. The collection of all such functionals $\mathcal{D}_{\varrho,S}$ for any finite $S \subset \mathbb{R}$ induces a functional $\mathcal{D}_{\varrho}$ on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}) \times \mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ which is additive in both arguments with respect to the canonical D-poset structure on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$.

Let $\alpha \in \mathbb{Q}, \alpha > 0$. Consider the function $q_\alpha : (\mathfrak{E}(\mathfrak{H}), \oplus_\alpha) \times (\mathfrak{E}(\mathfrak{H}), \oplus_\alpha) \to (\mathfrak{E}(\mathfrak{H}), \oplus) \times (\mathfrak{E}(\mathfrak{H}), \oplus), q_\alpha(E_1, E_2) := (Q_\alpha^{-1}(E_1), Q_\alpha^{-1}(E_2))$, where Q_α denotes the D-poset isomorphism introduced in Appendix A.3. The functional $\mathcal{D}_{\varrho,S,\alpha} := \mathcal{D}_{\varrho,S} \circ q_\alpha$ is a complex valued functional defined on $(\mathfrak{E}_S^{\otimes}(\mathfrak{H}), \oplus_\alpha) \times (\mathfrak{E}_S^{\otimes}(\mathfrak{H}), \oplus_\alpha)$. The functional $\mathcal{D}_{\varrho,S,\alpha}$ is additive in both arguments with respect to the D-poset structure $\oplus_\alpha$ on $\mathfrak{E}(\mathfrak{H})$.

The collection of all functionals $\mathcal{D}_{\varrho,S,\alpha}$ for any finite $S \subset \mathbb{R}$ and fixed α induces a bounded functional $\mathcal{D}_{\varrho,\alpha}$ on $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus_\alpha) \times (\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus_\alpha)$ which is additive in both arguments with respect to the D-poset structure $\oplus_\alpha$ on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$. All the functionals $\mathcal{D}_{\varrho,\alpha}$ are called DECOHERENCE FUNCTIONALS WITH RESPECT TO THE STATE ϱ.

Comparison with the decoherence weight defined in Definition 4.2 shows that it is the decoherence functional $\mathcal{D}_{\varrho,2}$ which coincides with d_ϱ from Definition 4.2 when restricted to homogeneous effect histories. So, the "physical" value of α is $\alpha = 2$.

In view of our above remarks and due to the pairwise isomorphy of all $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus_\alpha)$, we conclude that the value of $\alpha \in \mathbb{Q}^+$ can be chosen at will. Indeed, it can be easily seen by inspection that the Foulis-Bennett Theorem in Appendix A.4 and all subsequent definitions in this section are forminvariant under the D-poset isomorphisms Q_α. It has only to be kept in mind that the "physical" form of the decoherence weight corresponds to $\alpha = 2$. In the rest of this section we will work with the value $\alpha = 1$. All results obtained in the $\alpha = 1$ "representation" can easily be shifted isomorphically to the "physical" $\alpha = 2$ case.

The Foulis-Bennett Theorem allows for an intrinsic definition of the notion of Boolean sublattice of the D-poset $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$. [12]

Definition 4.4 *A sub-D-poset $(\mathfrak{B}, \oplus)$ of $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ is said to be a* BOOLEAN SUBLATTICE OF $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ *if $\mathfrak{B}$ satisfies the coherence law and the law of compatibility.*

We also say that a Boolean sublattice of $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ is a *Boolean lattice of effect histories*. The reader may wish to recall the definition of the notion of sub-D-poset given in Appendix A.2. It is important to remember that the universal bounds $0_{\mathfrak{B}}, 1_{\mathfrak{B}}$ of $\mathfrak{B}$ do not necessarily coincide with the universal bounds $0, 1$ of $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ and that the D-poset structure on $\mathfrak{B}$ induced by the D-poset structure on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ does depend on $1_{\mathfrak{B}}$ and $0_{\mathfrak{B}}$.

Remark 4.5 *Since $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ is a D-poset, $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ is in particular a partially ordered set. However, for two elements $e_1, e_2 \in \mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ the supremum $e_1 \vee e_2$ and the infimum $e_1 \wedge e_2$ not necessarily exist. That is, $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ is not a lattice. But there exists a partially defined join operation denoted*

[12] In [43] I have given a seemingly more artificial definition of the notion of "admissible Boolean sublattice." However, an application of the Foulis-Bennett Theorem shows that the every admissible Boolean lattice as defined in [43] is isomorphic as a D-poset to a Boolean sublattice of $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ as defined here. The isomorphism is denoted by $\mathfrak{M}$ in [43].

by $\vee$ and a partially defined meet operation denoted by $\wedge$. Strictly speaking a SUBLATTICE $\mathcal{L}$ OF $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ *is a subset $\mathcal{L} \subset \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ such that $\mathcal{L}$ endowed with the restrictions of $\vee$ and $\wedge$ to $\mathcal{L}$ is a lattice. It makes thus sense to speak of sublattices of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$. However, it is important to notice that a Boolean sublattice of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is not necessarily a sublattice of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ in this sense.*

Our target is to generalize Omnès' 'logical' rule and thus to single out the appropriate subsets of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ on which the decoherence functional $\mathcal{D}_{\varrho,1}$ induces a probability measure and on which a description and a reasoning involving inhomogeneous effect histories compatible with 'common sense' can be defined.

