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On p-sparse Schrödinger operators with quasiperiodic potentials

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Abstract. We apply a decomposition approach of Guille-Biel to p-sparse Schrödinger operators with quasiperiodic potentials. A general extension principle is presented. Applications include extensions of results for the almost Mathieu operator and Fibonacci-type operators.

1 Introduction

In a recent paper [6], Guille-Biel introduced the notion of a *p-sparse Schrödinger operator*, which provides a generalization of the standard one-dimensional discrete Schrödinger operator

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n), \tag{1.1}$$

namely, the following higher order difference operator

$$(H_p u)(n) = u(n+p) + u(n-p) + V(n)u(n), \tag{1.2}$$

where $p \in \mathbf{N}$, and V is a bounded, real-valued potential. She proposed an approach how to study spectral properties of the operator H_p and particularly studied the case where V depends on a parameter ω from a probability space Ω such that the family $\{V_\omega\}_{\omega\in\Omega}$ is ergodic with respect to the shift operator. This approach is based on a decomposition of H_p into an orthogonal sum such that the fiber operators are standard operators corresponding to the case $p = 1$. This decomposition approach was applied to the cases where V is periodic, of Anderson type, or coming from a substitution sequence. Some of the results for these classes of potentials could be extended to suitable p-sparse versions. Furthermore, it was conjectured in [6] that this approach should also be applicable to quasiperiodic V.

Our aim is to show that this is indeed the case and, moreover, there is a general extension principle. In fact, this extension is rather straightforward once it is realized that the potentials of the fiber oprators retain the quasiperiodic form. We will therefore study operators of the form

$$(H_p^{\lambda,\alpha,\omega}u)(n) = u(n+p) + u(n-p) + \lambda f(n\alpha + \omega)u(n) \tag{1.3}$$

in $l^2(\mathbf{Z})$, where f is a real-valued, bounded, measurable, 1-periodic function, $\lambda \in \mathbf{R}$, $\alpha \in (0,1)$ irrational and $\omega \in \Omega = \mathbf{T} = \mathbf{R}/\mathbf{Z} \cong [0,1)$.

Primary and best studied examples are the almost Mathieu operator and the Fibonacci operator, corresponding to $p = 1$ and $f(\cdot) = \cos(2\pi(\cdot))$, resp. $f(\cdot) = \chi_{[1-\alpha,1)}(\cdot \ mod\, 1)$ (and, usually, $\alpha = \frac{\sqrt{5}-1}{2}$). Many results have been obtained for these operators. The very nice review articles by Last [21], Jitomirskaya [11] and Sütő [27] give an excellent overview. Without attempting to summarize the results obtained so far, let us remark that in the almost Mathieu case the spectral properties of $H_1^{\lambda,\alpha,\omega}$ depend on the modulus of the coupling constant λ. For $|\lambda| < 2$ one has absolutely continuous spectrum (possibly along with some singular continuous spectrum), whereas, for $|\lambda| > 2$ one has sometimes pure point spectrum, sometimes purely singular continuous spectrum, but always zero-dimensional spectral measures [22, 12, 13]. At the self-dual point $|\lambda| = 2$, Gordon *et al.* have proven singular continuity of the spectral measures [8] for a.e. α, ω. For the Fibonacci operator, on the other hand, the spectral properties do not depend qualitatively on the coupling constant. Sütő [25, 26], Bellissard *et al.* [2] and Kaminaga [16] have shown that, for every $\lambda \neq 0$, the spectral measures are purely singular continuous (again, for a.e. ω). Quantitatively, however, some results do depend on λ, for example the polynomial upper bound for the norms of the transfer matrices corresponding to energies from the spectrum, as proven by Iochum *et al.* [9, 10], the lower bound for the Hausdorff dimensionality of the spectral measures, as proven by Jitomirskaya and Last [12, 13] (see also [3]), or the upper bound for the Hausdorff dimension of the spectrum, as proven by Raymond [24]. Furthermore, there are results on the Lebesgue measure of the spectrum in both cases, which is $|4-2|\lambda||$ in the almost Mathieu case (for a.e. α) [19, 20, 14] and zero in the Fibonacci case [2, 26].

We will be able to derive results for the p-sparse versions of almost Mathieu and Fibonacci-type operators from the results mentioned above by employing Guille-Biel's decomposition approach.

The organization of this article is as follows. In Section 2, we present the relevant part of the decomposition theory and prove the general extension principle. Applications of this principle are then given in Sections 3 and 4. Finally, Section 5 contains some concluding remarks.

2 Decomposition of the operator and the extension principle

Let p be fixed throughout this section and let H_p be a p-sparse Schrödinger operator. Define for $i \in \{0, \dots, p-1\}$ the following subspaces of $l^2(\mathbf{Z})$,

$$\mathcal{K}_i \equiv lin\{e_{mp+i} \ : \ m \in \mathbf{Z}\},$$

where $\{e_m\}_{m \in \mathbf{Z}}$ is the canonical orthonormal basis of $l^2(\mathbf{Z})$, i.e. $e_m(n) = \delta_{n,m}$. The following properties are obvious.

