

**Zeitschrift:** Helvetica Physica Acta

**Band:** 71 (1998)

**Heft:** 5

**Artikel:** Yangian  $Y(\mathfrak{sl}(2))$  in Coulomb problem

**Autor:** Zhang, Shengli

**DOI:** <https://doi.org/10.5169/seals-117122>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 16.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Yangian $Y(sl(2))$ in Coulomb Problem

By Shengli Zhang

Department of Applied Physics, Xian Jiaotong University, Xi'an 710049  
People's Republic of China.

(31.III.98)

*abstract* In this paper, the Yangian  $Y(sl(2))$  is shown existing in the system that a particle moves in Coulomb field. The generators of  $Y(sl(2))$  are constructed in terms of the angular momentum operators and so-called Yangian Runge-Lenz vector. The selection rule and matrix element of  $Y(sl(2))$  generators are calculated.

Yangian has drawn much attention in mathematical physics in recent years. It was introduced by Drinfeld [1]. Researchers found there is Yangian symmetry in some practicable physical models [2-6], such as long-range interaction models, one-dimensional Hubbard model, two-dimensional sigma model, and two-dimensional chiral model with or without topological terms. It is more significant that Yangian may play an important role in exploring the multiparton amplitudes in QCD and QED [7].

The Yangian  $Y(sl(2))$  will be studied in the Coulomb problem from the point of view of quantum mechanics in this paper. It is well-known that the dynamical symmetry group of the Coulomb problem is  $SO(4, 2)$ . And there is the  $SO(4)$  symmetry that is used to determine the energy level in this system [8], which is called energy symmetry. However, the angular momentum and Runge-Lenz vector operators, which are looked as  $SO(4)$  generators, become unclosed to the commutative product when the energy is not fixed. In above mentioned sense, those operators cannot form the  $so(4)$  algebra. It is necessary to approach what is the algebra of the operators.

We will research the algebra in the Coulomb problem. It will be seen that the angular

momentum and Runge-Lenz vector operators generate the subalgebra of the loop algebra corresponding to  $sl(2)$ , the limitation of Yangian  $Y(sl(2))$  when the quantum deformed parameter  $\alpha$  vanish. As the q-hydrogen atom [9], the  $Y(sl(2))$  will be realized in terms of the angular momentum  $\mathbf{L}$  and Runge-Lenz vector  $\mathbf{R}$ .

It is well known that the Hamiltonian of the Coulomb problem reads

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{e^2}{|\mathbf{r}|} \quad (1)$$

where  $\mu$  and  $e$  stand for the mass and electric charge of the electron respectively. It has been pointed out [7] that besides the angular momentum  $\mathbf{L}$ , the Runge-Lenz vector  $\mathbf{R}$  is another conserved quantity, which satisfy

$$[\mathbf{L}, H] = [\mathbf{R}, H] = 0 \quad (2)$$

and

$$\mathbf{R} = \frac{1}{2\mu e^2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{\mathbf{r}}{r}. \quad (3)$$

Both  $\mathbf{L}$  and  $\mathbf{R}$  generate the  $SO(4)$  for the fixed energy [8].

Drinfeld [1] defined the Yangian  $Y(sl(2))$  as being generated by the generators  $\{I_{\pm}, I_3\}$  and  $\{J_{\pm}, J_3\}$  with the commutation relation

$$\begin{aligned} [I_3, I_{\pm}] &= \pm I_{\pm}, \quad [I_+, I_-] = 2I_3 \\ [I_3, J_{\pm}] &= [J_3, I_{\pm}] = \pm J_{\pm}, \quad [I_{\pm}, J_{\mp}] = \pm 2J_3 \\ [I_3, J_3] &= [I_{\pm}, J_{\pm}] = 0 \end{aligned} \quad (4)$$

and

$$\begin{aligned} [J_3, [J_+, J_-]] &= \frac{\alpha^2}{4} I_3 (J_- I_+ - I_- J_+) \\ [J_{\pm}, [J_3, J_{\pm}]] &= \frac{\alpha^2}{4} I_{\pm} (J_{\pm} I_3 - I_{\pm} J_3) \\ 2[J_3, [J_3, J_{\pm}]] \pm [J_{\pm}, [J_{\pm}, J_{\mp}]] &= \frac{\alpha^2}{2} \{I_3 (J_{\pm} I_3 - I_{\pm} J_3) + I_{\pm} (I_- J_+ - J_- I_+)\} \end{aligned} \quad (5)$$

where  $\alpha$  is the deformation parameter.