Definition 4.6 *A Boolean sublattice $(\mathfrak{B}, \oplus)$ of $(\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H}), \oplus)$ is called* CONSISTENT W.R.T. ϱ *if (i) $0 \in \mathfrak{B}$, if (ii) $D_{\varrho,1}(b, b') \in \mathbb{C}$ for all $b, b' \in \mathfrak{B}$, if (iii) for every pair of disjoint elements $b_1, b_2 \in \mathfrak{B}$ (i.e., elements satisfying $b_1 \wedge_{\mathfrak{B}} b_2 = 0$) the* `consistency condition` $\mathrm{Re}\, \mathcal{D}_{\varrho,1}(b_1, b_2) = 0$ *is satisfied, and if (iv) $\mathcal{D}_{\varrho,1}(1_{\mathfrak{B}}, 1_{\mathfrak{B}}) \leq 1$.*

Theorem 4.7 *Let $(\mathfrak{B}, \oplus)$ be a consistent Boolean lattice of effect histories. Then the decoherence functional $\mathcal{D}_{\varrho,1}$ induces a probability functional $p_{\varrho,\mathfrak{B}}$ on $\mathfrak{B}$ by*

$$b \mapsto p_{\varrho,\mathfrak{B}}(b) \equiv \frac{\mathcal{D}_{\varrho,1}(b, b)}{\mathcal{D}_{\varrho,1}(1_{\mathfrak{B}}, 1_{\mathfrak{B}})}.$$

The proof is straightforward.

Definition 4.8 *An effect history proposition $e_1 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is said to* IMPLY *an effect history proposition $e_2 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ in the state ϱ if there exists a consistent Boolean sublattice $\mathfrak{B}$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ containing e_1 and e_2 and if the conditional probability $p_{\varrho,\mathfrak{B}}(e_2|e_1) \equiv \frac{p_{\varrho,\mathfrak{B}}(e_1 \wedge_{\mathfrak{B}} e_2)}{p_{\varrho,\mathfrak{B}}(e_1)}$ is well-defined and equal to one. We write $e_1 \Longrightarrow_{\varrho} e_2$. Two history propositions e_1 and e_2 are said to be* EQUIVALENT *if e_1 implies e_2 and vice versa. We write $e_1 \Longleftrightarrow_{\varrho} e_2$.*

Remark 4.9 *The so-defined notions of implication and equivalence of effect histories are certainly framework independent. However, there is a major difference between Definition* 4.8 *and the parallel notions in the standard consistent histories formalism. In the latter case the conditional probability $p_{\varrho}(k \mid h)$ is independent of the ϱ-consistent Boolean lattice chosen. In the consistent effect histories approach the situation is different. To understand this, consider two ϱ-consistent Boolean sublattices $\mathfrak{B}$ and $\mathfrak{B}'$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ both containing the effect history propositions e_1 and e_2. In general $e_1 \wedge e_2$ does not exist in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ and correspondingly in general $e_1 \wedge_{\mathfrak{B}} e_2 \neq e_1 \wedge_{\mathfrak{B}'} e_2$ (even if $e_1 \wedge e_2$ exists in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$). To wit, even if $e_1 \Longrightarrow e_2$, there may be a consistent Boolean lattice $\mathfrak{B}_1$ containing e_1 and e_2 such that $p_{\varrho,\mathfrak{B}_1}(e_2 \mid e_1)$ is not one or is not well-defined. However, it is reasonable to define $e_1 \Longrightarrow_{\varrho} e_2$ if there exists a consistent Boolean sublattice $\mathfrak{B}$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ containing e_1, e_2 and some further element $e_3 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ satisfying $e_1 \geq e_3$ and $e_2 \geq e_3$ such that $\frac{p_{\varrho,\mathfrak{B}}(e_3,e_3)}{p_{\varrho,\mathfrak{B}}(e_1,e_1)}$ is well-defined in $\mathfrak{B}$ and equal to one. But this is exactly what has been done in Definition* 4.8.