- The $\mathcal{K}_i$ are mutually orthogonal.
- $l^2(\mathbf{Z})$ is their orthogonal direct sum: $l^2(\mathbf{Z}) = \bigoplus_{i=0}^{p-1} \mathcal{K}_i$.
- For every i, $l^2(\mathbf{Z})$ is isometrically isomorphic to $\mathcal{K}_i$.
- For every i, $\mathcal{K}_i$ reduces H_p, and therefore, $H_p = \bigoplus_{i=0}^{p-1} \hat{H}_{p,i}$, where $\hat{H}_{p,i} \equiv H_{p|\mathcal{K}_i}$.

Define the operators $H_{p,i}$ by

$$(H_{p,i}u)(n) = u(n+1) + u(n-1) + V_{p,i}(n)u(n), \tag{2.1}$$

where $V_{p,i}(n) \equiv V(np+i)$. It is now straighforward to show that $\hat{H}_{p,i}$ and $H_{p,i}$ are unitarily equivalent, for a proof see [6]. We have thus obtained a representation of H_p as an orthogonal sum of standard discrete one-dimensional Schrödinger operators. If the potential V is such that the fiber potentials $V_{p,i}$ yield operators $H_{p,i}$ which are well studied already, this decomposition enables us to derive results for H_p.

Let us remark that we have presented a deterministic version of Guille-Biel's decomposition. Of course, p-sparse Schrödinger operators fit into the framework of random operators, but if the $p = 1$ case is studied well enough, we obtain results for the general case by pointwise extension (i.e. separately for every ω). We will illustrate this below.

Let now a quasiperiodic operator of the form (1.3) be given. We are going to state the extension principle in Theorem 1 below. This principle extends properties of H_1 which hold (at least) for a.e. ω to the higher order operators H_p, where the set of α's for which the property holds has to be changed. Let us say that a property $\mathcal{P}$ is called *stable under direct sums* iff for every pair of selfadjoint operators A_1, A_2 the following holds:

$$A_1 \text{ and } A_2 \text{ satisfy } \mathcal{P} \Rightarrow A_1 \oplus A_2 \text{ satisfies } \mathcal{P}.$$

Think, for example, of $\mathcal{P} - \sigma_\varepsilon(\cdot) = \emptyset$, $\varepsilon \in \{pp, sc, ac\}$.

Theorem 1 *Let $\mathcal{P}$ be a property which is stable under direct sums. Then, for every $\lambda \in \mathbf{R}$, the following holds. Let $S_\lambda \subseteq \mathbf{R}$ be such that*

$$\alpha \in S_\lambda \Rightarrow H_1^{\lambda,\alpha,\omega} \text{ has the property } \mathcal{P} \text{ for a.e. } \omega. \tag{2.2}$$

Then,

$$p\alpha \in S_\lambda \Rightarrow H_p^{\lambda,\alpha,\omega} \text{ has the property } \mathcal{P} \text{ for a.e. } \omega. \tag{2.3}$$

If $\mathcal{P}$ holds everywhere, rather than almost everywhere, in (2.2), then the same is true in (2.3).

Proof. Fix $\lambda \in \mathbf{R}$ and let $p\alpha \in S_\lambda$. The potentials of the fiber operators $H_{p,i}^{\lambda,\alpha,\omega}$ introduced above are given by

$$V_{p,i}^{\lambda,\alpha,\omega}(n) = \lambda f((np+i)\alpha + \omega) = \lambda f(n(p\alpha) + (i\alpha + \omega)).$$

By assumption, for every $i \in \{0, \ldots, p-1\}$, $H_{p,i}^{\lambda,\alpha,\omega}$ satisfies $\mathcal{P}$ for a.e. (resp., every) ω . Thus, for a.e. (resp., every) ω, the property $\mathcal{P}$ is satisfied by all $H_{p,i}^{\lambda,\alpha,\omega}$, $i \in \{0, \ldots, p-1\}$. Stability of $\mathcal{P}$ now implies that for those ω

$$\bigoplus_{i=0}^{p-1} H_{p,i}^{\lambda,\alpha,\omega}$$

satisfies $\mathcal{P}$, concluding the proof. □

Remark. The proof is so simple because it was easily seen that the fiber operators are also quasiperiodic. The situation is less simple in the case where the potential is generated by a substitution, compare [6].

3 Extensions of Fibonacci-type results

In this section, we consider operators of the form

$$(H_p^{\lambda,\alpha,\omega}u)(n) = u(n+p) + u(n-p) + \lambda\chi_J(n\alpha + \omega \bmod 1)u(n), \tag{3.1}$$

where J is a half-open interval in $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

Some results require certain number theoretical properties of α. We refer the reader to the monographs by Lang [18] and Khintchine [15] for the necessary background.

The following series of Corollaries can be obtained.