Equation (4) shows that  $I_a$  itself obeys the same commutation relations as  $L_a$ . Therefore, we can directly assign

$$I_3 = L_3, \quad I_{\pm} = L_{\pm} = L_1 \pm iL_2. \quad (6)$$

In order to realize  $Y(sl(2))$ , we must determine the generators  $J_3$  and  $J_{\pm}$ . These generators are called Yangian Runge-Lenz vector  $\mathbf{R}^Y$  in this paper. It will be shown that the usual Runge-Lenz vector is the limitation of this Yangian Runge-Lenz vector when  $\alpha \rightarrow 0$ . Our intention is to find this deformed Runge-Lenz vector, which play the role of generators  $J_3, J_{\pm}$

in Yangian  $Y(sl(2))$ , i.e.  $\mathbf{J} = \mathbf{R}^Y$ . Since the commutation relations of  $J_3$  and  $J_{\pm}$  with  $I_3$  and  $I_{\pm}$  are similar to those of  $I_3$  and  $I_{\pm}$  itself, we have

$$\mathbf{J} = \mathbf{R}^Y = \mathbf{R} + \alpha \mathbf{Q} \quad (7)$$

in which  $\mathbf{Q}$  is a vector operator to be found. This expression make the limitation of  $\mathbf{R}^Y$  to be  $\mathbf{R}$  under  $\alpha \rightarrow 0$ .

It can be verified that Runge-Lenz vector satisfy

$$\begin{aligned} [L_3, R_{\pm}] &= [R_3, L_{\pm}] = R_{\pm} \\ [L_3, R_3] &= [L_{\pm}, R_{\pm}] = 0, \quad [L_{\pm}, R_{\mp}] = \pm 2R_3, \end{aligned} \quad (8)$$

with

$$R_{\pm} = R_1 \pm iR_2.$$

Therefore, the generators  $\{Q_3, Q_{\pm}\}$  may also be determined by the angular momentum and Runge-Lenz vector. Through analysing the equations (5), (7), (8) and the commutation relation of angular momentum, we find that the generators  $\{Q_3, Q_{\pm}\}$  can be chosen as

$$\begin{aligned} Q_3 &= [\mathbf{L}^2, R_3] = \frac{1}{2}(L_+R_- - R_+L_-) \\ Q_{\pm} &= [\mathbf{L}^2, R_{\pm}] = \pm(L_3R_{\pm} - R_3L_{\pm}) \end{aligned} \quad (9)$$

From (7) and (9), the Yangian Runge-Lenz vector can be defined

$$\mathbf{R}^Y = \mathbf{R} + \alpha[\mathbf{L}^2, \mathbf{R}] \quad (10)$$

Because  $L_a$  and  $R_a$  are conserved for Hamiltonian (1), i.e. they obey the commutation relation (2), one easily asserts that the generators  $\{R_3^Y, R_{\pm}^Y\}$  are also conserved. They satisfy

$$[H, R_3^Y] = [H, R_{\pm}^Y] = 0 \quad (11)$$

Therefore, there is Yangian  $Y(sl(2))$  symmetry in the Coulomb problem.

In following part, the operators (6) and (10) are verified obeying the commutation relation of  $Y(sl(2))$ , i.e. equations (4) and (5). Because of the definition (6), in which  $L_a$  ( $a = 1, 2, 3$ ) are usual angular momentum, the operators  $L_3$  and  $L_{\pm}$  must satisfy the first two formulas of (4) obviously. Using (6), (7) and (8), one directly has  $[L_3, R_{\pm}^Y] = \pm R_{\pm}^Y$ . The other formulas of (4) can also verified similarly. Now, let us consider the commutation relations of the generators  $\mathbf{R}^Y$ . After tedious calculation by making use of (7), (8) and

$$[R_a, R_b] = -\frac{i2}{\mu e^4} H \varepsilon_{abc} L_c$$

we have

$$[R_+^Y, R_-^Y] = \frac{4H\alpha^2}{\mu e^4} L_3(2\mathbf{L}^2 + 1) + 2(\alpha^2 - \frac{2H}{\mu e^4}) L_3 \quad (12)$$

Since  $[L_3, R_3^Y] = 0$  and

$$[\mathbf{L}^2, R_3^Y] = R_-^Y L_+ - L_- R_+^Y,$$

which can be proved by means of directly calculation, it is derived that

$$[R_3^Y, [R_+^Y, R_-^Y]] = \frac{8\alpha^2 H}{\mu e^4} L_3 (R_-^Y L_+ - L_- R_+^Y). \quad (13)$$