The Generalized Rule of Interpretation

The generalized "universal rule of interpretation" of quantum mechanics can now simply be formulated as

Rule 2 *Propositions about quantum mechanical systems should solely be expressed in terms of effect history propositions. Every description of an isolated quantum mechanical system should be expressed in terms of finite effect history propositions belonging to a common consistent Boolean sublattice of effect histories. Every reasoning relating several propositions should be expressed in terms of the logical relations induced by the probability measure from Theorem* 4.7 *in that Boolean algebra.*

It is clear that Rule 2 is indeed a generalization of Rule 1.

5 Conclusion

In this review we have discussed the basic ideas underlying the notion of generalized quantum mechanical observable as POV measure and the standard formulation of the consistent histories approach to quantum machanics based on the standard notion of quantum mechanical observable. We have seen that the consistent histories formalism admits a generalization covering also POV measures. We conclude that these two modern developments in our understanding of quantum mechanics are not mutually exclusive but, on the contrary, are mutually compatible.

A Miscellaneous Definitions and Results

A.1 Posets and Lattices

Definition A.1 *Let $\mathcal{P}$ be a nonempty set. A binary relation $\leq$ on $\mathcal{P}$ is called a* PARTIAL ORDER *if for all $x, y, z \in \mathcal{P}$ the following conditions are satisfied*

- $x \leq x$ *(reflexivity)*
- *if $x \leq y$ and $y \leq x$, then $x = y$* *(antisymmetry)*
- *if $x \leq y$ and $y \leq z$, then $x \leq z$* *(transitivity)*

In this case the pair $(\mathcal{P}, \leq)$ is called a PARTIALLY ORDERED SET *or a* POSET. *A poset is said to have* UNIVERSAL BOUNDS *if there exist two elements $0, 1 \in \mathcal{P}$ such that $0 \leq x \leq 1$ for all $x \in \mathcal{P}$.*

If $x \leq y$ for some elements x, y in a poset, then we also say that x is smaller than y, that x is contained in y or that y is greater than x.

Definition A.2 *Let $\mathcal{P}$ be a poset. A non-zero element $a \in \mathcal{P}$ is called an* ATOM *if $y \leq a$ implies $y = 0$ or $y = a$. $\mathcal{P}$ is said to be* ATOMIC *if for every non-zero element $x \in \mathcal{P}$ there is an atom a with $a \leq x$.*

Definition A.3 *A* LATTICE *is a poset $\mathcal{L}$ any two of whose elements x, y have a greatest lower bound or "meet" denoted by $x \wedge y$ and a least upper bound or "join" denoted by $x \vee y$. A lattice is called* COMPLETE *if each of its subsets $\mathcal{L}_1$ has a greatest lower bound and a least upper bound. A lattice $\mathcal{L}$ is said to be a σ-*LATTICE *if each countable subset of $\mathcal{L}$ has a greatest lower bound and a least upper bound.*

Definition A.4 *We say that a lattice $\mathcal{L}$ is* DISTRIBUTIVE *if for all $x, y, z \in \mathcal{L}$ the following identity is satisfied*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{12}$$

Lemma A.5 *A lattice $\mathcal{L}$ is* DISTRIBUTIVE *if and only if for all $x, y, z \in \mathcal{L}$ the following identity is satisfied*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{13}$$

Definition A.6 *A lattice $\mathcal{L}$ is* MODULAR *if for all $x, y, z \in \mathcal{L}$*

$$x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ whenever } x \leq y. \tag{14}$$

Let $\mathcal{L}$ be a lattice with universal bounds. An element $x' \in \mathcal{L}$ is said to be a COMPLEMENT *of $x \in \mathcal{L}$ if $x \wedge x' = 0$ and $x \vee x' = 1$. If any element of $\mathcal{L}$ has at least one complement, then $\mathcal{L}$ is said to be* COMPLEMENTED. *A lattice $\mathcal{L}$ is said to be* RELATIVELY COMPLEMENTED *if all its intervals (i.e. sublattices of the form $a \leq x \leq b$, with $a, b \in \mathcal{L}$) are complemented.*

Definition A.7 *A mapping $\perp: \mathcal{P} \to \mathcal{P}$ is said to be an* ORTHOCOMPLEMENTATION *on a poset $\mathcal{P}$ with universal bounds if for all $x, y \in \mathcal{P}$*

- $(x^\perp)^\perp = x$,
- *if $x \leq y$, then $y^\perp \leq x^\perp$,*
- $x \vee x^\perp = 1$.

In this case the pair $(\mathcal{P}, \perp)$ is called an ORTHOCOMPLEMENTED POSET. *A lattice $\mathcal{L}$ with an orthocomplementation is called an* ORTHOLATTICE.

Definition A.8 *An ortholattice $\mathcal{L}$ which satisfies for all $x, y \in \mathcal{L}$*

$$x \vee (x^\perp \wedge y) = y, \text{ whenever } x \leq y. \tag{15}$$

is called ORTHOMODULAR.