Corollary 1 *If $\lambda \neq 0$, then, for every $p \in \mathbf{N}$, α irrational and $\omega \in \Omega$, $\sigma_{ac}(H_p^{\lambda,\alpha,\omega}) = \emptyset$.*

Proof. By Kotani [17] and Last-Simon [23], absence of absolutely continuous spectrum holds for all fiber operators. Since absence of absolutely continuous spectrum is stable under direct sums, the assertion follows from Theorem 1. □

Corollary 2 *Let (a_n) be the coefficients in the continued fraction expansion of $p\alpha$. If $\limsup a_n \geq 4$, then, for every $\lambda \in \mathbf{R}$, $\sigma_{pp}(H_p^{\lambda,\alpha,\omega}) = \emptyset$ holds for a.e. $\omega \in \Omega$.*

Proof. By Kaminaga [16] (see also [5]), the assumptions of Theorem 1 are satisfied. □

Remarks.

1. The last two corollaries can be generalized to the case where J is a finite union of half-open intervals. The potential can even take different values on these intervals. In the assumption of Corollary 2 the condition has to be changed to $\limsup a_n \geq 4\times$ *the number of intervals*, compare [5].
2. We see that, as a rule, quasiperiodic potentials taking finitely many values seem to yield purely singular continuous spectrum. No exception to this rule is known yet. That is, it is still open if there exist $p, J, \lambda, \alpha, \omega$ such that $\sigma_{pp}(H_p^{\lambda,\alpha,\omega}) \neq \emptyset$.

Corollary 3 *If $J = [1 - p\alpha, 1)$, then, for every $\lambda \in \mathbf{R}$, $\sigma_{pp}(H_p^{\lambda,\alpha,\omega}) = \emptyset$ holds for a.e. $\omega \in \Omega$.*

Proof. Again by [16], the assumptions of Theorem 1 are satisfied. □

Corollary 4 *If $J = [1 - p\alpha, 1)$, then, for every $\lambda \neq 0, \omega \in \Omega$, $\sigma(H_p^{\lambda,\alpha,\omega})$ has Lebesgue measure zero.*

Proof. Bellissard *et al.* have shown that if the length of the interval J coincides with the (irrational) rotation number α, then, for every $\lambda \neq 0, \omega \in \Omega$, $\sigma(H_1^{\lambda,\alpha,\omega})$ has Lebesgue measure zero [2]. This is clearly a property which is stable under direct sums. By assumption, the length condition is obeyed by the fiber operators. Thus, Theorem 1 can be applied. □

4 Extensions of almost Mathieu results

In this section, we shall apply Theorem 1 to the case $f(\cdot) = \cos(2\pi(\cdot))$. We therefore consider the operators

$$(H_p^{\lambda,\alpha,\omega}u)(n) = u(n+p) + u(n-p) + \lambda\cos(2\pi(n\alpha+\omega))u(n). \tag{4.1}$$

The following series of Corollaries can be obtained.

Corollary 5 *If* $|\lambda| > 2$, *then, for every* $p \in \mathbf{N}$, α *irrational and* $\omega \in \Omega$, $\sigma_{ac}(H_p^{\lambda,\alpha,\omega}) = \emptyset$.

Proof. An already classical result gives absence of absolutely continuous spectrum for the fiber operators almost everywhere in Ω, see, e.g., [11] for references. Again, the result by Last-Simon [23] extends this to all ω. Apply Theorem 1. □

Corollary 6 *If* $|\lambda| < 2$, *then* $\sigma_{pp}(H_p^{\lambda,\alpha,\omega}) = \emptyset$ *for every* p, α *and* ω.

Proof. The assertion follows from [4] and Theorem 1. □

Another general result on the absence of eigenvalues is given in

Corollary 7 *If* $p\alpha$ *is a Liouville number, then* $\sigma_{pp}(H_p^{\lambda,\alpha,\omega}) = \emptyset$ *for every* λ, ω.

Proof. Avron and Simon [1] proved absence of eigenvalues in the case $p = 1$ by verifying Gordon's condition [7] for Liouville frequencies. Theorem 1 then yields the result. □

Remark. The results contained in Corollaries 5 and 7 provide explicit examples with purely singular continuous spectrum.

Corollary 8 *If* $|\lambda| = 2$, *then, for every* p *and a.e.* α, ω, *the spectrum of* $H_p^{\lambda,\alpha,\omega}$ *is purely singular continuous and has Lebesgue measure zero.*

Proof. Apply Theorem 1 together with [8, 20]. □

5 Concluding remarks

We have seen how the extension principle stated in Theorem 1 provides a mechanism for producing results for p-sparse operators from results in the standard case which hold (at least) almost everywhere. This is particularly nice because the spectral theoretical machinery is much more developed for the classical case $p = 1$. It is far from obvious how to obtain results of the type presented in Sections 3 and 4 directly, that is, by applying higher order methods to H_p instead of considering the decomposition introduced by Guille-Biel.

The list of applications we have presented serves rather as an illustration of the usefulness and applicabilty of Guille-Biel's decomposition along with the extension principle and we have by no means aimed at completeness. In particular the set of almost Mathieu results in the literature provides much more possibilities to formulate further Corollaries for p-sparse versions, but this would be quite pointless.

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