Comparing with the first equation of (5), we see that the generators (6) and (10) indeed satisfy the first commutation relations of  $Y(sl(2))$ . In equatuion (13), the hamiltonian  $H$  can be looked upon as a constant. This is due to its commuting with all generators of  $Y(sl(2))$  in our discussion. Similarly, we also determine the other commutation relations of the generators  $\mathbf{R}^Y$  as follows,

$$[R_3^Y, R_\pm^Y] = \pm \frac{-2H}{\mu e^4} L_\pm \pm \frac{2\alpha^2 H}{\mu e^4} L_\pm (2\mathbf{L}^2 + 1) \pm \alpha^2 L_\pm \quad (14)$$

And we obtain

$$[R_\pm^Y, [R_3^Y, R_\pm^Y]] = \frac{8\alpha^2 H}{\mu e^4} L_\pm (R_\pm^Y L_3 - L_\pm R_3^Y). \quad (15)$$

We have applied the equation the commutation relations between the angular momentum and the Runge-Lenz vector, and

$$[\mathbf{L}^2, R_\pm^Y] = 2(R_\pm^Y L_3 - L_\pm R_3^Y)$$

in derivating (15). It can been seen that the commutation relation (15) corresponds to the second equation of (5). If we use the commutation relations (12) and (14), the last commutation relation of  $Y(sl(2))$  can be determined as follows:

$$\begin{aligned} & 2[R_3^Y, [R_3^Y, R_\pm^Y]] \pm [R_\pm^Y, [R_\pm^Y, R_\mp^Y]] \\ &= \frac{4\alpha^2 H}{\mu e^4} \{L_3 (R_\pm^Y L_3 - L_\pm R_3^Y) + L_\pm (L_- R_+^Y - R_-^Y L_+)\} \end{aligned} \quad (16)$$

When we consider our realization (6) and (10) for fixed energy, the Hamiltonian in (13), (15) and (16) can be replaced by the energy eigenvalue as the discussion of energy level symmetry in usual hydrogen atom. We see that the energy plays the similar role as the deformation parameter in above mentioned case. In the limitation  $E_n \rightarrow 0$ , i.e.  $n \rightarrow \infty$ , the Yangian  $Y(sl(2))$  becomes a subalgebra of the loop algebra  $L(sl(2))$ . Therefore, the symmetrical algebraic structure in the Coulumb problem is different at the energy critical point at which transition of the state of the system happen from bounded states to free states. The reduced algebra satisfy

$$\begin{aligned} & [L_3, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_3 \\ & [L_3, R_\pm^Y] = [R_3^Y, L_\pm] = \pm R_\pm^Y, \quad [L_\pm, R_\mp^Y] = \pm 2R_3^Y \\ & [L_3, R_3^Y] = [L_\pm, R_\pm^Y] = 0 \end{aligned}$$

and

$$[R_3^Y, [R_+^Y, R_-^Y]] = 0$$

$$[R_{\pm}^Y, [R_3^Y, R_{\pm}^Y]] = 0 \quad (17)$$

$$2[R_3^Y, [R_3^Y, R_{\pm}^Y]] \pm [R_{\pm}^Y, [R_3^Y, R_{\mp}^Y]] = 0.$$

It should be pointed out that above equations hold true only for the case  $E = 0$ . The deformation parameter  $\alpha$  does not vanish. On the other hand, if the deformation parameter is zero, the above-mentioned Yangian  $Y(sl(2))$  also reduces to the same subalgebra of the loop algebra  $L(sl(2))$ .

Now, let us study the selection rule of the generator of  $Y(sl(2))$ . It is well known that the spherical harmonics  $Y_{lm}(\theta, \phi) = |lm\rangle$  are the eigenstates of  $L_z$  and  $\mathbf{L}^2$ . From the equation (6), we have

$$I_3|lm\rangle = m|lm\rangle$$

$$I_{\pm}|lm\rangle = [(l \mp m)(l \pm m + 1)]^{1/2}|lm\rangle \quad (18)$$

Furthermore, using (4) and (7), we can derive

$$\langle l'm'|R_{\pm}^Y|lm\rangle = \delta_{m'm\pm 1}\langle l'm \pm 1|R_{\pm}^Y|lm\rangle \quad (19)$$

and

$$\langle l'm'|R_3^Y|lm\rangle = \delta_{m'm}\langle l'm|R_3^Y|lm\rangle \quad (20)$$