Remark A.9 *Any modular ortholattice is orthomodular.*

Definition A.10 *A* BOOLEAN LATTICE *is a complemented distributive lattice.*

Throughout this paper we will use the terms *Boolean lattice* and *Boolean algebra* synonymously For a justification we refer the reader to the monograph by Birkhoff [71], Section I.10.

Remark A.11 *Complements are unique in distributive lattices.*

Theorem A.12 (Stone) *Every Boolean lattice is isomorphic to the algebra of clopen subsets of some topological space.*

A.2 D-posets and Orthoalgebras

A.2.1 Basic Definitions

Definition A.13 *A* DIFFERENCE POSET *or* D-POSET *is a partially ordered set D with greatest element* 1 *and with a partial binary operation $\ominus : D_2 \to D$, where $D_2 \subset D \times D$, such that*

- *$b \ominus a$ is defined if and only if $a \leq b$ for all $a, b \in D$,*
- *$b \ominus a \leq b$ for all $a \leq b$,*
- *$b \ominus (b \ominus a) = a$, for all $a \leq b$,*
- *$a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

Difference posets have been introduced by Kôpka and Chovanec [72] and have been further studied in [73, 74, 75, 76, 77, 41].

Proposition A.14 *Let $(D, \ominus)$ be a D-poset. Then*

- *D has universal bounds, given by* 1 *and* $0 := 1 \ominus 1$*;*
- *$a \ominus 0 = a$, for all $a \in D$;*
- *$a \ominus a = 0$, for all $a \in D$;*
- *If $a, b \in D$ with $a \leq b$, then $b = a$ if and only if $b \ominus a = 0$;*
- *If $a, b \in D$ with $a \leq b$, then $a = 0$ if and only if $b \ominus a = b$;*
- *If $a, b, c \in D$ with $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $a \leq c \ominus (b \ominus a)$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$ and $(c \ominus (b \ominus a)) \ominus a = (c \ominus b)$;*
- *If $a, b, c \in D$ with $b \leq c$ and with $a \leq c \ominus b$, then $b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.*

Definition A.15 *A set D with two special elements $0, 1 \in D$ supplied with a partially defined associative and commutative operation $\oplus : D_2' \to D$, where $D_2' \subset D \times D$, is called an* EFFECT ALGEBRA *if*

- *For every $a \in D$ there exists a unique $a' \in D$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$,* **[orthosupplementation law]**
- *If $1 \oplus b$ is defined, then $b = 0$, for all $b \in D$.* **[Zero-One law]**

An effect algebra D is called an ORTHOALGEBRA *if furthermore*

- *If $b \oplus b$ is defined, then $b = 0$, for all $b \in D$.* **[consistency law]**

Effect algebras have been introduced by Foulis and Bennett [78, 79]. Whenever $a \oplus b$ is well-defined for $a, b \in D$, then we write $a \perp b$. Let $(D, \ominus)$ be a D-poset. Define

$$a \oplus b := 1 \ominus ((1 \ominus a) \ominus b),$$

whenever the right hand side is well-defined. Then $\oplus$ is a well-defined partial binary operation on D and $(D, \oplus)$ is an effect algebra. Conversely, let $(D, \oplus)$ be an effect algebra. Define

$$b \ominus a := (a \oplus b')',$$

whenever the right hand side is well-defined. Then $\ominus$ is a well-defined partially binary operation on D. Further, define $a \leq b$ for $a, b \in D$ if there exists $c \in D$ such that $c \perp a$ and $a \oplus c = b$. Then $(D, \ominus)$ is a D-poset. Therefore the notions of D-poset and effect algebra are equivalent and we will use *both* terms synonymously in this work.

Definition A.16 *Let $(D, \oplus)$ be a D-poset. A* PROBABILITY MEASURE ON *D is a map $p : D \longrightarrow \mathbb{R}^+$ satisfying $p(1) = 1$ and $p(a \oplus b) = p(a) + p(b)$, whenever $a \oplus b$ is well-defined.*

A.2.2 Sub-D-Posets

Let $(D, \ominus)$ be a D-poset. Then it is natural to say that a subset $S \subset D$ is a sub-D-poset of D if

- $1 \in S$;
- $b \ominus a \in S$, for all $a, b \in S$ with $a \leq b$.

However, we need a more general notion of sub-D-poset. Consider a subset S of D. Then the partial order on D induces a partial order on S. We assume that S possesses universal bounds 0_S and 1_S not necessarily coinciding with the universal bounds $0, 1$ of D. In this case we define a partial binary operation $\ominus_S : D_2 \cap (S \times S) \longrightarrow D$ by

$$b \ominus_S a := 0_S \oplus (b \ominus 0_S) \ominus (a \ominus 0_S),$$

for all $a, b \in S$ with $a \leq b$.

Definition A.17 *Let S be a subset of the D-poset $(D, \ominus)$ with universal bounds $0_S, 1_S$. Then the pair $(S, \ominus_S)$ is called a* SUB-D-POSET *of D if $b \ominus_S a \in S$ for all $a, b \in S$ with $a \leq b$.*

It is easy to verify that a sub-D-poset of a D-poset is a D-poset in its own right. The operation $\oplus_S$ dual to $\ominus_S$ is given by

$$\begin{aligned} a \oplus_S b &:= 1_S \ominus_S ((1_S \ominus_S a) \ominus_S b) \\ &= (a \ominus 0_S) \oplus (b \ominus 0_S) \oplus 0_S, \end{aligned}$$

whenever the right hand sides are well defined. In the main text we shall drop the subscript S in the symbols $\ominus_S$ and $\oplus_S$ and simply write $(S, \ominus)$ for a sub-D-poset of $(D, \ominus)$. This abuse of notation can be justified by the observation that $\ominus_S$ is unambiguously determined by $\ominus$ and by the universal bounds of S.

A.2.3 Directed Systems and Direct Limits

Definition A.18 *Let $(\mathfrak{T}, \leq)$ be a partially ordered set. A* $\mathfrak{T}$-DIRECTED SYSTEM *of D-posets is a family $D_{\mathfrak{T}} := \{D_t, t \in \mathfrak{T}\}$ of D-posets supplied with a family of morphisms $f_{ts} : D_t \to D_s, t, s \in \mathfrak{T}$, defined iff $t \leq s$, such that*

- *$f_{tt} = id_{D_t}$, for all $t \in \mathfrak{T}$;*
- *If $t \leq s \leq r$ in $\mathfrak{T}$, then $f_{sr} f_{ts} = f_{tr}$.*

Let $D_{\mathfrak{T}}$ be a $\mathfrak{T}$-directed system of D-posets. Then a D-poset $\mathfrak{L}$ supplied with a family of morphisms $\{f_t : D_t \to \mathfrak{L}\}_{t \in \mathfrak{T}}$ is called the DIRECT LIMIT *of $D_{\mathfrak{T}}$ if*

- *If $t \leq s$ in $\mathfrak{T}$, then $f_s f_{ts} = f_t$;*
- *If D is a D-poset supplied with a set of morphisms $\{g_t : D_t \to D, t \in \mathfrak{T}\}$, then there exists a unique morphism $g : \mathfrak{L} \to D$, such that $g f_t = g_t$, for all $t \in \mathfrak{T}$.*

The direct limit of a directed system of D-posets always exists [41].

A.3 The D-poset $\mathfrak{E}(\mathfrak{H})$

The set $\mathfrak{E}(\mathfrak{H})$ of all effect operators on a Hilbert space $\mathfrak{H}$ [with the scalar product denoted by $(\cdot, \cdot)$] can be organized into a D-poset by defining a partial ordering $\leq$ and a partial binary operation $\ominus$ on D by $A \leq B$ if $(Ax, x) \leq (Bx, x)$ for all $x \in \mathfrak{H}$ and $C = B \ominus A$ if $(Bx, x) - (Ax, x) = (Cx, x)$ for all $x \in \mathfrak{H}$. This D-poset structure can be alternatively characterized by the partial sum $\oplus$, where $E_1 \oplus E_2$ is defined if $E_1 + E_2 \leq 1$ by $E_1 \oplus E_2 := E_1 + E_2$. We refer to the so defined D-poset structure on $\mathfrak{E}(\mathfrak{H})$ as the CANONICAL D-POSET STRUCTURE.

The D-poset structure on $\mathfrak{E}(\mathfrak{H})$ is not unique. It is possible to define a countably infinite family of D-poset structures on $\mathfrak{E}(\mathfrak{H})$. Let α be a rational number with $\alpha > 0$ and define

$$A \oplus_\alpha B := \left(A^{1/\alpha} + B^{1/\alpha}\right)^\alpha, \text{ for all } A, B \in \mathfrak{E}(\mathfrak{H}) \text{ satisfying } A^{1/\alpha} + B^{1/\alpha} \leq 1.$$

That these expressions are well-defined is a consequence of the work of Langer [80]. In particular it follows from Proposition 2 in [80] that E^α is well-defined and that E^α is itself an effect operator for all $E \in \mathfrak{E}(\mathfrak{H})$ and all $\alpha \in \mathbb{Q}, \alpha > 0$. The pair $(\mathfrak{E}(\mathfrak{H}), \oplus_\alpha)$ is a D-poset for every $\alpha > 0$. However, all these D-poset structures are isomorphic. The D-poset isomorphisms are given by $Q_\alpha : (\mathfrak{E}(\mathbb{H}), \oplus) \to (\mathfrak{E}(\mathbb{H}), \oplus_\alpha), u \mapsto u^\alpha$.