respectively. The above two formulas just indicate the selection rules of the generators  $\mathbf{R}^Y$ . From the commutation relation  $[I_+, J_+] = [I_-, J_-] = 0$ , we can determine the recurrence relation of the matrix elements of generators  $\mathbf{R}^Y$  as:

$$\langle l'm + 1|R_+^Y|lm\rangle = \sqrt{\frac{(l' - m)(l' + m + 1)}{(l + m)(l - m + 1)}}\langle l'm|R_+^Y|lm - 1\rangle$$

$$\langle l'm - 1|R_-^Y|lm\rangle = \sqrt{\frac{(l' + m)(l' - m + 1)}{(l - m)(l + m + 1)}}\langle l'm|R_-^Y|lm + 1\rangle$$

Making use of  $[L_+, R_-^Y] = [R_+^Y, L_-] = 2J_3$ , we also obtain

$$2\langle l'm|R_3^Y|lm\rangle = \sqrt{(l' + m)(l' - m + 1)}\langle l'm - 1|R_-^Y|lm\rangle$$

$$- \sqrt{(l - m)(l + m + 1)}\langle l'm|R_-^Y|lm + 1\rangle$$

$$= \sqrt{(l + m)(l - m + 1)}\langle l'm|R_+^Y|lm - 1\rangle - \sqrt{(l' - m)(l' + m + 1)}\langle l'm + 1|R_+^Y|lm\rangle$$

Equation (18) shows that the generators  $\mathbf{I}$  represent the transition in same quantum number  $l$ . And equations (19) and (20) indicate that the generators  $\mathbf{R}^Y$  may play the role of transition between states with different angular quantum number  $l$ , which selection rule is similar to the generators  $\mathbf{I}$  for the magnetic quantum number  $m$ . It must be pointed out the above conclusion is universal for  $Y(sl(2))$ , not confined to the realization of (6) and (10).

It is well known that the Hilbert space of the Coulomb problem is given by the wave function

$$|nlm\rangle = R_{nl}(r)Y_{lm}(\theta, \phi)$$

where

$$R_{nl}(r) = -\left\{ \left( \frac{2}{na_0} \right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3} \right\}^{1/2} e^{-\frac{r}{na_0}} \left( \frac{2r}{na_0} \right)^l L_{n+l}^{2l+1} \left( \frac{2r}{na_0} \right)$$

and  $L_{n+l}^{2l+1}$  stands for the associated Laguerre polynomial. Using our realization (6) and (10), we have derived the commutator brackets (12) and (14). From the commutation relations (12), (14) and selection rules (19) and (20), one has

$$\begin{aligned} & \langle n'l'm | R_+^Y | n''l''m-1 \rangle \langle n''l''m-1 | R_-^Y | nlm \rangle - \langle n'l'm | R_-^Y | n''l''m+1 \rangle \langle n''l''m+1 | R_+^Y | nlm \rangle \\ &= \delta_{nn'} \delta_{ll'} m \left\{ \frac{4\alpha^2 E_n}{\mu e^4} [2l(l+1)+1] + 2(\alpha^2 - \frac{2E_n}{\mu e^4}) \right\} \end{aligned} \quad (21)$$

and

$$\begin{aligned} & \langle n'l'm \pm 1 | R_3^Y | n''l''m \pm 1 \rangle \langle n''l''m \pm 1 | R_{\pm}^Y | nlm \rangle - \langle n'l'm \pm 1 | R_{\pm}^Y | n''l''m \rangle \langle n''l''m | R_3^Y | nlm \rangle \\ &= \pm \delta_{nn'} \delta_{ll'} \sqrt{(l \mp m)(l \pm m+1)} \left\{ -\frac{2\alpha^2 E_n}{\mu e^4} + \frac{2E_n}{\mu e^4} [2l(l+1)+1] + \alpha^2 \right\} \end{aligned} \quad (22)$$

Since  $R_+^{Y+} = R_-^Y$ , the following relation

$$\langle n'l'm-1 | R_-^Y | nlm \rangle = \langle nlm | R_+^Y | n'l'm-1 \rangle,$$

holds true. Therefore, we can obtain

$$\langle n'l'm+1 | R_+^Y | nlm \rangle = \delta_{nn'} \delta_{ll'} \left\{ \frac{2\alpha^2 E_n}{\mu e^4} [2l(l+1)+1] + [\alpha^2 - \frac{2E_n}{\mu e^4}] \right\}^{1/2} \sqrt{(l-m)(l+m+1)} \quad (23)$$

and

$$\langle n'l'm-1 | R_-^Y | nlm \rangle = \delta_{nn'} \delta_{ll'} \left\{ \frac{2\alpha^2 E_n}{\mu e^4} [2l(l+1)+1] + [\alpha^2 - \frac{2E_n}{\mu e^4}] \right\}^{1/2} \sqrt{(l+m)(l-m+1)} \quad (24)$$