Let in the sequel $\mathfrak{T}$ denote the set of all finite subsets of $\mathbb{R}$ partially ordered by set inclusion. For every $t \in \mathbb{R}$ set $\mathfrak{E}(\mathfrak{H})_t := \mathfrak{E}(\mathfrak{H})$ and for every $T = \{t_1, ..., t_n\} \in \mathfrak{T}$ set $\mathfrak{E}(\mathfrak{H})_T := \mathfrak{E}\left(\otimes_{t \in T} \mathfrak{H}_t\right)$ where $\mathfrak{H}_t := \mathfrak{H}$ for all $t \in \mathbb{R}$. Then it has been shown in [41] that for every $T \subset S \in \mathfrak{T}$ there exists a morphism $f_{TS} : \mathfrak{E}(\mathfrak{H})_T \to \mathfrak{E}(\mathfrak{H})_S$ such that $\{\mathfrak{E}(\mathfrak{H})_T, T \in \mathfrak{T}\}$ supplied with $\{f_{TS}, T \subset S \in \mathfrak{T}\}$ is a $\mathfrak{T}$-directed system. Let, e.g., $T = \{t_1, t_3\} \subset S = \{t_1, t_2, t_3\}$, then $f_{TS}(A \otimes B) = A \otimes 1 \otimes B$. Therefore the direct limit of $\{(\mathfrak{E}(\mathfrak{H})_T, \oplus), T \in \mathfrak{T}\}$ exists and will be denoted by $(\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H}), \oplus)$. $\mathfrak{E}(\mathfrak{H})_{\mathfrak{T}}$ can be constructed as follows: consider the disjoint union $\cup_{T \in \mathfrak{T}} \mathfrak{E}(\mathfrak{H})_T$ and call two elements

h_1, h_2 of $\cup_{T\in\mathfrak{T}}\mathfrak{E}(\mathfrak{H})_T$ EQUIVALENT if there exist $T_1, T_2, T_{12} \in \mathfrak{T}$ such that $h_1 \in T_1 \subset T_{12}$, $h_2 \in T_2 \subset T_{12}$ and such that $f_{T_1T_{12}}(h_1) = f_{T_2T_{12}}(h_2)$. Then $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is the quotient space of $\cup_{T\in\mathfrak{T}}\mathfrak{E}(\mathfrak{H})_T$ by the such defined equivalence relation. It is easy to extend the D-poset structures on $\mathfrak{E}(\mathfrak{H})_T$, for $T \in \mathfrak{T}$ to a D-poset structure on $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$.

A.4 An Alternative Characterization of Boolean Lattices

Boolean lattices can be characterized as special D-posets. The following Theorem is due to Foulis and Bennett [79]

Theorem A.19 *Let $(\mathcal{B}, \oplus)$ be a D-poset such that for all $a, b, c \in \mathcal{B}$:*

- *If $a \oplus b$, $b \oplus c$ and $a \oplus c$ are defined, then $(a \oplus b) \oplus c$ is defined.* **[coherence law]**
- *For all $a, b \in \mathcal{B}$, there exist $c, d, e \in \mathcal{B}$ such that $d \oplus e$ and $c \oplus (d \oplus e)$ are defined, $a = c \oplus e$ and $b = d \oplus e$.* **[law of compatibility]**

Then $\mathcal{B}$ can be organized into a Boolean algebra in one and only one way, so that

- *$0 \leq b \leq 1$ for all $b \in \mathcal{B}$,*
- *$a \oplus b$ is defined if and only if $a \wedge b = 0$,*
- *$a \wedge b = 0$ implies $a \oplus b = a \vee b$.*

Conversely, let $\mathcal{B}'$ denote a Boolean lattice, then a partial binary operation $\oplus_{\mathcal{B}'}$ is defined by $a \oplus_{\mathcal{B}'} b := a \vee b$ whenever $a \wedge b = 0$. The pair $(\mathcal{B}', \oplus_{\mathcal{B}'})$ is then a D-poset satisfying the coherence law and the law of compatibility.

We will call a D-poset satisfying both the coherence law and the law of compatibility a BOOLEAN D-POSET.