Furthermore, we have

$$\langle nlm | R_3^Y | n'l'm \rangle = \delta_{nn'} \delta_{ll'} m \left\{ \frac{2\alpha^2 E_n}{\mu e^4} [2l(l+1)+1] + [\alpha^2 - \frac{2E_n}{\mu e^4}] \right\}^{1/2} \quad (25)$$

This means that the  $Y(sl(2))$  generators realized by (6) and (10) also represent the transition within the states with same principal quantum number  $n$  and angular quantum number  $l$ . It must be pointed out above conclusion is consistent with the general discussion in the last paragraph.

When the quantum parameter  $\alpha$  vanishes, the generator (10) reduced to the usual Runge-Lenz vector, i.e.  $\mathbf{R}^Y \rightarrow \mathbf{R}$ . The commutation relation (13), (15) and (16) become

$$\begin{aligned} & [R_3, [R_+, R_-]] = 0 \\ & [R_{\pm}, [R_3, R_{\pm}]] = 0 \\ & 2[R_3, [R_3, R_{\pm}]] \pm [R_{\pm}, [R_{\pm}, R_{\mp}]] = 0. \end{aligned} \quad (26)$$

Equations (25), (8) and the commutation relation of angular momentum  $\mathbf{L}$  show that our realization of Yangian  $Y(sl(2))$  indeed reduce to the sub-algebra of loop algebra  $L(sl(2))$ . Correspondingly, the selection rule and matrix element given as above formulas are become those of usual Runge-Lenz vector in quantum mechanics [8]. It must be pointed out that equations (26) holds true for all energy eigenstate. This is different from the equations (17), which holds only for vanish energy. In (17), the generators are also different from this expression.

In this paper, we deform the Runge-Lenz vector to so-called Yangian Runge-Lenz vector in Coulomb problem. It is shown that the usual angular momentum and Yangian Runge-Lenz vector construct the Yangian  $Y(sl(2))$ . They are also the conservative quantities of the Coulomb problem. When the quantum deformation parameter  $\alpha$  vanishes, the Yangian Runge-Lenz reduces to the usual Runge-Lenz vector. For the state with fixed energy, the energy plays the role as a deformation parameter in our realization of the  $Y(sl(2))$  generators. It has been seen that the algebraic structures of the bounded state, free state and the critical point transiting from a bounded state to a free state, are different, which are Yangian  $Y(sl(1,1))$ ,  $Y(sl(2))$  and the loop algebra  $L(sl(2))$  (more pricisely, the sub-algebra of  $L(sl(2))$ ) [4] respectively. The selection rule and matrix elements of the  $Y(sl(2))$  generators for the eigenstates of the Coulomb problem have also been discussed.

## References

- [1] V. G. Drinfeld, Sov. Math. Dokl. **32** (1985) 254; Proc. ICM-86 (Berkeley, 1987) p. 798.
- [2] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard and Pasquier, Phys. Rev. Lett., **69** (1992) 2021; Y. Kato, Y. Kuramoto, Phys. Rev. Lett., **74** (1995) 1222; K. Hikami, J. Phys. **A28** (1995) L131.
- [3] D. B. Uglov and V. E. Korepin, Phys. Lett., **A190** (1994) 238; F. Gohmann and V. Inozemtsev, Phys. Lett., **A214** (1996) 161.
- [4] D. Bernard, Int. J. Mod. Phys., **B7** (1993) 3517.
- [5] D. Bernard, Commun. Math. Phys. **137** (1991) 191; K. Schoutens, Phys. Lett., **B331** (1994) 335.
- [6] S. L. Zhang, Int. J. Theor. Phys., bf 36 (1997) 2997.
- [7] W.A. Bardeen, Prog. Theor. Phys. Supp. No.123 (1996) 1; L. N. Lipatov, Talk in 1997 Nankai Symposium, (Tianjin 1997).
- [8] L. I. Schiff, Quantum Mechanics, (McGraw-Hill, 1968).
- [9] M. Kibler and T. Negadi, J. Phys., **A24** (1991) 2583; S. L. Zhang, Int. J. Theor. Phys., **34** (1995) 2217.