If $(\mathcal{B}, \oplus)$ is a Boolean sub-D-poset of $(\mathfrak{E}(\mathfrak{H}), \oplus)$, then $(Q_\alpha(\mathcal{B}), \oplus_\alpha)$ is a Boolean sub-D-poset of $(\mathfrak{E}(\mathfrak{H}), \oplus_\alpha)$ such that for all $a, b \in \mathcal{B}$

- $Q_\alpha(a \wedge b) = Q_\alpha(a) \wedge_\alpha Q_\alpha(b)$;
- $Q_\alpha(a \vee b) = Q_\alpha(a) \vee_\alpha Q_\alpha(b)$;
- $Q_\alpha(a') = Q_\alpha(a)'$.

That is, the D-poset isomorphism Q_α lifts to an isomorphism between the corresponding Boolean lattices.

A.5 Hilbert Lattices

It is well-known that there is a one-to-one correspondence between closed subspaces of a Hilbert space $\mathfrak{H}$ and projection operators on $\mathfrak{H}$. In this work we will freely switch between both pictures and identify each projection operator with the subspace onto which it projects. We denote by $\mathcal{P}(\mathfrak{H})$ the set of all projection operators on the Hilbert space $\mathfrak{H}$. For all $p_1, p_2 \in \mathcal{P}(\mathfrak{H})$ one defines

- $p_1 \leq p_2$ if p_1 projects on a subspace of the range of p_2, ($\leq$ defines a partial order on $\mathcal{P}(\mathfrak{H})$),
- the join $p_1 \vee p_2$ of p_1 and p_2 to be the projection operator which projects on the smallest closed subspace of $\mathfrak{H}$ which contains the subspaces $p_1\mathfrak{H}$ and $p_2\mathfrak{H}$,
- the meet $p_1 \wedge p_2$ of p_1 and p_2 to be the projection operator which projects on the intersection of $p_1\mathfrak{H}$ and $p_2\mathfrak{H}$ and
- the orthocomplementation $\neg p_1$ of p_1 to be the projection operator which projects on the orthogonal subspace of $p_1\mathfrak{H}$ in $\mathfrak{H}$.

Analogously, for any family $\{p_i\}_i$ we define the meet $\wedge_i p_i$ to be the projection operator projecting onto $\cap_i p_i\mathfrak{H}$ and the join $\vee_i p_i$ the be the projection operator onto the closure of $\cup_i p_i\mathfrak{H}$. Then we have

Theorem A.20 *$\mathcal{P}(\mathfrak{H})$ is a complete, atomic, orthomodular lattice. The atoms are the projection operators onto the one-dimensional subspaces of $\mathfrak{H}$. If $\mathfrak{H}$ is finite dimensional, then $\mathcal{P}(\mathfrak{H})$ is modular.*

B The Tensor Product Form of the Standard Decoherence Functional

In this appendix we give some remarks on the proof of Theorem 3.15.

Since the state ϱ is a density operator, there exists an orthonormal basis $\{|e_i^\varrho\rangle\}_i$ of $\mathfrak{H}$ and positive numbers $\{\omega_i\}$ such that $\sum_i \omega_i = 1$ such that ϱ can be written as $\varrho = \sum_i \omega_i |e_i^\varrho\rangle\langle e_i^\varrho|$. Now let h, k be homogeneous finite histories and let $m := \#\mathfrak{s}(h)$ and $n := \#\mathfrak{s}(k)$.

For every $k \in \mathbb{N}$, denote by $\otimes_{al}^k \mathfrak{H}$ the k-fold algebraic tensor product of $\mathfrak{H}$, the set of all finite sums of homogeneous vectors of the form $\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k$, where $\phi_j \in \mathfrak{H}$. Define an operator $S_k : \otimes_{al}^k \mathfrak{H} \to \otimes_{al}^k \mathfrak{H}$ by $S_k(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k) := \phi_2 \otimes \cdots \otimes \phi_k \otimes \phi_1)$ and extend by linearity. Moreover, define an operator $R_k : \otimes_{al}^k \mathfrak{H} \to \otimes_{al}^k \mathfrak{H}$ by $R_k(\phi_1 \otimes \phi_2 \otimes \cdots \phi_{k-1} \otimes \phi_k) := \phi_k \otimes \phi_{k-1} \otimes \cdots \otimes \phi_2 \otimes \phi_1)$ and extend by linearity. In the sequel we assume without loss of generality $\mathfrak{s}(h) = \mathfrak{s}(k)$. This can always be achieved by inserting the $1 \in \mathcal{P}(\mathfrak{H})$ appropriately. In particular, $m = n$. We denote the unit operator on $\otimes^k \mathfrak{H}$ by 1_k.

A straightforward computation now shows that $\mathrm{d}_\varrho(h, k)$ can be written as

$$\mathrm{d}_\varrho(h, k) = \mathrm{tr}_{\otimes^{2n}\mathfrak{H}} \left((h \otimes k)(U^\dagger_{t_1,t_2,\ldots,t_n} \otimes U^\dagger_{t_1,t_2,\ldots,t_n}) \mathfrak{Y}_\varrho (U_{t_1,t_2,\ldots,t_n} \otimes U_{t_1,t_2,\ldots,t_n}) \right),$$

where

$$U_{t_1,t_2,\ldots,t_n} := U(t_0,t_1) \otimes U(t_0,t_2) \otimes \cdots \otimes U(t_0,t_n),$$

with some fiducial time t_0 and where

$$\mathfrak{Y}_\varrho = (R_n \otimes 1_n)(1_{2n-1} \otimes \varrho) S_{2n} (R_n \otimes 1_n).$$

Inserting resolutions of the identity for 1_{2n-1} shows that there exist $2n$ orthonormal bases $\{|e^k_{i_k}\rangle\}$, $k \in \{1, \ldots, 2n\}$, of $\mathfrak{H}$ such that

$$\mathfrak{Y}_\varrho = \sum_{i_1,\ldots,i_{2n}} \omega_{i_{2n}} \Big\{ |e^n_{i_n}\rangle\langle e^{n-1}_{i_{n-1}}| \otimes |e^{n-1}_{i_{n-1}}\rangle\langle e^{n-2}_{i_{n-2}}| \otimes \cdots \otimes |e^1_{i_1}\rangle\langle e^{2n}_{i_{2n}}| \otimes$$
$$\otimes |e^{n+1}_{i_{n+1}}\rangle\langle e^n_{i_n}| \otimes |e^{n+2}_{i_{n+2}}\rangle\langle e^{n+1}_{i_{n+1}}| \otimes \cdots \otimes |e^{2n}_{i_{2n}}\rangle\langle e^{2n-1}_{i_{2n-1}}| \Big\}.$$

The orthonormal bases $\{|e^j_{i_j}\rangle\}$, $j \in \{1, \ldots, 2n-1\}$ are completely arbitrary, whereas necessarily $|e^{2n}_i\rangle = |e^\varrho_i\rangle$ for all i.

Obviously, $\mathfrak{Y}_\varrho$ is a trace class operator on $\otimes^{2n}\mathfrak{H}$ if $\mathfrak{H}$ is finite dimensional.

Now, using the abbreviation $\mathcal{U}_n := U_{t_1,t_2,\ldots,t_n}$, $\mathsf{d}_\varrho(h,k)$ can also be written as

$$\mathsf{d}_\varrho(h,k) = \sum_{i_1,\ldots,i_{2n}} \omega_{i_{2n}} \Big\{ \langle e^{n-1}_{i_{n-1}} \otimes e^{n-2}_{i_{n-2}} \otimes \cdots \otimes e^{2n}_{i_{2n}} | \mathcal{U}_n h \mathcal{U}_n^\dagger | e^n_{i_n} \otimes e^{n-1}_{i_{n-1}} \otimes \cdots \otimes e^1_{i_1}\rangle \times$$
$$\times \langle e^n_{i_n} \otimes e^{n+1}_{i_{n+1}} \otimes \cdots \otimes e^{2n-1}_{i_{2n-1}} | \mathcal{U}_n k \mathcal{U}_n^\dagger | e^{n+1}_{i_{n+1}} \otimes e^{n+2}_{i_{n+2}} \otimes \cdots \otimes e^{2n}_{i_{2n}}\rangle \Big\}.$$

By choosing $e^{n-j}_{n-j} = e^{n+j}_{i_{n+j}}$ for all $j \in \{0, \ldots, n-1\}$, one immediately sees from this representation that $\mathsf{d}_\varrho(h,k) = \mathsf{d}_\varrho(k,h)^*$ and that also the extension d_ϱ of d_ϱ satisfies $d_\varrho(h,k) = d_\varrho(k,h)^*$ and $d_\varrho(h,h) \geq 0$ for arbitrary inhomogeneous histories $h, k \in \mathcal{P}^\otimes_{fin}(\mathfrak{H})$.

If $\mathfrak{H}$ is infinite dimensional, then one can perform the same manipulations but has to be aware that $\mathfrak{Y}_\varrho$ will not be of trace class. In [45] it is shown that $\mathfrak{Y}_\varrho$ is a bounded operator on $\otimes^{2n}\mathfrak{H}$. So, the sum defining $\mathsf{d}_\varrho(h,k)$ will diverge for some inhomogeneous and infinite dimensional h and k. We conclude that d_ϱ can be extended to a functional $d_\varrho : \mathcal{P}^\otimes_{fin}(\mathfrak{H}) \times \mathcal{P}^\otimes_{fin}(\mathfrak{H}) \to \mathbb{C} \cup \{\infty\}$ which is orthoadditive in each argument in its finite sector. The details can be found in [45].

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